# Thermodynamics and statistical mechanics of induced Liouville gravity

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(Received 8 February 1999; published 25 June 1999)

In this paper we describe Liouville gravity which is induced by a set of quantum fields (constituents) and represents a two-dimensional analogue of Sakharov's induced gravity. The important feature of the considered theory is the presence of massless constituents which are responsible for the appearance of the induced Liouville field. The role of the massive constituents is only to induce the cosmological constant. We consider the instanton solutions of the Euclidean Liouville gravity with negative and zero cosmological constants, some instantons being interpreted as two-dimensional anti–de Sitter ($AdS_2$) black holes. We study the thermodynamics of all the solutions and conclude that their entropy is completely determined by the statistical-mechanical entropy of the massless constituents. This shows, in particular, that the constituents of the induced gravity are the true degrees of freedom of $AdS_2$ black holes. Special attention is also paid to the induced Liouville gravity with a zero cosmological constant on a torus. We demonstrate the equivalence of its thermodynamics to the thermodynamics of BTZ black holes and comment on computations of the BTZ black hole entropy. [S0556-2821(99)01514-3]

PACS number(s): 04.60.Kz, 05.30.−d, 04.70.Dy

## I. INTRODUCTION

The microscopic explanation of the Bekenstein-Hawking entropy [1,2] of black holes is one of the most intriguing problems of theoretical physics. Although there are several approaches to its resolution (for a review of some of them see [3,4]) this problem is a subject of intensive study and continues to inspire new ideas.

In particular, it was realized recently that at least some black holes may be macroscopically equivalent to two-dimensional systems described by Liouville theory. The statistical-mechanical entropy of these systems can be computed by means of a conformal field theory and coincides with the black hole entropy.

This observation was first made for extremal black holes [5] and was based on dualities in string models. However, in the most explicit and simple form it appeared in the work by Strominger [6] concerning Bañados-Teitelboim-Zanelli (BTZ) black holes [7] (for a detailed analysis of these computations with the large list of references see [8]). The BTZ black holes have the same thermodynamic characteristics as dual Liouville theory defined at asymptotic infinity. The reason for such a relation arises from a specific property of three-dimensional gravity with a negative cosmological constant. It is equivalent to Chern-Simons theory which has only boundary degrees of freedom that are described by Liouville theory [9,10].

Liouville theory which is dual to the BTZ black hole can be also defined at the black hole horizon [11]. Moreover, close to the horizon one can find a Liouville-like description of black holes in an arbitrary dimension [12,13]. This description becomes possible because the gravitational action for spherically symmetric metrics is the action for a two-dimensional dilatonic gravity. In the region near the horizon this gravity is equivalent to Liouville theory [13] and one can use the conformal theory to calculate the entropy.

These results may have the following interpretation [14]. On the level of thermodynamics black holes are equivalent to Liouville field theory. Liouville field, which is purely classical, is a collective excitation of some quantum constituents which are described by 2D conformal field theory. To put it in another way, Liouville theory is an effective theory of the constituents. The constituents are those microscopic degrees of freedom which explain the thermodynamic entropy related to Liouville field and thus reproduce the black hole entropy in a statistical-mechanical way.

Remarkably, this mechanism is basically the same as the mechanism of the generation of the black hole entropy in Sakharov's induced gravity [15–18]. According to Sakharov's idea [19], the gravitational field is a collective excitation of the matter constituents and the Einstein action is the low-energy effective action of the constituents. The equations for the metric $g_{\mu\nu}$ are

$$\langle \hat{T}_{\mu\nu}(g) \rangle = 0. \tag{1.1}$$

Here $\hat{T}_{\mu\nu}$ is the stress-energy tensor of constituent fields on the background with the metric $g_{\mu\nu}$ and its average is taken in some quantum state.

It should be noted that in Sakharov's induced gravity the microscopic states of a black hole are related to the constituents which live on the physical space-time. In the Liouville description of black holes the microscopic degrees of freedom live on a dual two-dimensional space-time. For this rea-

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son the statistical origin of the Bekenstein-Hawking entropy in the two approaches is different.

Nevertheless, the similarity of both approaches suggests that they may be connected. To see whether there is any connection one must first understand better how Sakharov's mechanism of induced gravity works in the case of Liouville theory. Studying induced Liouville gravity is the subject of this paper.

