The $\overline{B}\rightarrow X_s\gamma$ photon spectrum

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The photon energy spectrum in inclusive weak radiative $\overline{B}\rightarrow X_s\gamma$ decay is computed to order $\alpha_s^2\beta_0$. This result is used to extract a value for the heavy quark effective theory (HQET) parameter $\overline{\Lambda}$ from the average $\langle 1-2E_\gamma/m_B\rangle$, and a value of the parameter λ_1 from $\langle (1-2E_\gamma/m_B)^2\rangle$. An accurate measurement of $\langle 1-2E_\gamma/m_B\rangle$ can determine the size of the nonperturbative contributions to the $\Upsilon(1S)$ mass which cannot be absorbed into the b quark pole mass. [S0556-2821(99)00217-9]

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Comparison of the measured weak radiative $\overline{B}\rightarrow X_s\gamma$ decay rate with theory is an important test of the standard model. In contrast with the decay rate itself, the shape of the photon spectrum is not expected to be sensitive to new physics, but it can, nevertheless, provide important information. First of all, studying the photon spectrum is important for understanding how precisely the total rate can be predicted in the presence of an experimental cut on the photon energy [1], which is important for a model independent interpretation of the resulting decay rate. Secondly, moments of the photon spectrum may be used to measure the heavy quark effective theory (HQET) parameters which determine the quark pole mass and kinetic energy [2,3], much like the shape of the lepton energy [4] or hadronic invariant mass [5] spectrum in semileptonic $\overline{B}\rightarrow X_c l\overline{\nu}$ decay. The main purpose of this paper is to present the order $\alpha_s^2\beta_0$ piece of the two-loop correction to the photon spectrum and to study its implications. A calculation to this order is required for a meaningful comparison of the HQET parameters extracted from $\overline{B}\rightarrow X_s\gamma$ with those from other processes.

To leading order in small weak mixing angles, the effective Hamiltonian is

$$H_{\rm eff}=-\frac{4G_F}{\sqrt{2}}V_{tb}V_{ts}^*\sum_{i=1}^{8}C_i(\mu)O_i\,, \tag{1}$$

where G_F is the Fermi constant, V_{ij} are elements of the Cabibbo-Kobayashi-Maskawa matrix, $C_i(\mu)$ are Wilson coefficients evaluated at a subtraction point μ, and O_i are the dimension six operators

$$O_1=(\overline{c}_{L\beta}\gamma^\mu b_{L\alpha})(\overline{s}_{L\alpha}\gamma_\mu c_{L\beta}),$$

$$O_2=(\overline{c}_{L\alpha}\gamma^\mu b_{L\alpha})(\overline{s}_{L\beta}\gamma_\mu c_{L\beta}),$$

$$O_3=(\overline{s}_{L\alpha}\gamma^\mu b_{L\alpha})\sum_q(\overline{q}_{L\beta}\gamma_\mu q_{L\beta}),$$

$$O_4=(\overline{s}_{L\alpha}\gamma^\mu b_{L\beta})\sum_q(\overline{q}_{L\beta}\gamma_\mu q_{L\alpha}),$$

$$O_5=(\overline{s}_{L\alpha}\gamma^\mu b_{L\alpha})\sum_q(\overline{q}_{R\beta}\gamma_\mu q_{R\beta}),$$

$$O_6=(\overline{s}_{L\alpha}\gamma^\mu b_{L\beta})\sum_q(\overline{q}_{R\beta}\gamma_\mu q_{R\alpha}),$$

$$O_7=\frac{e}{16\pi^2}m_b\overline{s}_{L\alpha}\sigma^{\mu\nu}b_{R\alpha}F_{\mu\nu},$$

$$O_8=\frac{g}{16\pi^2}m_b\overline{s}_{L\alpha}\sigma^{\mu\nu}T^a_{\alpha\beta}b_{R\beta}G^a_{\mu\nu}. \tag{2}$$

In Eq. (2), e is the electromagnetic coupling, g is the strong coupling, m_b is the b quark mass, $F_{\mu\nu}$ is the electromagnetic field strength tensor, $G^a_{\mu\nu}$ is the strong interaction field strength tensor, and T^a is a color $SU(3)$ generator. The sums over q include $q=u,d,s,c,b$ and the subscripts L,R denote left and right handed fields. The Wilson coefficients have been calculated to next-to-leading order (NLO) [6–8]. Using $\alpha_s(m_Z)=0.12$, and the convention that the covariant derivative is $D_\mu=\partial_\mu+igA^a_\mu T^a+ieQA_\mu$ (where Q is the fermion's electric charge), the values we need are $C_2(m_b)=1.13$, $C_7(m_b)=-0.306$, $C_8(m_b)=-0.168$ [6].

