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Solving four-dimensional field theories with the Dirichlet fivebrane

Robert de Mello Koch
Department of Physics, Brown University, Providence, Rhode Island 02912

João P. Rodrigues
Department of Physics and Centre for Non Linear Studies, University of the Witwatersrand, Wits 2050, South Africa

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The realization of $\mathcal{N}=2$ four-dimensional super Yang-Mills theories in terms of a single Dirichlet fivebrane in type-IIB string theory is considered. A classical brane computation reproduces the full quantum low-energy effective action. This result has a simple explanation in terms of mirror symmetry. [S0556-2821(99)02012-3]

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A particularly fruitful approach to the study of supersymmetric quantum field theories has been to realize these theories as a limit of string or M theory where gravitational effects decouple. There are two complementary approaches to this problem—the geometric engineering [1] approach and the Hanany-Witten brane setup [2]. To study $\mathcal{N}=2$ super Yang-Mills theories in $3+1$ dimensions within the geometric engineering approach, one typically compactifies type-IIA or B string theory on a Calabi-Yau threefold. The full nonperturbative solution of the $\mathcal{N}=2$ super Yang-Mills theory is then obtained by invoking mirror symmetry. In the Hanany-Witten approach, one typically considers a web of branes in a flat space. In order to study $\mathcal{N}=2$ super Yang-Mills theory in $3+1$ dimensions, one considers two parallel solitonic fivebranes with a number of Dirichlet fourbranes stretched between them [3]. In this approach, all perturbative and nonperturbative corrections to the field theory are coded into the shape of the branes. The solution of these theories is performed by lifting to M theory. After the lift, the original type-IIA brane setup is reinterpreted as a single fivebrane in M theory, wrapping the Seiberg-Witten curve Σ. The relationship between these two approaches has been explained in [4]. In this report, we will study $\mathcal{N}=2$ super Yang-Mills theory using the Hanany-Witten approach.

Up to now, the description of the M theory fivebrane relevant for $\mathcal{N}=2$ super Yang-Mills theory has been in terms of 11-dimensional supergravity, which is a valid description of M theory at low energy [5]. A number of holomorphic quantities [6] including the exact low-energy effective action [5] can be recovered using the supergravity description. The supergravity description corresponds to a strong-coupling description of the original type-IIA setup. However, one expects the field theory to emerge in the opposite limit, where the string theory is weakly coupled [7]. This limit is not captured by the supergravity approximation, so that one expects that the supergravity approach will only be capable of reproducing field theory quantities which are protected by supersymmetry.

In this note, we will provide a direct construction in string theory which realizes the $\mathcal{N}=2$ super Yang-Mills theory in terms of a single Dirichlet fivebrane wrapping the Seiberg-Witten curve. We will be mainly concerned with two important issues: how a matrix description is obtained and how the string theory configurations described in this article are related to the original type-IIA brane setup [3]. In particular the single D5 in type-IIB string theory will be seen to be related by T duality to what has been described in the literature as the ''magnetic'' IIA brane configuration. We will then show how the D5 provides a strongly coupled, low-energy description of weakly coupled IIA string theory in the original brane setup.

We will start with a brane construction consisting of a number of Dirichlet fourbranes suspended between Dirichlet sixbranes in type-IIA string theory on $R^9\times S^1$. The coordinate x^7 is compact, with radius R_7. In the classical approximation, the sixbranes are located at $x^8=x^9=0$ and at some fixed x^6. The world volume coordinates for the sixbranes are $x^0,x^1,x^2,x^3,x^4,x^5,x^7$. The fourbranes are located at $x^8=x^9=0$ and at some fixed values of x^4,x^5,x^7. The fourbranes have world volume coordinates x^0,x^1,x^2,x^3,x^6. Since the fourbranes stretch between the two sixbranes, the x^6 coordinate is restricted to a finite interval. This brane configuration is related to the configuration studied in [8] by T duality along x^1,x^2,x^3. The supersymmetries preserved by the fourbranes are of the form [9] $\epsilon_L Q_L+\epsilon_R Q_R$ where $\epsilon_L=\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_6\epsilon_R$. Thus, the fourbrane breaks one half of the supersymmetry. The sixbranes preserve supersymmetries of the form $\epsilon_L Q_L+\epsilon_R Q_R$ where $\epsilon_L=\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_7\epsilon_R$, which breaks half of the remaining supersymmetry. This leaves a total of $\mathcal{N}=2$ supersymmetry in $3+1$ dimensions. The super Yang-Mills theory we wish to study is realized on the world volume of the fourbrane. The coordinates of the field theory are x^0,x^1,x^2,x^3. The essential ingredient allowing a solution of the field theory, is the realization that by performing a T duality along x^7 one obtains a single Dirichlet fivebrane in type-IIB string theory. This Dirichlet fivebrane has the form $R^4\times\Sigma$ where R^4 is parametrized by the world volume coordinates x^0,x^1,x^2,x^3, and Σ is a surface in the four-dimensional space parametrized by the coordinates x^4,x^5,x^6,x^7. The requirement that we preserve $\mathcal{N}=2$ supersymmetry and reproduce the required asymptotic brane geometry implies that Σ is the Seiberg-Witten curve [3].