The important feature of the considered models of two-dimensional induced gravity is the presence of massless constituents which induce Liouville action. The massive constituents serve only to induce a finite cosmological constant. Thus, as distinct from higher dimensions, all the dynamics in Liouville induced gravity is due to massless fields. This fact has a crucial consequence for the statistical interpretation of the thermodynamics of the instanton solutions of the theory.

We consider models with negative and zero cosmological constants. The theory with negative cosmological constant is a sort of gravity on anti–de Sitter $AdS_2$ space-time. Studying this theory is motivated by various reasons, one of which is its relation to $AdS_2$ string theories arising as near-horizon limits of different four- and five-dimensional black holes (for a recent discussion of $AdS_2$ gravity in this context see [20]). Liouville induced gravity with a vanishing cosmological constant is interesting because it is that theory which is thermodynamically equivalent to BTZ black holes.

The paper is organized as follows. In Sec. II we use Sakharov's mechanism to induce Liouville gravity by massless and massive quantum constituent fields. The considered quantum models are free from ultraviolet divergences and, as a result, the induced theory is finite and well-defined. All the solutions of Liouville gravity with the negative cosmological constant are locally $AdS_2$. In the Euclidean theory they can be of three types: elliptic, hyperbolic, and parabolic. We remind the reader the form of these solutions in Sec. III. The instantons of the elliptic type can be interpreted as $AdS_2$ black holes. Thermodynamics and statistical mechanics of these black holes are studied in Sec. IV. We demonstrate that in the physical processes the changes of the entropy of a $AdS_2$ black hole coincide with the corresponding changes of the entanglement (statistical-mechanical) entropy of the massless constituents. Therefore, the constituents of the induced gravity are the true internal degrees of freedom of $AdS_2$ black holes. In Sec. V we comment on thermodynamics and statistical mechanics of the hyperbolic and parabolic solutions. Finally, in Sec. VI we consider induced Liouville gravity with a vanishing cosmological constant on a torus and demonstrate the equivalence of its thermodynamics to the thermodynamics of the BTZ black hole under identifying the central charges of both theories. A brief discussion is given in Sec. VII.

## II. INDUCED LIOUVILLE GRAVITY

Induced Liouville gravity (ILG) can be constructed from models with different constituent field species, similar to the construction of induced Einstein gravity [16–18]. To illustrate the idea we consider here the simplest model. It consists of noninteracting scalar and spinor fields, with some of the scalar fields being massless. The numbers of massive and massless scalars are $N_s$ and $N$, respectively, and the number of massive spinor fields is $N_d$. It is assumed that some massive scalars are nonminimally coupled and the corresponding constants are denoted as $\xi_s$. The effective gravitational action of the fields propagating on a background with the metric $g_{\mu\nu}$ is

$$\Gamma=\sum_s W_s+\sum_d W_d+NW_s^0, \tag{2.1}$$

$$W_s=\frac{1}{2}\log\det(-\nabla^2+\xi_s R+m_s^2),$$

$$W_d=-\log\det(\gamma^\mu\nabla_\mu+m_d), \tag{2.2}$$

$$W_s^0=\frac{1}{2}\log\det(-\nabla^2). \tag{2.3}$$

Here $R$ is the scalar curvature of the background. It is not difficult to show that $\Gamma$ is free from ultraviolet divergences if the following constraints are satisfied:

$$N_s+N-2N_d=0, \quad \sum_s m_s^2-2\sum_d m_d^2=0, \tag{2.4}$$

$$N_s+N+N_d-6\sum_s \xi_s=0. \tag{2.5}$$

Constraints (2.4) ensure the finiteness of the induced cosmological constant, while condition (2.5) guarantees the finiteness of the induced Newton constant.

Suppose now that masses $m_i$ have the order of magnitude of a typical mass $M$. The low-energy limit of the theory is realized when the curvature of the background geometry is small compared to $M^2$. In this limit contributions of the massive constituents to the induced action can be expanded in powers of the curvature. On the other hand, the contributions of the massless constituents $W_s^0$ can be calculated exactly.