For the photon energy, E_γ, not too close to its maximal value, the photon spectrum $d\Gamma/dE_\gamma$ for weak radiative B decay has a perturbative expansion in the strong interaction fine structure constant α_s. It is known at order α_s and the main purpose of this letter is to present the order $\alpha_s^2\beta_0$ [so-called Brodsky-Lepage-Mackenzie (BLM) [9]] contribution. It is well known that the part of the order α_s^2 piece proportional to the one-loop beta function, $\beta_0=11-2n_f/3$ usually provides a reliable estimate of the full order α_s^2 piece. This part of the order α_s^2 contribution is straightforward to compute using the method of Smith and Voloshin [10].

Using the dimensionless variable,[1] $x_b=2E_\gamma/m_b$, the photon energy spectrum in $\overline{B}\rightarrow X_s\gamma$ takes the form

[1]Later we will introduce a dimensionless photon energy variable normalized by the B meson mass, $x_B=2E_\gamma/m_B$.

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dx_b}\Big|_{x_b<1}=A_0(x_b)+\frac{\alpha_s(m_b)}{\pi}A_1(x_b)+\left(\frac{\alpha_s(m_b)}{\pi}\right)^2\beta_0A_2(x_b)+\cdots, \qquad (3)$$

where

$$\Gamma_0=\frac{G_F^2|V_{tb}V_{ts}^*|^2\alpha_{\rm em}C_7^2}{32\pi^4}m_b^5, \qquad (4)$$

is the contribution of the tree level matrix element of O_7 to the $B\to X_s\gamma$ decay rate, and

$$A_p(x_b)=\sum_{i\leqslant j}a_p^{ij}(x_b)\left(\frac{C_i(m_b)C_j(m_b)}{C_7(m_b)^2}\right). \qquad (5)$$

The sums over i,j in Eq. (5) give the contributions of the various operators in Eq. (2) to the photon energy spectrum.

It is important to note that since the coefficients in $H_{\rm eff}$ are known only to NLO accuracy, the BLM calculation of the O_1-O_8 contribution to the photon spectrum is only meaningful away from the endpoint. At the endpoint, order α_s^2 contributions to the matrix elements are the same order as the unknown NNLO running [where $\alpha_s\ln(m_W/m_b)$ is counted as $\mathcal{O}(1)$]. Neglecting the small contribution to A_0 from O_1-O_6 discussed in the next paragraph, at least one gluon must be in the final state to populate the spectrum for $x_b<1$, so it is consistent to combine the α_s^2 matrix elements with the NLO Wilson coefficients. (Strictly speaking, we should for consistency only use the β_0 part of the NLO running of the operators with the BLM calculation, but for simplicity, we will use the full NLO result. The difference between these two approaches is small.) Thus, powers of α_s in Eq. (3) and elsewhere reflect the perturbation expansion of the matrix elements only, and not of the Wilson coefficients.

At zeroth order in the strong coupling, the spectrum for $x_b<1$ arises from matrix elements of the four-quark operators O_1-O_6 in Eq. (2). Of these, O_1 and O_2 include two charm quarks in the final state, and therefore they contribute to the photon spectrum only for lower values of x_b than what we consider in this paper. These contributions are divergent in perturbation theory and the divergence can be absorbed into the definition of the quark to photon fragmentation function, $D^{q\to\gamma}(x)$, which depends on an infrared scale Λ. $D^{q\to\gamma}(x)$ is calculable in the leading logarithmic approximation [11,12]. There is some data on $D^{q\to\gamma}(x)$, however, the experimental errors are still quite large [13]. This fragmentation contribution to the coefficients $a_0^{ij}(x)$ vanishes as $x_b\to1$, and it is small in the region of large x_b, $0.65<x_b$, which we consider in this paper.