We will focus on the simplest case of pure gauge theory. The Riemann surface relevant for a configuration of k fourbranes in the original IIA setup is [3]

$$t^2-2B(v)t+1=0,$$

$$B(v)=v^k+u_2 v^{k-2}+u_3 v^{k-3}+\cdots+u_k,$$

$$t=\exp(-s/\tilde{R}_7)=\exp[-(x^6+ix^7)/\tilde{R}_7], \quad v=x^4+ix^5,$$

$$\tilde{R}_7=l_s^2/R_7. \tag{1}$$

This curve corresponds to a fivebrane with two asymptotic sheets connected by k tubes. The two asymptotic sheets are T dual to the sixbranes in the above IIA setup, while the tubes are T dual to the fourbranes. Our first task is to provide a classical description of a fivebrane with this geometry. Since we are interested in describing the world volume of the fourbranes in the original IIA description, it is most natural to use the world volume coordinates x^0,x^1,x^2,x^3,x^6,x^7. This is different from the approach followed in [5].

Consider the low-energy description of the Dirichlet fivebrane, which is known to be a 5+1-dimensional super Yang-Mills theory [10]. The bosonic part of the fivebrane Lagrangian is

$$\mathcal{L}=\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}+D_\mu X^I D^\mu X^I+[X^I,X^J]^2), \tag{2}$$

where $I=4,5,8,9$ and $\mu,\nu=0,1,2,3,6,7$. The X^I are $k\times k$ dimensional matrices.

The classical configuration corresponding to a Dirichlet fivebrane wrapped on the Seiberg-Witten curve is $X^8=X^9=0$ with X^4 and X^5 simultaneously diagonal. The eigenvalues x_i^4 and x_i^5 of X^4 and X^5 depend on x^6 and x^7 as we now explain. Once a value for x^6 and x^7 is given, Eq. (1) may be solved for the k roots v_i. The real part of v_i then gives x_i^4, while the imaginary part gives x_i^5. Clearly, once we specify a point on the membrane world volume, x_i^4 and x_i^5 range over the correct coordinates to be identified either with a given tube connecting the two parallel sheets of the fivebrane, or with one of the k ''circular'' arcs on these parallel sheets themselves. The k^2-k off-diagonal entries in X^4 and X^5 describe the fivebrane self-interaction arising from open strings stretching between these tubes. These off-diagonal entries, as well as the gauge field, are set to zero in the classical configuration which we study.

We remark that the above fivebrane solution is obtained by identifying the eigenvalues of the fivebrane's matrix coordinates with the zeroes (in v) of the Seiberg-Witten curve. This is familiar from the collective field approach to the large-N limit of matrix models, where the collective field describing the density of eigenvalues can be identified with the density of zeros of a suitable polynomial [11].

We now show how the exact Seiberg-Witten [12] low-energy effective action is reproduced. The terms in the matrix theory Lagrangian giving rise to the scalar kinetic term of the four-dimensional field theory are ($m=0,1,2,3$)

$$\mathcal{L}_{\mathrm{kin}}=\int d^2s\,\mathrm{Tr}(\partial_m Y\partial^m Y^\dagger)=\int d^2s\,\partial_m y_i\partial^m\overline{y_i}, \tag{3}$$

where y_i are the diagonal elements of $Y=X^4+iX^5$. Later we will use the fact that y_i are simply the roots $v(t)$ described by Eq. (1).[1] For concreteness, we now consider $N_c=2$ in which case we have ($y_1^2=y_2^2=v^2\equiv y^2$)

$$y^2=-u+\cosh(s/\tilde{R}_7). \tag{4}$$

To evaluate the integral in Eq. (3) we can proceed as follows: Since the only x^m dependence in y is contained in u, we may rewrite Eq. (3) as

$$\mathcal{L}_{\mathrm{kin}}=\partial_m u\,\partial^m\overline{u}\int_\Sigma \lambda\wedge\overline{\lambda},$$

$$\lambda=\frac{\partial}{\partial u}\left[\sqrt{\cosh\left(\frac{s}{\tilde{R}_7}\right)-u}\right]ds\equiv\frac{\partial\tilde{\lambda}}{\partial u}. \tag{5}$$