It is convenient to represent the induced action in the form

$$\Gamma=\Gamma^m+NW, \tag{2.6}$$

$$\Gamma^{\rm m}=\sum_s W_s+\sum_d W_d+NW^0_{s,\rm div}, \tag{2.7}$$

$$W\equiv W_s^0-W^0_{s,\rm div}. \tag{2.8}$$

Here $W^0_{s,\rm div}$ is the divergent part of the action of massless fields, so that $W$ is the "renormalized" action. Note that because we are dealing with ultraviolet finite theories the functionals $\Gamma^{\rm m}$ are free from the divergences. The divergences of the massive fields in $\Gamma^{\rm m}$ are canceled by the term $NW^0_{s,\rm div}$.

Let us consider fields given on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. The metric on $\mathcal{M}$ will be denoted $g_{\mu\nu}$. In the general case, $g_{\mu\nu}$ can have Lorentzian or Euclidean signature. In what follows we assume that $\mathcal{M}$ is a Euclidean manifold. In the low energy limit of the theory the action of the massive constituents $\Gamma^m$ can be expanded in powers of the curvature. We keep only the leading (cosmological) term in this decomposition and approximate $\Gamma^m$ as

$$\Gamma^m = \int_{\mathcal{M}} \sqrt{g}\, d^2x\, \lambda. \tag{2.9}$$

Here the cosmological constant $\lambda$ is

$$\lambda = -\frac{1}{8\pi}\left( \sum_s m_s^2 \ln m_s^2 - 2\sum_d m_d^2 \ln m_d^2 \right). \tag{2.10}$$

In what follows, we consider only the models where the cosmological constant is negative or zero. The curvature corrections to expression (2.9) are suppressed by powers $RM^{-2}$.

The above approximation is not applicable to the action $W$ of massless fields. In fact, this functional is the well-known Polyakov action [21] which can be computed exactly. Consider the conformal map of $\mathcal{M}$ onto a space $\bar{\mathcal{M}}$ with the metric $\bar{g}_{\mu\nu} = \exp(-2\sigma) g_{\mu\nu}$. The actions on $\mathcal{M}$ and $\bar{\mathcal{M}}$ are related as

$$W[g] = W[\bar{g}] - \frac{1}{24\pi}\left[ \int_{\mathcal{M}} \sqrt{g}\, d^2x (R\sigma - (\nabla\sigma)^2) + \int_{\partial\mathcal{M}} h^{1/2} dy (2K\sigma + 3n^\mu \sigma_{,\mu}) \right]. \tag{2.11}$$

Here $n^\mu$ is a unit vector normal to the boundary $\partial\mathcal{M}$ of $\mathcal{M}$; $K$ and $h$ are the extrinsic curvature and the metric on $\partial\mathcal{M}$. The functional $W[\bar{g}]$ is the effective action computed on the background $\bar{\mathcal{M}}$ with the metric $\bar{g}$. It is convenient to assume that $\bar{\mathcal{M}}$ is locally flat.

As follows from the above analysis, the induced gravitational action after subtracting a boundary term depending on $n^\mu \sigma_{,\mu}$ can be written in the form

$$\Gamma[g] = I_L[g,\phi] + N W[\bar{g}], \tag{2.12}$$

$$I_L[g,\phi] = -\frac{1}{8\pi}\int_{\mathcal{M}} \sqrt{g}\, d^2x \left( (\nabla\phi)^2 + \frac{2}{\gamma} R\phi + \frac{\mu}{\gamma^2} \right) - \frac{1}{2\pi\gamma}\int_{\partial\mathcal{M}} h^{1/2} dy K\phi. \tag{2.13}$$

Here we put

$$\phi = \frac{2}{\gamma}\sigma, \tag{2.14}$$

$$\gamma = \sqrt{\frac{12}{N}}, \quad \mu = \frac{96\pi}{N}|\lambda|. \tag{2.15}$$

The action $I_L$ can be also represented as a functional on the flat space $\bar{\mathcal{M}}$

$$I_L[\phi,g] = \bar{I}_L[\phi] = -\frac{1}{2\pi}\int_{\bar{\mathcal{M}}} d^2y \left( (\bar{\nabla}\phi)^2 + \frac{\mu}{\gamma^2} e^{\gamma\phi} \right) - \frac{\beta}{2\pi\gamma}(\phi_+ - \phi_-). \tag{2.16}$$

Here $\beta$ is the circumference length of the boundary, and $\phi_+$, $\phi_-$ are the values of $\phi$ on the external and internal parts of the boundary. If the internal boundary is absent $\phi_- = 0$ in Eq. (2.16).