A very important B decay background to the $\bar{B}\to X_s\gamma$ photon spectrum is from nonleptonic $b\to c\bar{u}d$ and $b\to u\bar{u}d$ decays, where a massless quark in the final state radiates a photon. Such backgrounds due to the operators $(\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)$ and $(\bar{u}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)$ are shown in Fig. 1 (using $|V_{ub}/V_{cb}|=0.1$). We used the Duke-Owens

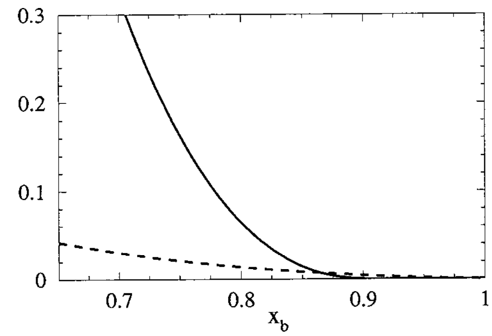

FIG. 1. B decay background to the photon spectrum due to the operators $(\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)$ (solid curve) and $(\bar{u}_L\gamma^\mu b_L)\times(\bar{d}_L\gamma_\mu u_L)$ (dashed curve).

parameterization of the fragmentation function [14], setting $\Lambda=1.3$ GeV and $Q^2=m_b^2$. (This value of Λ is motivated by a fit to the ALEPH data [13].) The uncertainty of this result is sizable, since the Λ-dependence is large and m_b may not be large enough to justify keeping only the leading logarithms. Close to maximal x_b, the resummed fragmentation function may predict too large a suppression of the photon spectrum, since the lightest exclusive final states dominate there. The background from $b\to c\bar{u}d$ ($b\to u\bar{u}d$) is more than 50% of the 77 contribution to $(1/\Gamma_0)d\Gamma/dx_b$ below $x_b\sim0.75$ ($x_b\sim0.65$).[2] Therefore, we will concentrate on the region $x_b>0.65$; to measure the $B\to X_s\gamma$ photon spectrum at lower values of x_b would not only require excluding final states with charm with very good efficiency, but also demanding a strange quark in the final state. Note that for $B\to X_d\gamma$, the fragmentation contribution from $b\to u\bar{u}d$ is larger than the short distance piece unless x_b is very close to 1.

Neglecting the strange quark mass, a_1^{88} is also divergent in perturbation theory. This divergence can also be absorbed into the definition of fragmentation functions. In the leading logarithmic approximation [15],

$$a_1^{88}(x)=\left(\frac{4\pi}{3\alpha_{\rm em}}\right)[D^{s\to\gamma}(x)+D^{g\to\gamma}(x)], \qquad (6)$$

where $D^{s\to\gamma}(x)$ and $D^{g\to\gamma}(x)$ are the strange quark to photon and gluon to photon fragmentation functions, which have large uncertainties. In the region $x_b>0.65$, the a_1^{88} contribution to the photon spectrum $(1/\Gamma_0)d\Gamma/dx_b$ is less than 0.01. Given the uncertainty in a_1^{88} and its small magnitude, it does not appear useful to calculate a_2^{88}.

Experimentally, because of backgrounds, only $\bar{B}\to X_s\gamma$ photons with large energies can be detected. The present experimental cut is $E_\gamma>2.1$ GeV at CLEO [1], which cor-

[2]Note that these backgrounds are steeply falling functions of x_b, and are indeed negligible in the present CLEO region of $E_\gamma>2.1$ GeV. The tree level contribution of the operators O_3-O_6 in Eq. (2) to the photon spectrum is about a fifth of the $b\to u\bar{u}d$ background.

responds to $x_b>0.875$ with $m_b=4.8$ GeV. In the large x_b region, the most important contribution to the sum in Eq. (5) come from the 77 term, with moderate corrections from the 22, 78, and 27 terms. The other contributions (88, 28, and the ones involving O_1 and O_3-O_6) are very small, and will be neglected in this paper.

Simple analytic expressions for a_1^{77} and a_1^{78} are available [16,17]:

$$a_1^{77}(x)=\frac{(2x^2-3x-6)x+2(x^2-3)\ln(1-x)}{3(1-x)}, \tag{7}$$

$$a_1^{78}(x)=\frac{8}{9}\left(\frac{4+x^2}{4}+\frac{1-x}{x}\ln(1-x)\right). \tag{8}$$

