Four-fermion field theories and the Chern-Simons field: A renormalization group study

V. S. Alves,* M. Gomes, S. V. L. Pinheiro,* and A. J. da Silva
Instituto de Física, USP, Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil

(Received 15 January 1999; published 25 June 1999)

In 2+1 dimensions, we consider the model of an N-flavor, two-component fermionic field interacting through a Chern-Simons field in addition to a four-fermion self-interaction which consists of a linear combination of the Gross-Neveu- and Thirring-like terms. The four-fermion interaction is not perturbatively renormalizable and the model is taken as an effective field theory in the region of low momenta. Using the Zimmerman procedure for reducing coupling constants, it is verified that, for small values of the Chern-Simons parameter, the origin is an infrared-stable fixed point but changes to ultraviolet stable as α becomes bigger than a critical α_c. Composite operators are also analyzed and it is shown that a specific four-fermion interaction has an improved ultraviolet behavior as N increases. [S0556-2821(99)00914-5]

PACS number(s): 11.10.Gh, 11.10.Hi, 11.10.Kk

Fermionic quartic interactions have been very important for the clarification of conceptual aspects as well as for the applications of quantum field theory. Illustrative examples of such a dual role are provided by the Thirring and Nambu–Jona-Lasinio models. However, perturbative studies of the models have been hampered by the fact that only in two dimensions are they renormalizable. If the number of flavors is high enough, a better ultraviolet behavior is achieved in the context of the $1/N$ expansion which turns out to be renormalizable up to $4-\epsilon$ dimensions [1–3]. Various studies have been performed using such a scheme [4].

On the other hand, for small N we may consider the models as effective field theories [5], reliable at low energies, as has indeed been done in their phenomenological applications [6]. Besides that, recent studies [7,8] pointed out that in 2+1 dimensions yet another complementary direction would be available. Through interaction with a Chern-Simons (CS) field [9] fermionic fields could change their operator dimension in such way as to improve the ultraviolet behavior of the perturbative expansion. In [8] this conjecture was investigated for the case of the Gross-Neveu model coupled to a Chern Simons field and considering $N=1$. Although an improvement does occur for the basic field, we found that quartic composite operators do not share this property. This means that the behavior of these operators is not affected by the CS field. Nevertheless, from the characteristics of the $1/N$ expansion and also from nonperturbative investigations based in the Schwinger Dyson equation we may expect the existence of relevant four-fermion interactions when $N\neq 1$. In fact, nonperturbative studies point towards the existence of critical values of N where mass generation occurs and basic properties of the theories are drastically changed [10,2]. It is therefore reasonable to expect substantial changes in these theories as N increases, even at the perturbative level.

In this Brief Report, pursuing the work of [8], we will present some results on four-fermion theories coupled to a CS field when N is small but $\neq 1$. The basic field ψ, belonging to the two-dimensional representation of the Lorentz group, has now both Lorentz and SU(N) indices which we will sometimes indicate by Greek and Latin letters, respectively. Our first observation is that, as a result of the Fierz identity [11],

$$\left(1+\frac{2}{N}\right)(\bar{\psi}\psi)^2+(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)+(\bar{\psi}\lambda^a\psi)(\bar{\psi}\lambda^a\psi)=0, \tag{1}$$

$$2\left(2+\frac{1}{N}\right)(\bar{\psi}\psi)^2+\frac{2}{N}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)+(\bar{\psi}\gamma^\mu\lambda^a\psi)(\bar{\psi}\gamma_\mu\lambda^a\psi)=0, \tag{2}$$

where λ^a, $a=1,\ldots,N^2-1$, are the generators of SU(N); there are only two independent, Lorentz and SU(N) scalar quartic self-interactions. Therefore, without losing generality, we may restrict our study to the theory described by the Lagrangian

$$\mathcal{L}=\frac{1}{2\pi\alpha}\varepsilon^{\mu\nu\alpha}\,\partial_\mu A_\nu A_\alpha+\bar{\psi}(i\partial\!\!\!/-m)\psi+\bar{\psi}\gamma^\mu\psi A_\mu-G_1(\bar{\psi}\psi)\times(\bar{\psi}\psi)-G_2(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)+\frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{3}$$