Up to the boundary term, $\bar{I}_L[\phi]$ is the canonical Euclidean Liouville action.[1] The Liouville theory is known from the last century as a theory of negatively curved surfaces. A review of some its properties can be found in [22,23]. The important feature of Eq. (2.16) is that it describes a classical conformal theory with the central charge

$$c = \frac{12}{\gamma^2}, \tag{2.17}$$

which in our model is just the number of massless constituents

$$c = N, \tag{2.18}$$

see Eq. (2.15). The latter fact is not surprising. The massless constituents of our model are conformally invariant in two dimensions. Under quantization the conformal symmetry acquires a central extension due to the conformal anomaly. The central charge $c$ corresponds to this anomaly.

## III. SOLUTIONS TO THE LIOUVILLE THEORY

Equation (2.16) shows that the Liouville field is the only dynamical variable of induced Liouville gravity.[2] By varying $\bar{I}_L$ with a fixed boundary value of $\phi$, one finds the equation

$$e^{-\gamma\phi}\bar{\Delta}\phi = -\frac{\mu}{2\gamma}. \tag{3.1}$$

It follows from this equation that the physical metric $g_{\mu\nu} = e^{\gamma\phi}\bar{g}_{\mu\nu}$ corresponds to a space with constant negative curvature[3] $R = -\mu/2$. This space is locally a two-

[1]Strictly speaking, $\bar{I}_L[\phi]$ differs from the standard definition by a sign, see [23].

[2]To avoid the confusion, let us note that $\phi$ is the dynamical variable only in the classical theory, in quantum Liouville theory $\phi$ appears in the conformal gauge but its contribution is compensated by the contribution of corresponding ghosts.

[3]It should be noted that $R = -\mu/2$ results in further restrictions on the parameters of the constituent fields of Polyakov induced gravity. Namely, the condition of the large masses, $M^2 \gg R$, becomes $M^2 \gg 48\pi|\lambda|/N$ where $M$ is a typical scale for masses of the fields and $\lambda$ is given by Eq. (2.10). One can construct models where this condition is satisfied.

dimensional Lobachevsky space $H_2$. The corresponding solution in the Lorentzian space-time is locally anti–de Sitter $(\mathrm{AdS}_2)$.

There is also another approach to the variational problem. One can start with the functional (2.12) and consider Liouville field and metric $g_{\mu\nu}$ as independent variables. Then the equations of motion obtained from Eq. (2.12) by varying[4] $\phi$ and $g_{\mu\nu}$, respectively, are

$$R=\gamma\Delta\phi, \tag{3.2}$$

$$G_{\mu\nu}=0, \tag{3.3}$$

$$G_{\mu\nu}\equiv\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}|\nabla\phi|^2+\frac{2}{\gamma}(g_{\mu\nu}\Delta\phi-\nabla_\mu\nabla_\nu\phi)+\frac{\mu}{2\gamma^2}g_{\mu\nu}. \tag{3.4}$$

The trace of Eq. (3.3) results in the relation

$$\Delta\phi=-\frac{\mu}{2\gamma}, \tag{3.5}$$

which coincides with Liouville equation (3.1). This justifies considering the more general variational problem where $\phi$ and the metric are independent fields.

When $\phi$ obeys (3.1),

$$G_{\mu\nu}=-4\pi\langle\hat{T}_{\mu\nu}\rangle, \tag{3.6}$$

where $\hat{T}_{\mu\nu}$ is the quantum stress-energy tensor of the constituents computed in the considered approximation (3.2). Thus, Eq. (3.3) is equivalent to relation (1.1) of the Sakharov's induced gravity. It should be noted that boundary conditions for $\phi$ which are required to solve for it from Eq. (3.2) are related to the choice of the quantum state.[5]

There are three types of solutions of the Euclidean Liouville equations (3.1): elliptic, parabolic, and hyperbolic (see, e.g., [23]). These solutions correspond to different coordinate maps on the Lobachevsky space $H_2$.