Neglecting the small A_0 term in Eq. (3), we can calculate the shape of the photon spectrum away from $x=1$ to order $\alpha_s^2\beta_0$ accuracy knowing the effective Hamiltonian to order α_s (NLO) only. At order $\alpha_s^2\beta_0$, we find that a_2^{77} and a_2^{78} are given by

$$\begin{aligned}a_2^{77}(x)=\frac{1}{18}\Bigg(&\frac{38x^3-93x^2+6x-36}{4(1-x)}\\&-\frac{6x^4-31x^3+24x^2-30x+18}{2x(1-x)}\ln(1-x)\\&+3(3-x^2)\frac{3\ln^2(1-x)+2L_2(x)}{2(1-x)}\Bigg),\end{aligned} \tag{9}$$

$$\begin{aligned}a_2^{78}(x)=\frac{1}{9}\Bigg(&\frac{19x^2-24x+88}{12}\\&-\frac{3x^3-12x^2+56x-32}{6x}\ln(1-x)\\&-(1-x)\frac{3\ln^2(1-x)+2L_2(x)}{x}\Bigg),\end{aligned} \tag{10}$$

where $L_2(z)=-\int_0^z dt\ln(1-t)/t$ is the dilogarithm. The strange quark mass is neglected throughout this paper; it only enters the final results quadratically, as $m_s^2/[m_b^2(1-x_b)]$.

The functions of a_1^{22} and a_1^{27} are known in the literature [16,17], and we agree analytically with those results. The order $\alpha_s^2\beta_0$ contributions, a_2^{22} and a_2^{27}, are computed numerically. We find it most useful to present simple approximations to these functions:

$$\begin{aligned}a_1^{22}(x)\simeq&-0.0842+0.3333x-0.2005x^2+0.0227x^3\\&+\left(\frac{m_c}{m_b}-\frac{1.4}{4.8}\right)(-0.454+0.061x),\\a_2^{22}(x)\simeq&-0.1272+0.3957x-0.3227x^2+0.0952x^3\\&-0.0180\ln(1-x)+\left(\frac{m_c}{m_b}-\frac{1.4}{4.8}\right)[-0.155\\&-0.106x+0.106\ln(1-x)],\end{aligned} \tag{11}$$

and

$$\begin{aligned}a_1^{27}(x)\simeq&-0.1064+0.4950x-0.4361x^2+0.0373x^3\\&+\left(\frac{m_c}{m_b}-\frac{1.4}{4.8}\right)(-1.207+2.901x),\\a_2^{27}(x)\simeq&-0.0156+0.0463x+0.3467x^2-0.3045x^3\\&+0.0027\ln(1-x)+\left(\frac{m_c}{m_b}-\frac{1.4}{4.8}\right)\\&\times[-1.523+2.538x-0.448\ln(1-x)].\end{aligned} \tag{12}$$

These approximations are accurate to within 1% in the region $x_b>0.6$ for $m_c/m_b=1.4/4.8$. Note that the perturbation series in α_s is particularly badly behaved for the 27 contribution. The 27 contribution is also very sensitive to m_c/m_b. Changing m_c/m_b from 1.4/4.8 to 1.2/4.6 or 1.6/5.0 [a variation of approximately $\delta(m_c/m_b)=\pm0.03$] modifies a_1^{27} and a_2^{27} dramatically. The approximation Eq. (12) is accurate at the 20% level when m_c/m_b changes in this range. The 22 contribution only varies by 25% in the previously mentioned range of m_c/m_b, and the approximation in Eq. (11) is accurate to 1% over this region. Roughly 2/3 of the 22 contribution is from absorptive parts corresponding to real intermediate states.

The coefficients a_2^{ij} are determined by calculating the order $\alpha_s^2 n_f$ piece and making the identification, $-2n_f/3\to\beta_0$. There is a subtlety in applying this method to weak radiative B decay. There is a contribution of order $\alpha_s^0 n_f$ from the tree level $b\to s\gamma q\bar{q}$ matrix elements of O_3-O_6, coming from Feynman diagrams where the photon couples to the bottom or strange quarks. It is not associated with a term of order $\alpha_s^0\beta_0$. To avoid adding an analogous spurious order $\alpha_s^2\beta_0$ contribution to a_2^{27} and a_2^{22}, only diagrams where the photon couples to the charm quark were included in the calculation of the matrix element of O_2.

Part of the $\bar{B}\to X_s\gamma$ matrix element of O_2 is not adequately calculated in perturbation theory. It corresponds to the process $\bar{B}\to J/\psi X_s$ followed by the decay $J/\psi\to\gamma+(\text{light hadrons})$. There will be large corrections to the part of the charm quark loop where the $c\bar{c}$ are almost on-shell and have the same velocity. In this region, there are large "Coulombic QCD corrections" that produce the J/ψ state. However, cutting this small part of the $c\bar{c}$ phase space out of our calculation of the matrix element of O_2 has a negligible effect. Hence, at the order of perturbation theory to which we are working, calculating the $c\bar{c}$ loop, while removing J/ψ's from the data would be a consistent approximation.