Actually, evading possible infrared divergences, throughout this paper we will work in a Landau gauge obtained by formally letting $\xi\to 0$. As the canonical dimension of ψ is 1, both G_1 and G_2 have dimension -1 in mass unity. The model is therefore nonrenormalizable, the degree of superficial divergence of a generic graph with N_A and N_F bosonic and fermionic external lines, and with V_1 and V_2 Gross-Neveu- and Thirring-like vertices, being equal to

$$d(\gamma)=3-N_A-N_F+V_1+V_2. \tag{4}$$

To validate our calculations we shall treat Eq. (3) as an effective theory, suppressing the high momentum contributions to the Feynman amplitudes. This is conveniently done by introducing a dimensional parameter Λ through the definitions $G_1=g_1/\Lambda$ and $G_2=g_2/\Lambda$ and restricting the calcula-

*On leave of absence from Universidade Federal do Pará.

tion by requiring that $p \ll \Lambda$. In this implementation g_1 and g_2 must then be considered as the perturbative couplings.

To regulate Feynman integrals, we use the following ''dimensional regularization'' recipe. Initially, the algebra of the Dirac matrices and contractions of the ε Levi-Cività symbols are performed in $2+1$ dimensions using

$$\gamma^{\mu}\gamma^{\nu}=g^{\mu\nu}-i\varepsilon^{\mu\nu\rho}\gamma_{\rho} \tag{5}$$

and

$$\varepsilon^{\mu\nu\rho}\varepsilon_{\rho\sigma\lambda}=\delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda}-\delta^{\mu}_{\lambda}\delta^{\nu}_{\sigma}. \tag{6}$$

After this step, the integrals are promoted to d dimensions and carried out according to the usual rules [12]. Singularities appear as poles at $d=3-\epsilon$ which should then be removed. To this end, to each loop integral we incorporate the factor μ^{ϵ} where the massive parameter μ plays the role of the renormalization point. The renormalized amplitude is given by the constant term (i.e., the ϵ-independent one) of the Laurent expansion of the resulting expression. This ''dimensional regularization'' method does not require an extension of the Levi-Cività symbol outside $2+1$ dimensions and thus is very convenient for practical calculations. One should be aware that slight modifications of these rules may change the finite part (for example, using $\gamma^{\mu}\gamma^{\alpha}\gamma_{\mu}=2-\mathrm{d}$ instead of -1) of the outcome. However, our results will not be affected since we will be dealing only with the simple pole part of the amplitudes (double poles only appear at higher orders, i.e., in the computation of graphs with three or more loops). Actually, with the restrictions mentioned the method has been applied and tested in a variety of problems in $2+1$ dimensions [7,13].

The vertex functions so defined approximately satisfy a renormalization group equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\mu\frac{\partial}{\partial\mu}+\beta_1\frac{\partial}{\partial g_1}+\beta_2\frac{\partial}{\partial g_2}-\gamma N_F\right]\Gamma^{(N)}(p_1,\ldots,p_N)\approx 0, \tag{7}$$

where, as a consequence of the Coleman-Hill theorem [14], a term proportional to the derivative of the α parameter is absent. The coefficients γ and β_i can be calculated by substituting the two- and four-point functions into Eq. (7).

To fix γ notice that, up to two loops, only graphs which are second order in α may contribute to the wave function renormalization (i.e., linearly divergent graphs with two external fermionic lines). There are three graphs (the same as in Fig. 2 of [8]) and a direct computation gives

$$\gamma=-\frac{N+1}{24}\alpha^2. \tag{8}$$

Notice that for $N=1$ this result agrees with [8], as it should.