Note that $\tilde{\lambda}$ is a meromorphic one form, with a double pole at $t=\infty$, and that λ is a holomorphic one form. Choose a symplectic basis of the first homology class of Σ denoted α,β. Now, using the Riemann bilinear identity for Abelian differentials of the first kind [13], Eq. (3) can be expressed as

$$\mathcal{L}_{\mathrm{kin}}=\frac{1}{2i}(\partial_m a\,\partial^m\overline{a_D}-\partial_m\overline{a}\,\partial^m a_D)=\mathrm{Im}(\partial_m a\,\partial^m\overline{a_D}),$$

$$a=\oint_\alpha\frac{ds}{\tilde{R}_7}\sqrt{\cosh\left(\frac{s}{\tilde{R}_7}\right)-u}=\oint_\alpha dx\frac{\sqrt{x-u}}{\sqrt{x^2-1}},$$

$$a_D=\oint_\beta\frac{ds}{\tilde{R}_7}\sqrt{\cosh\left(\frac{s}{\tilde{R}_7}\right)-u}=\oint_\beta dx\frac{\sqrt{x-u}}{\sqrt{x^2-1}}. \tag{6}$$

The classical brane solution has reproduced the correct scalar kinetic terms of the exact low energy effective action. $\tilde{\lambda}$ is in fact the Seiberg-Witten differential, remarkably in its original form [12]. In view of the comments following Eq. (3) it is clear that our approach gives the Seiberg-Witten differential, in the general case, as $\lambda_{\mathrm{SW}}=v(t)dt/t$. This agrees nicely with the known results [14].

The relationship between our brane setup and the one used in [3][2] is most easily demonstrated by considering the $\mathcal{N}=2$ super Yang-Mills theory on $R^3\times S^1$. Consider the system described earlier of parallel Dirichlet sixbranes with N_c Dirichlet fourbranes stretching between them. We take x^1 to be compact with radius R_1, x^7 to be compact with radius R_7 and $g_s=R_{11}l_s^{-1}$.[3] The sixbranes wrap the compact x^1 and x^7 directions; the fourbranes wrap x^1. Performing a T duality along the cycle of length R_7, one obtains a single Dirichlet fivebrane as discussed above. This Dirichlet fivebrane wraps the Seiberg-Witten curve Σ and the compact x^1 direction and has string coupling $g_s=R_{11}/R_7$. Now perform a T duality

[1]We do not worry about dimensions in the present discussion. This will be discussed in a later section. We have also set $\Lambda=1$.

[2]Related ideas were considered in [15,16].

[3]R_{11} should be thought of as a parameter, which we could have also chosen as g_s. The lift to M theory is described later.

along the cycle of length R_1. We obtain a single Dirichlet fourbrane in type-IIA string theory, wrapped on the Seiberg-Witten curve Σ. After this second T duality, x^1 is compact with radius l_s^2/R_1, x^7 is compact with radius l_s^2/R_7, and $g_s = R_{11}l_s/R_1R_7$. Lifting to M theory, we obtain a single M-theory fivebrane wrapping the Seiberg-Witten curve Σ and x^{11}. The x^1 direction is compact with radius $\sqrt{R_7 l_p^3/R_1R_{11}}$, the x^7 direction is compact with radius $\sqrt{R_1 l_p^3/R_7R_{11}}$ and x^{11} is compact with radius $\sqrt{R_{11}l_p^3/R_1R_7}$. Using the 11-dimensional Lorentz invariance of M theory, we may reinterpret x^7 as the dimension which grows at strong coupling. Since the Planck length of the theory is held fixed, the string tension is transformed as

$$l_s^2 \to l_s'^2 = (R_{11}l_s^2)/R_1, \tag{7}$$

as explained in [17]. In this case, we obtain two parallel solitonic fivebranes in IIA string theory, with N_c Dirichlet fourbranes stretched between them. The string coupling is $g_s' = (R_1 l_s')/(R_{11}R_7)$. The solitonic fivebranes and the Dirichlet fourbranes both wrap the x^{11} direction which has a radius $l_s'^2/R_7$.

This compactification has been considered in [16][4] where it was argued that the link between the two solitonic fivebranes with N_c fourbranes (the ''electric'' IIA brane configuration) and the single Dirichlet fourbrane wrapping the Seiberg-Witten curve (the ''magnetic'' IIA brane configuration) is in fact a mirror transform. In the present context, this can independently be seen as follows: under T duality along x^{11}, the electric IIA brane configuration is mapped into two solitonic fivebranes with N_c Dirichlet threebranes stretched between them. Our starting brane configuration, consisting of N_c Dirichlet fourbranes stretched between two Dirichlet sixbranes is mapped into two Dirichlet fivebranes with N_c Dirichlet threebranes stretched between them under T duality along x^1. According to [2], these IIB brane configurations are related by mirror symmetry. The mirror transform maps the Coulomb branch of the electric theory to the Higgs branch of the magnetic theory [18,2], which does not receive string loop corrections [18] so that the classical calculation is exact. This provides a simple explanation for why the classical Dirichlet fivebrane is capable of reproducing the full quantum effective action of the $\mathcal{N}=2$ gauge theory.