The sum of the 77, 22, 78, and 27 contributions is plotted in Fig. 2 in the region $0.65<x_b<0.9$ [using $\alpha_s(m_b)=0.22$ and $\beta_0=25/3$]. For very large x, other effects that we have not calculated become important. There are both nonperturbative and perturbative terms that are singular as $x\to1$. They sum into a shape function that modifies the spectrum in this region [18]. Unfortunately, at the present time, it is not possible to make a model independent estimate of these effects. Therefore, we do not plot the perturbation theory predictions

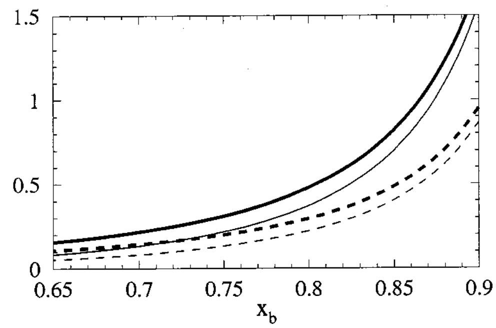

FIG. 2. The sum of the 77, 22, 78, and 27 contributions to $(1/\Gamma_0)d\Gamma/dx_b$ at order α_s (thick dashed curve) and $\alpha_s^2\beta_0$ (thick solid curve). The thin curves show the 77 contribution only. The scale is the same as in Fig. 1.

for $x_b>0.9$. In the plotted region, the 22, 78, and 27 terms make a moderate correction to the dominant 77 contribution to $(1/\Gamma_0)d\Gamma/dx$, which is shown in Fig. 2 with the thin curves. Although the order $\alpha_s^2\beta_0$ corrections are important away from $x_b=1$, their influence on the decay rate integrated over a region that includes $x_b=1$ is small.

The b quark mass can be eliminated in favor of the B meson mass by a change of variables to

$$x_B=2E_\gamma/m_B. \tag{13}$$

Using $m_b=m_B-\bar{\Lambda}+(\lambda_1+3\lambda_2)/(2m_b)+\cdots$, the photon spectrum becomes

$$\frac{d\Gamma}{dx_B}=\left(1+\frac{\bar{\Lambda}}{m_B}+\cdots\right)\frac{d\Gamma}{dx_b}\bigg|_{x_b=x_B(1+\bar{\Lambda}/m_B+\cdots)}. \tag{14}$$

For x_B within a region of order $\Lambda_{\rm QCD}/m_B$ of unity (its maximal value), nonperturbative effects are very important. However, for integrals of x_B over a large enough range, these nonperturbative effects are small.

An important integral of this type is

$$\overline{(1-x_B)}|_{x_B>1-\delta}=\frac{\int_{1-\delta}^1 dx_B(1-x_B)(d\Gamma/dx_B)}{\int_{1-\delta}^1 dx_B(d\Gamma/dx_B)}. \tag{15}$$

The parameter $\delta=1-2E_\gamma^{\rm min}/m_B$ has to satisfy $\delta>\Lambda_{\rm QCD}/m_B$; otherwise, nonperturbative effects are not under control. It is straightforward to show that

$$\overline{(1-x_B)}|_{x_B>1-\delta}=\frac{\bar{\Lambda}}{m_B}+\left(1-\frac{\bar{\Lambda}}{m_B}\right)\langle 1-x_b\rangle|_{x_b>1-\delta} -\frac{\bar{\Lambda}}{m_B}\delta(1-\delta)\frac{1}{\Gamma_0}\frac{d\Gamma}{dx_b}\bigg|_{x_b=1-\delta}+\cdots, \tag{16}$$

where

$$\langle 1-x_b\rangle|_{x_b>1-\delta}=\int_{1-\delta}^1 dx_b(1-x_b)\frac{1}{\Gamma_0}\frac{d\Gamma}{dx_b}. \tag{17}$$

Note that all terms but the first one in Eq. (16) have perturbative expansions which begin at order α_s. The ellipses denote contributions of order $(\Lambda_{\rm QCD}/m_B)^3$, $\alpha_s(\Lambda_{\rm QCD}/m_B)^2$, and α_s^2 terms not enhanced by β_0, but it does not contain contributions of order $(\Lambda_{\rm QCD}/m_B)^2$ or additional terms[3] of order $\alpha_s(\Lambda_{\rm QCD}/m_B)$. Terms in the operator product expansion proportional to $\lambda_{1,2}/m_b^2$ enter precisely in the form, so that they are absorbed in m_B in Eq. (16) [3]. There are also nonperturbative corrections suppressed by $(\Lambda_{\rm QCD}/m_c)^2$ instead of $(\Lambda_{\rm QCD}/m_b)^2$ [19]. These do not contribute to Eq. (16).