Analogously, β_1 and β_2 can be determined from the momentum-independent residues in the four-point vertex functions. In this calculation, it should be observed that the μ dependence of the pole part arises through the expansion of the $\mu^{\epsilon}=1+\epsilon\ln\mu+O(\epsilon^2)$ factors, introduced for each loop momentum integral. We denote the Fourier transform of $\langle 0|\,\mathrm{T}\psi_{\alpha_1 a_1}(x_1)\psi_{\alpha_2 a_2}(x_2)\overline{\psi}_{\alpha_3 a_3}(y_1)\overline{\psi}_{\alpha_4 a_4}(y_2)|0\rangle$ by $\Gamma^{(4)}_{\alpha_1 a_1,\alpha_2 a_2;\alpha_3 a_3,\alpha_4 a_4}$, where Lorentz and SU($N$) indices are represented by Greek and Latin letters, respectively. We found that, up to third order in g_1, g_2, and α, the μ dependence of the four-point function is given by

$$\begin{aligned}&\mu\text{-dependent part of } \Gamma^{(4)}_{\alpha_1 a_1,\alpha_2 a_2;\alpha_3 a_3,\alpha_4 a_4}(p_i=0)\\&=-2i\alpha^2\ln\mu\left\{\left[\frac{g_1}{\Lambda}\left(\frac{7}{2}+3N\right)+\frac{g_2}{\Lambda}(9+3N)\right](\Delta\otimes\Delta)\right.\\&\quad+\left[\frac{g_1}{\Lambda}\left(\frac{5}{2}+\frac{N}{3}\right)-\frac{g_2}{\Lambda}\left(\frac{2}{3}+\frac{N}{3}\right)\right]\\&\quad\left.\times(\Gamma\otimes\Gamma)\right\}_{\alpha_1 a_1,\alpha_2 a_2,\alpha_3 a_3,\alpha_4 a_4},\end{aligned} \tag{9}$$

where we adopted the simplified notation

$$(\Delta\otimes\Delta)_{\alpha_1 a_1,\alpha_2 a_2;\alpha_3 a_3,\alpha_4 a_4}=\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\delta_{a_1 a_3}\delta_{a_2 a_4}-\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}\delta_{a_1 a_4}\delta_{a_2 a_3}, \tag{10}$$

$$(\Gamma\otimes\Gamma)_{\alpha_1 a_1,\alpha_2 a_2;\alpha_3 a_3,\alpha_4 a_4}=\gamma^{\mu}_{\alpha_1\alpha_3}\gamma_{\mu\alpha_2\alpha_4}\delta_{a_1 a_3}\delta_{a_2 a_4}-\gamma^{\mu}_{\alpha_1\alpha_4}\gamma_{\mu\alpha_2\alpha_3}\delta_{a_1 a_4}\delta_{a_2 a_3}, \tag{11}$$

for the Gross-Neveu and Thirring vertices, respectively.

Substituting the above expression into Eq. (7), using Eq. (8), and equating to zero the coefficients of the Gross-Neveu and Thirring vertices we determine β_1 and β_2 to be

$$\beta_1=g_1-\frac{43+37N}{6}g_1\alpha^2-2(9+3N)g_2\alpha^2, \tag{12}$$

$$\beta_2=g_2-\left(5+\frac{2N}{3}\right)g_1\alpha^2+\frac{1}{2}\left(\frac{7}{3}+N\right)g_2\alpha^2. \tag{13}$$

Since the Gross-Neveu and Thirring interactions were taken as independent, these expressions are valid only if $N>1$. They show that the renormalization group fixed points, defined through the vanishing of β_1 and β_2, will require

$$\alpha^2=\alpha_c^2=\frac{6[-17N-18+(\Theta)^{1/2}]}{(3541+1900N+255N^2)}, \tag{14}$$

with $\Theta=3865+2512N+544N^2$. However, to better understand the nature of this result it is convenient to use the systematic procedure devised by Zimmermann [15] which allows us to consider just one constant, g_1 let us say, as independent. The other coupling is then fixed so as to have just one β function in the renormalization group equation. Such a scheme has been applied in a variety of circumstances, including cases of nonrenormalizable models treated as effective theories [16]. We thus suppose that $g_2=\rho_0 g_1$ where ρ_0 is a constant such that $\beta_2=\rho_0\beta_1$, which gives

$$\rho_0=\frac{-25-20N+(\Theta)^{1/2}}{36(3+N)}. \tag{15}$$

In this situation,

$$\beta_1=g_1\left\{1-\alpha^2\left[\frac{43+37N}{6}+(18+6N)\rho_0\right]\right\}. \tag{16}$$

From this equation we conclude that the origin is an infrared fixed point stable or unstable accordingly $\alpha<\alpha_c$ or $\alpha>\alpha_c$. At $\alpha=\alpha_c$, $\beta_1=0$ and the theory is approximately scale invariant.