(1) *Elliptic solutions.* The metric has the form[6]

$$ds^2=e^{\gamma\phi}\overline{ds^2}=\frac{16}{\mu}\frac{1}{(1-\rho^2)^2}\overline{ds^2}, \tag{3.7}$$

$$\overline{ds^2}=\rho^2d\tau^2+d\rho^2, \quad 0\leqslant\tau\leqslant2\pi. \tag{3.8}$$

We specify the flat metric by the boundary condition

$$0<\rho\leqslant\rho_+. \tag{3.9}$$

Solution (3.7) can be also written in the form

$$ds^2=g(x)d\tau^2+\frac{1}{g(x)}dx^2, \tag{3.10}$$

$$g(x)=\frac{\mu}{4}x^2+\frac{1}{2}x, \tag{3.11}$$

$$x=\frac{8}{\mu}\frac{\rho^2}{1-\rho^2}. \tag{3.12}$$

This solution has the topology of a disk and can be interpreted as a black hole instanton. The horizon of the black hole is located at $x=0$ $(\rho=0)$. The normalization of the time coordinate $\tau$ is chosen so that the corresponding surface gravity constant is $g'(0)/2=1$. One can make the surface gravity equal to another constant by rescaling $\tau$.

The black hole solutions in the Lorentzian metric have the anti–de Sitter geometry and we will call them $\mathrm{AdS}_2$ black holes for brevity.

(2) *Hyperbolic solutions*. A hyperbolic solution depends on an integration constant $m$ and has the form

$$ds^2=e^{\gamma\phi}\overline{ds^2}=\frac{4}{\mu}\frac{m^2}{\rho^2\sin^2(m\ln\rho)}\overline{ds^2}. \tag{3.13}$$

The flat metric $\overline{ds^2}$ is given by Eq. (3.8). The hyperbolic solution has the topology of a cylinder. The flat space $\mathcal{M}$ is defined by the boundary condition $\rho_-\leqslant\rho\leqslant\rho_+$, where

$$e^{-\pi/m}<\rho_-<\rho_+<1. \tag{3.14}$$

The metric (3.13) can be also written in the form (3.10) with

$$g=\frac{\mu}{4}x^2+\frac{4m^2}{\mu}, \tag{3.15}$$

$$x=-\frac{4m}{\mu}\cot(m\ln\rho). \tag{3.16}$$

(3) *Parabolic solutions*. Parabolic solutions can be obtained from the hyperbolic ones in the limit $m\to0$ and look as follows:

$$ds^2=e^{\gamma\phi}\overline{ds^2}=\frac{4}{\mu}\frac{1}{(\rho\ln\rho)^2}\overline{ds^2}, \tag{3.17}$$

where $\overline{ds^2}$ is determined by Eq. (3.8). The coordinate $\rho$ is subject to the boundary condition $\rho_-\leqslant\rho\leqslant\rho_+$, where $\rho_->0$ and $\rho_+<1$. In the limit when $\rho_-=0$ the curvature of the solution has a delta-functionlike singularity. The metric (3.17) can be also written in the form (3.10) with

$$g=\frac{\mu}{4}x^2, \tag{3.18}$$

$$x=-\frac{4}{\mu\ln\rho}. \tag{3.19}$$

As follows from Eq. (3.18) these solutions are analogous to extremal black holes.

[4]The variational procedure implies that the metric and $\phi$ are fixed on the boundary. The boundary term which depends on $n^\mu\sigma_{,\mu}$ was removed from induced action (2.12) in order to obey this requirement.

[5]A detailed discussion of this problem can be found for instance in Ref. [24].

[6]We consider only the solutions which are free from conical singularities.

Note that Liouville solutions of all three types have the same asymptotic behavior at large $x$. It is also worth pointing out that each Liouville solution can be interpreted as a boundary theory of a three-dimensional AdS-gravity and corresponds to a particular three-dimensional object [14].[7] Elliptic solutions with conical singularities correspond to massive particles in $AdS_3$, hyperbolic and parabolic solutions may be related to nonextreme and extreme three-dimensional black holes, respectively. The nonextreme black holes correspond to a particular Liouville theory on a torus.