Using our results, $\langle 1-x_b\rangle|_{x_b>1-\delta}$ in Eq. (17) is known to order $\alpha_s^2\beta_0$. Writing

$$\langle 1-x_b\rangle|_{x_b>1-\delta}=B_0(\delta)+\frac{\alpha_s(m_b)}{\pi}B_1(\delta) +\left(\frac{\alpha_s(m_b)}{\pi}\right)^2\beta_0B_2(\delta)+\cdots, \tag{18}$$

B_p have decompositions analogous to Eq. (5),

$$B_p(\delta)=\sum_{i\leqslant j}b_p^{ij}(\delta)\left(\frac{C_i(m_b)C_j(m_b)}{C_7^2(m_b)}\right). \tag{19}$$

Neglecting $B_0(\delta)$, Eqs. (7) and (9) yield for the dominant 77 contribution

$$b_1^{77}(\delta)=\frac{\delta}{54}[-9\delta^3+14\delta^2+72\delta-54+12(\delta^2-3\delta-6)\ln\delta], \tag{20}$$

$$\begin{aligned}b_2^{77}(\delta)=\frac{1}{2592}[&-369\delta^4+116\delta^3+1800\delta^2-3852\delta+408\pi^2\\&+12\delta(9\delta^3+34\delta^2-102\delta+66)\ln\delta\\&-216\delta(\delta^2-3\delta-6)\ln^2\delta\\&-144(\delta^3-3\delta^2-6\delta+17)L_2(1-\delta)].\end{aligned} \tag{21}$$

Our prediction for $\langle 1-x_b\rangle|_{x_b>1-\delta}$ is shown in Fig. 3 as a function of δ, both at order α_s and $\alpha_s^2\beta_0$. The bad behavior of the perturbation expansion would improve somewhat by evaluating the strong coupling at a smaller scale than m_b, such as $m_b\sqrt{\delta}$, the maximal available invariant mass of the hadronic final state. This bad behavior may also be related to the renormalon ambiguity [20] in $\bar{\Lambda}$.

[3]There are actually additional contributions formally of order $\alpha_s(\Lambda_{\rm QCD}/m_B)$ coming from the expansion of m_c/m_b in the 22 and 27 terms. Although the 27 term is very sensitive to the value of m_c/m_b, this $\bar{\Lambda}$-dependence is negligible for $\overline{(1-x_B)}|_{x_B>1-\delta}$.

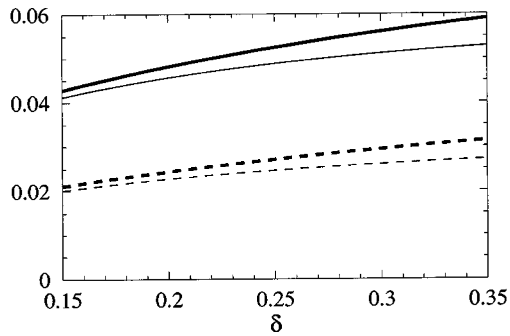

FIG. 3. The sum of the 77, 22, 78, and 27 contributions to $\langle 1-x_b\rangle|_{x_b>1-\delta}$ at order α_s (thick dashed curve) and $\alpha_s^2\beta_0$ (thick solid curve). The thin curves show the 77 contribution only.

A determination of $\bar{\Lambda}$ is straightforward using Eq. (16). The left hand side is directly measurable, while $\langle 1-x_b\rangle|_{x_b>1-\delta}$ and $(1/\Gamma_0)d\Gamma/dx_b|_{x_b=1-\delta}$ in the second and third terms on the right hand side can be read off from Figs. 3 and 2, respectively. Using the CLEO data in the region, $E_\gamma>2.1$ GeV [1], we obtain the central values $\bar{\Lambda}_{\alpha_s^2\beta_0}\simeq 270$ MeV and $\bar{\Lambda}_{\alpha_s}\simeq 390$ MeV. We have indicated the order kept in the perturbation expansion to determine $\bar{\Lambda}$, since a value of $\bar{\Lambda}$ extracted from data can only be used consistently in predictions valid to the same order in α_s. These values are consistent with the ones obtained from a fit to the $\bar{B}\to X_c l\bar{\nu}$ lepton spectrum [4], and from the CLEO fit [21] to the $\bar{B}\to X_c l\bar{\nu}$ hadron mass distribution [5].