We want now to go back to the question posed at the beginning of this paper: namely, if $N>1$, does the coupling with the CS field improve the ultraviolet behavior of quartic operators? If this were the case, one could use this quartic interaction to perturb the model of fermionic particles interacting just through a CS field. We thus consider $g_1=g_2=0$ and study the renormalization behavior of integrated operators of canonical dimension 4. Specifically, we define (symbolically)

$$\Delta_1=\int d^3x\,\bar{\psi}D^2\psi,\quad \Delta_2=\int d^3x(\bar{\psi}\psi)^2,$$

$$\Delta_3=\int d^3x(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{17}$$

where $D^2=D_\mu D^\mu$ and $D_\mu=\partial_\mu-i\sqrt{\alpha}A_\mu$ is the covariant derivative. The renormalized integrated operators are obtained by removing poles so that, up to second order in α, in momentum space the renormalized amplitude with the insertion of the operator Δ_i is

$$\Gamma_{\Delta_i}=(1-\tau)I_{\Delta_i}=(\delta_{ij}+z_{ij})\Gamma'_{\Delta_j}, \tag{18}$$

where τ is the operator for the pole part [8], I_{Δ_i} is the dimensionally regularized integral, Γ'_{Δ_i} is the μ-independent part of Γ_{Δ_i}, and the matrix z is given by

$$z=2\alpha^2\ln[\mu]\begin{pmatrix}-\frac{1}{3} & 0 & 0\\ 0 & 7/2+3N & 5/2+N/3\\ (1+N/4)\frac{1}{6\pi\alpha} & 9+3N & -2/3-N/3\end{pmatrix}. \tag{19}$$

With this understanding we may write $\Delta_{iR}=(\delta_{ij}+z_{ij})\Delta'_j$, where Δ'_j is the finite part corresponding to $\Gamma'_{\Delta i}$.

Although the operators Δ_i in Eqs. (17) are not multiplicative renormalized, we can find new operators having such a property by taking adequate linear combinations $\bar{\Delta}_i=C_{ij}\Delta_j$. The new renormalized operators are then linear combinations of the old ones, $\bar{\Delta}_{iR}=C_{ij}\Delta_{jR}$.

The specific form of the matrix C is not actually relevant but it is such that $\bar{\Delta}_{iR}=(\delta_{ij}+Z_{ij})\bar{\Delta}_j$ where Z is a diagonal matrix. We found

$$Z=2\alpha^2\ln\mu\;\mathrm{Diagonal}\left(-1/3,\;\frac{1}{12}(-\sqrt{\Theta}+17+16N),\;\frac{1}{12}(\sqrt{\Theta}+17+16N)\right). \tag{20}$$

We are now in a position to calculate the anomalous dimension for these operators. Indeed, from the above results and noticing that they satisfy

$$\left(\mu\frac{\partial}{\partial\mu}-\gamma N_F+\gamma_{\bar{\Delta}_{iR}}\right)\Gamma^{(N_F)}_{\bar{\Delta}_{iR}}=0, \tag{21}$$

we arrive at

$$\gamma_{\bar{\Delta}_{1R}}=\frac{7-N}{12}\alpha^2,\qquad \gamma_{\bar{\Delta}_{2R}}=\frac{1}{6}(\sqrt{\Theta}-17N-18)\alpha^2, \tag{22}$$

and

$$\gamma_{\bar{\Delta}_{3R}}=-\frac{1}{6}(\sqrt{\Theta}+17N+18)\alpha^2=-\frac{\alpha^2}{\alpha_c^2}. \tag{23}$$