## IV. ELLIPTIC SOLUTIONS

### A. Thermodynamics of $AdS_2$ black holes

Let us now study black holes corresponding to elliptic solutions. It is perhaps necessary at this stage to explain in what sense we expect constant curvature solutions to behave like black holes. The key assumption that we make is that Liouville field $\phi$ is an observable quantity, despite the fact that it is not, strictly speaking, a fundamental field. It is rather introduced for calculational convenience to make the effective action local and defined formally by the relationship $\phi=(\gamma\Delta)^{-1}R(g)$. In effect, the observability of $\phi$ requires matter to couple nonlocally to the metric $g$ (i.e., to $\Delta^{-1}R$). This does not occur at the classical level, but in the full quantum effective action one does expect such terms to appear. It is on this basis that we feel justified in assuming that $\phi$ is an observable field, analogous to a dilaton. Henceforth we treat Liouville action as if it were a dilaton gravity theory of the general form considered extensively in a variety of references [25].[8]

We will interpret the induced Euclidean action (2.12) considered on the elliptic solution (3.7) as $T$ times the free energy of the corresponding black hole. The black hole is completely characterized by the temperature $T$ measured at the boundary $x=x_+$,

$$T=(2\pi\sqrt{g(x_+)})^{-1/2}. \tag{4.1}$$

This condition defines $x_+$ in terms of $T$:

$$x_+=\frac{1}{\pi\mu T}(\sqrt{(4\pi T)^2+\mu}-4\pi T). \tag{4.2}$$

The Liouville action on the elliptic solution is

$$I_L(x_+)=-\frac{\mu}{2\gamma^2}x_+. \tag{4.3}$$

To calculate the induced action (2.12) one also has to know the value of the Polyakov action on the disk of the radius $\rho_+$. The dependence of this action on $\rho_+$ can be found out by making a scaling transformation (i.e., a conformal transformation with constant factor) to the disk of unit radius

$$NW[\overline{g}(\rho_+)]=-\frac{2}{\gamma^2}\ln\rho_+ + C. \tag{4.4}$$

Here $C$ is a constant corresponding to the action on the unit disk. It does not depend on $T$ and can be omitted. From Eqs. (3.12), (4.3), and (4.4) we obtain, for (2.12),

$$\Gamma=-\frac{1}{\gamma^2}\left(\frac{\mu}{2}x_+ + \ln\frac{\mu x_+}{\mu x_+ + 8}\right). \tag{4.5}$$

This result can be immediately expressed in terms of the temperature on the boundary by using Eqs. (4.1) and (4.2). By neglecting in Eq. (4.5) a numerical constant we find the free energy of the black hole:

$$F^{BH}(T)=T\Gamma(x_+)|_{x_+=x_+(T)}=-\frac{1}{\gamma^2}\left[\frac{1}{2\pi}\sqrt{(4\pi T)^2+\mu}\right.$$
$$\left.+2T\ln(\sqrt{(4\pi T)^2+\mu}-4\pi T)\right]. \tag{4.6}$$

Thus, the black hole entropy defined by the standard relation, is

$$S^{BH}(T)=-\frac{\partial F^{BH}}{\partial T}=\frac{2}{\gamma^2}\ln(\sqrt{(4\pi T)^2+\mu}-4\pi T). \tag{4.7}$$

(Here all numerical constants were omitted.) The energy corresponding to this solution is

$$E^{BH}(T)=-\frac{1}{2\pi\gamma^2}\sqrt{(4\pi T)^2+\mu}. \tag{4.8}$$

The variations of the energy and the entropy are related by the first law

$$dE^{BH}=TdS^{BH}. \tag{4.9}$$

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[7]It is interesting to note that in a 4D gravity there also exist three different families of Schwarzschild-(anti) de Sitter solutions. These solutions are given by metric $ds^2=-A\,dt^2+dr^2/A+r^2\,d\Omega^2$, where $A=K-(2m/r)+\Lambda r^2/3$, $d\Omega^2=d\theta^2+\mathrm{sinn}^2\theta\,d\phi^2$. For an elliptic case, $K=1$ and $\mathrm{sinn}\,\theta=\sin\theta$. For a hyperbolic case, $K=-1$ and $\mathrm{sinn}\,\theta=\sinh\theta$. For a parabolic case, $K=0$ and $\mathrm{sinn}\,\theta=\theta$. The elliptic solution (with $K=1$) corresponds to a usual black hole. The hyperbolic solution describes a black hole moving with a superluminar velocity, while the hyperbolic solution describes a black hole moving with the velocity of light.