At the present time, this extraction of $\bar{\Lambda}$ has large uncertainties. The potentially most serious one is from both nonperturbative and perturbative terms that are singular as $x\to 1$ and sum into a shape function that modifies the spectrum near the endpoint. A model independent determination of these effects is not available at the present time; however, it may be possible to address this issue using lattice QCD [22]. For sufficiently large δ, these effects are not important. They have been estimated in Refs. [23,24] using phenomenological models for the shape function. We have implicitly neglected these effects throughout our analysis. The validity of this can be tested experimentally by checking whether the value of $\bar{\Lambda}$ extracted from Eq. (16) is independent of δ in some range. This would also improve our confidence that the total decay rate in the region $x_B>1-\delta$ can be predicted in perturbative QCD without model dependence.

The value of $\bar{\Lambda}$ at order α_s has a sizable scale dependence: lowering the scale such that α_s changes from 0.22 to 0.3 reduces the value of $\bar{\Lambda}_{\alpha_s}$ by about 40 MeV. At order $\alpha_s^2\beta_0$, this scale dependence is much smaller. Uncertainties due to the unknown order $(\Lambda_{\rm QCD}/m_B)^3$ terms in the operator product expansion (OPE) [24] are largely uncorrelated to those in the analyses of the lepton energy or hadron mass spectra in $\bar{B}\to X_c l\bar{\nu}$ [25]. The effect of the boost from the B rest frame into the $\Upsilon(4S)$ is small for $\overline{(1-x_B)}|_{x_B>1-\delta}$ [23].

The upsilon expansion [26] yields parameter free predictions for $\overline{(1-x_B)}|_{x_B>1-\delta}$ in terms of the $\Upsilon(1S)$ meson mass. The analog of Eq. (16) is

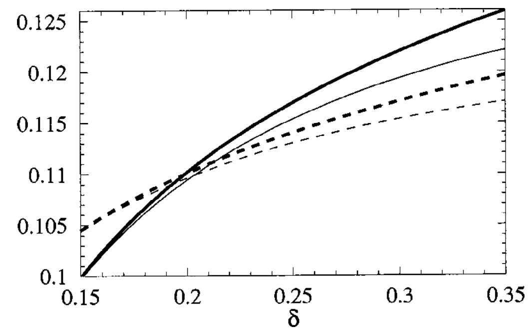

FIG. 4. Prediction for $\overline{(1-x_B)}|_{x_B>1-\delta}$ in the upsilon expansion at order ϵ (thick dashed curve) and $(\epsilon^2)_{\rm BLM}$ (thick solid curve). The thin curves show the 77 contribution only.

$$\overline{(1-x_B)}|_{x_B>1-\delta}=1-\frac{m_\Upsilon}{2m_B}[1+0.011\epsilon+0.019(\epsilon^2)_{\rm BLM} - \langle 1-x_b\rangle|_{x_b>(2m_B/m_\Upsilon)(1-\delta)}], \tag{22}$$

where $\epsilon\equiv 1$ denotes the order in the upsilon expansion. For $E_\gamma>2.1$ GeV, this relation gives 0.111, whereas the central value from the CLEO data is around 0.093.[4] In Fig. 4 we plot the prediction for $\overline{(1-x_B)}|_{x_B>1-\delta}$ as a function of δ, both at order ϵ and $(\epsilon^2)_{\rm BLM}$. The perturbation expansion is much better behaved than the one shown in Fig. 3. The most important uncertainty in this approach is the size of nonperturbative contributions to the $\Upsilon(1S)$ mass other than those which can be absorbed into the b quark mass. These have been neglected in Eq. (22). If the nonperturbative contribution to the $\Upsilon(1S)$ mass, Δ_Υ, were known, it could be included by replacing m_Υ by $m_\Upsilon-\Delta_\Upsilon$. For example, $\Delta_\Upsilon=+300$ MeV increases $\overline{(1-x_B)}$ by 21%, so measuring $\overline{(1-x_B)}$ with such accuracy will have important implications for the physics of quarkonia as well as for B physics.