Thus, in the infrared-stable region, $\alpha<\alpha_c$, there are two operators ($\bar{\Delta}_{1R}$ and $\bar{\Delta}_{3R}$) whose dimensions decrease with N. The anomalous dimension $\gamma_{\bar{\Delta}_{1R}}$ has a very small variation, implying that the ultraviolet behavior of the corresponding operator is not improved in a meaningful way. The situation is much better concerning the second operator. By conveniently choosing α near α_c, the operator dimension of $\bar{\Delta}_{3R}$ may become as near 3 as we want and, for all practical purposes, the interaction behaves like a renormalizable one. This operator is therefore a natural candidate for implementing a consistent perturbation scheme around the conformal invariant theory of fermions interacting through a Chern-Simons field. Of course, higher order corrections may modify the above results. Thus, increasing the parameter Λ will require the inclusion of new interactions and in principle new couplings will be needed. However, we may conjecture that in this phase, following Zimmermann's procedure, it will also be possible to fix the new couplings as definite functions of just one four-fermion coupling.

This work was supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (Capes).

[1] D. Gross, in *Methods in Field Theory*, Proceedings of the Les Houches Summer School of Theoretical Physics, Les Houches, France, 1975, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976); G. Parisi, Nucl. Phys. **B100**, 368 (1975); B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. Lett. **62**, 1433 (1989).
[2] M. Gomes, R. S. Mendes, R. F. Ribeiro, and A. J. da Silva, Phys. Rev. D **43**, 3516 (1991).
[3] N. V. Krasnikov and A. B. Kyatkin, Mod. Phys. Lett. A **6**, 1315 (1991).
[4] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. D **39**, 3088 (1989); L. Del Debbio, S. J. Hands, and J. C. Mehegan, Nucl. Phys. **B502**, 269 (1997); S. Hands, ''Fixed Point Four-Fermi Theories,'' hep-lat/9706018; J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991); K. Klimenko, Z. Phys. C **57**, 175 (1993); Theor. Math. Phys. **94**, 393 (1993).
[5] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995); G. P. Lepage, in Proceedings of TASI-89, Boulder, CO, edited by T. DeGrand and D. Toussiant (World Scientific, Singapore, 1990); G. P. Lepage *et al.*, Phys. Rev. D **46**, 4052 (1992); M. Luke and A. V. Manohar, *ibid.* **55**, 4129 (1997).
[6] S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992); W. A. Bardeen, C. T. Hill, and M. Lindner, Phys. Rev. D **41**, 1647 (1990); M. Atance and L. Cortés, *ibid.* **54**, 4973 (1996).
[7] W. Chen and M. Li, Phys. Rev. Lett. **70**, 884 (1993); W. Chen, Nucl. Phys. **B435**, 689 (1995).
[8] V. S. Alves, M. Gomes, S. V. L. Pinheiro, and A. J. da Silva, Phys. Rev. D **B59**, 045002 (1999).
[9] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); J. F. Schonfeld, Nucl. Phys. **B185**, 157 (1981).
[10] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).
[11] M. Gomes, V. O. Rivelles, and A. J. da Silva, Phys. Rev. D **41**, 1363 (1990).
[12] J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).
[13] L. V. Avdeev, G. V. Grigoryev, and D. I. Kazokov, Nucl. Phys. **B382**, 561 (1992); G. W. Semenoff, P. Sodano, and Yong-Shi Wu, Phys. Rev. Lett. **62**, 715 (1989); W. Chen, G. W. Semenoff, and Y. S. Wu, Phys. Rev. D **46**, 5521 (1992).
[14] S. Coleman and B. Hill, Phys. Lett. **159B**, 184 (1985).
[15] W. Zimmermann, Commun. Math. Phys. **97**, 211 (1985); R. Oehme and W. Zimmermann, *ibid.* **97**, 569 (1985).
[16] L. A. W. Toro, Z. Phys. C **56**, 635 (1992); M. Atance and L. Cortés, Phys. Rev. D **56**, 3611 (1997).