The electrical Bogomol'nyi-Prasad-Sommerfield (BPS) state, which is a fundamental string stretching between two Dirichlet fourbranes in the electric IIA brane configuration, becomes, in the single Dirichlet fivebrane description, a Dirichlet three brane wrapping x^1 (with radius R_1), x^7 (with radius $\tilde{R}_7$), and stretching between the two tubes of the fivebrane. Its mass is given by

$$m = 2\pi R_1 2\pi \tilde{R}_7 \frac{2|a(u)|}{\alpha}\frac{1}{g_s l_s^4} = \frac{R_1}{R_{11}l_s^2}\frac{8\pi^2|a(u)|}{\alpha}$$
$$= \frac{1}{l_s'^2}\frac{8\pi^2|a(u)|}{\alpha}. \tag{8}$$

We let $y = (1/\alpha)v$, with α a parameter with the dimensions of L^{-2}, so that y is a displacement and v has the usual dimensions of a Higgs field. We have also used $g_s = R_{11}/R_7$ for the D5 string coupling and Eq. (7). The magnetic BPS state, which is a Dirichlet two brane in the electric IIA brane setup stretching across the hole between the two Dirichlet fourbranes and the two solitonic fivebranes, becomes a Dirichlet threebrane in the single D5 description and has a mass

$$m = 2\pi R_1 \tilde{R}_7 \frac{|a_D(u)|}{\alpha}\frac{1}{g l_s^4} = \frac{R_1}{R_{11}l_s^2}\frac{2\pi|a_D(u)|}{\alpha}$$
$$= \frac{1}{g_s' l_s'^3}\frac{2\pi\tilde{R}_7|a(u)|}{\alpha}, \tag{9}$$

where g_s' is the string-coupling constant in the D4 plus Neveu-Schwarz fivebrane (NS5) setup.

We now discuss the region of applicability of the D5 picture. $\mathcal{N}=2$ super Yang-Mills theory has a characteristic mass scale Λ, and therefore one way to approach the problem is to consider scaling limits $\Lambda R_1 = \epsilon^{\alpha_1}, \ldots$, in the limit as $\epsilon \to 0$. There are four parameters R_1, R_{11}, $\tilde{R}_7$, and l_s. One constraint amongst them is that the size of the x^{11} coordinate in the original D4-NS5 set up is given by

$$r_{11} = l_s'^2/R_7 = (R_{11}\tilde{R}_7)/R_1. \tag{10}$$

If we wish to describe $\mathcal{N}=2$ super Yang-Mills theory on R^4, we may then choose r_{11} to grow in a specified manner. Other possible constraints depend crucially on how lengths are obtained from the field theory Higgs field [i.e., the parameter α in Eqs. (8) and (9)] and a possible specification relating field-theoretic masses to D5-M5 descriptions. Clearly, there is a preferred choice for α: $\alpha = l_s'^{-2}$, where l_s' is the string coupling constant in the D4-NS5 set up. For this choice, the BPS masses (8) and (9) are immediately field-theoretic masses. For this choice of α there are no further constraints among the parameters.

We then have for the single D5 string coupling constant g_s

$$g_s = (R_{11}\tilde{R}_7)/l_s^2 = (r_{11}R_1)/l_s^2, \tag{11}$$

where we used Eq. (10) in the second equality above. We require $l_s \to 0$ so that the Yang-Mills description of the D5 is valid, and r_{11} and R_1 to be large; as a result the D5 will be strongly coupled. This is to be expected since r_{11}, which is the size of the world volume coordinate x^{11} in the D4-NS5 picture becomes the M-theory circle on the magnetic side,

[4]Note however that the coordinate x^6 is not compact in our case, i.e., we are not considering the elliptic case.

and therefore decompactification in the Coulomb phase will in general correspond to strongly coupled magnetic descriptions.

Finally note that the second equality in Eq. (11) is independent of $\tilde{R}_7$ and R_{11}, and therefore $\tilde{R}_7$ can be made small by suitably adjusting R_{11}. Since $\tilde{R}_7$ is the *M*-theory circle in the electric picture, the type-IIA string theory in the D4-NS5 set up is weakly coupled. We also remark that the above fivebrane analysis provides a fivebrane world-volume description of the type-IIB setup considered in the first of [1].

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