The variance of the photon energy distribution can be used to determine λ_1 [3,24]. The analog of Eq. (16) in this case is

$$\overline{(1-x_B)^2}|_{x_B>1-\delta}-[\overline{(1-x_B)}|_{x_B>1-\delta}]^2 = -\frac{\lambda_1}{3m_B^2}+\frac{\beta^2}{3}+\left(1-\frac{2\bar{\Lambda}}{m_B}\right)\langle(1-x_b)^2\rangle\bigg|_{x_b>1-\delta} - \frac{\bar{\Lambda}}{m_B}\delta^2(1-\delta)\frac{1}{\Gamma_0}\frac{d\Gamma}{dx_b}\bigg|_{x_b=1-\delta}+\cdots, \tag{23}$$

[4]It is interesting to note that including the CLEO data point in the $1.9\ \text{GeV}<E_\gamma<2.1$ GeV bin, the experimental central value of $\overline{(1-x_B)}$ over the region $E_\gamma>1.9$ GeV is 0.117, whereas the upsilon expansion predicts 0.120.

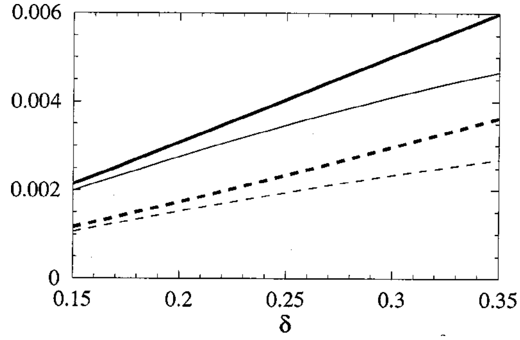

FIG. 5. The sum of the 77, 22, 78, and 27 contributions to $\langle(1-x_b)^2\rangle|_{x_b>1-\delta}$ at order α_s (thick dashed curve) and $\alpha_s^2\beta_0$ (thick solid curve). The thin curves show the 77 contribution only.

where $\beta\simeq 0.064$ is the magnitude of the velocity of the B meson in the $\Upsilon(4S)$ rest frame, and only the leading β-dependence has been kept. The ellipses denote terms of order $(\Lambda_{\rm QCD}/m_B)^3$, $\alpha_s(\Lambda_{\rm QCD}/m_B)^2$, and α_s^2 terms not enhanced by β_0. Our prediction for $\langle(1-x_b)^2\rangle|_{x_b>1-\delta}$ is shown in Fig. 5. Note that unlike the case of $\overline{(1-x_B)}|_{x_B>1-\delta}$, the effect of the boost is very important in Eq. (23). Using the CLEO data in the region $E_\gamma>2.1$ GeV, we obtain the central value $\lambda_1\simeq-0.1$ GeV^2, with large experimental errors. The uncertainty in this value of λ_1 due to $\bar\Lambda$ is small. Nonperturbative effects from the cut on E_γ [24], and the unknown higher order contributions to Eq. (23) are expected to have a larger impact on the determination of λ_1 than the corresponding effects have on the determination of $\bar\Lambda$ from Eq. (16).

In summary, we calculated order $\alpha_s^2\beta_0$ corrections to the shape of the photon energy spectrum in weak radiative $\bar B\to X_s\gamma$ decay. The dominant 77 contribution is given by simple analytic formulas in Eqs. (7) and (9). The other terms relevant in the region $x_b>0.65$ are the 22 and 27 contributions given in Eqs. (11) and (12), and the 78 term given in Eqs. (8) and (10). The HQET parameter $\bar\Lambda$ can be extracted from the average $\langle 1-2E_\gamma/m_B\rangle$ using Eq. (16), and it can also be used to test whether the nonperturbative contribution to the $\Upsilon(1S)$ mass is small. The CLEO data in the region $E_\gamma>2.1$ GeV implies the central values $\bar\Lambda_{\alpha_s}\simeq 390$ MeV and $\bar\Lambda_{\alpha_s^2\beta_0}\simeq 270$ MeV at order α_s and $\alpha_s^2\beta_0$, respectively. Possible contributions to the total decay rate from physics beyond the standard model are unlikely to affect this determination of $\bar\Lambda$. In the future, checking the δ-independence of the extracted value of $\bar\Lambda$, and comparing the experimental and theoretical shapes of the photon spectrum for $x_b<0.9$ can provide a check that nonperturbative effects and backgrounds are under control. This would also improve our confidence that the total decay rate in the region $x_B>1-\delta$ can be predicted model independently, and used to search for signatures of new physics with better sensitivity.

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