[8]It is worth noting that there exists at least one other theory, namely, Jackiw-Teitelboim gravity, in which constant curvature black hole solutions have been analyzed in some detail. In that case, the black holes can be interpreted in terms of dimensionally reduced $2+1$ Einstein gravity (the BTZ black holes [7]). Interestingly, the Jackiw-Teitelboim theory was originally motivated by the connection between constant curvature metrics and induced Liouville gravity. The dilaton was considered to be a physically irrelevant Lagrange multiplier field needed to enforce the constant curvature equation. Once the dilaton is taken seriously, it becomes clear that Jackiw-Teitelboim gravity and induced Liouville gravity are quite different theories, despite the fact that the solutions to both involve constant curvature metrics.

It should be noted that the given thermodynamical system is not stable because its heat capacity is negative for all values of $T$:

$$c(T)=T\frac{\partial S^{BH}}{\partial T}=-\frac{1}{\gamma^2}\frac{8\pi T}{\sqrt{(4\pi T)^2+\mu}}. \tag{4.10}$$

The energy of this black hole decreases when its temperature grows. Increasing the temperature corresponds to moving the boundary closer to the black hole horizon ($x_+\to 0$).

As follows from Eq. (4.10) the heat capacity is increasing at small temperatures and one may speculate that in this region the black hole may be in a quasiequilibrium state. Thus at low temperatures our description of the black hole in terms of a canonical ensemble may be justified. On the other hand, at high temperatures $T$ the system is very unstable and one should expect a phase transition. Other possible phases of the theory may be related to other solutions of ILG.

### B. Black hole statistical mechanics

#### *1. Black hole canonical ensemble in terms of the constituents*

We now show how the black hole can be described in terms of statistical mechanics of the constituents of ILG. To this aim let us consider the constituents which propagate in the static region of the black hole, outside the horizon, and compare the canonical ensembles of the black hole and the constituents.

It is well known that the description of statistical-mechanics in the presence of a Killing horizon meets difficulties due to the divergences of the density of states near the horizon [3]. The standard method to proceed in this situation is to introduce a cutoff near the horizon at some proper distance $\epsilon$. This cutoff can be considered as an inner boundary. Then the regularized free energy $F(\epsilon,T)$ of the constituents has the standard definition

$$e^{-F/T}=\mathrm{Tr}\, e^{-(\hat{H}-\mathcal{E}_v)/T}, \tag{4.11}$$

where $T$ is the temperature of the system. The operator $\hat{H}$ is normally ordered total Hamiltonian of all the constituents, and $\mathcal{E}_v$ is the zero-point energy. In our approach the massive constituents are very heavy and contribute only to the cosmological constant. It means that in the considered approximation all the effect of these fields is in the vacuum energy $\mathcal{E}_v$. Thus, one can rewrite Eq. (4.11) as

$$e^{-F/T}=e^{\mathcal{E}_v/T}(\mathrm{Tr}\, e^{-\hat{H}_0/T})^N=e^{(\mathcal{E}_v-NF_0)/T}, \tag{4.12}$$

where $\hat{H}_0$ and $F_0$ are, respectively, the normally ordered Hamiltonian and the statistical-mechanical free energy of a single massless constituent.

It is also instructive to represent the free energy (4.11) of the constituents in another form, in terms of the induced Euclidean action $\Gamma(x_+,\epsilon,T)$. The Euclidean theory is formulated on the elliptic solution (3.7) with an inner boundary at $\rho_-$ determined by the cutoff,

$$\rho_-\simeq e^{-\gamma\phi_-/2}\epsilon, \tag{4.13}$$

where $\phi_-$ is the value of the Liouville field at this boundary ($\phi_-$ is a finite constant at $\epsilon\to 0$). The both functionals are related as

$$\Gamma(x_+,\epsilon,T)+C=F(\epsilon,T)T^{-1}, \tag{4.14}$$

where $C$ is a possible finite numerical constant which accounts for, according to [26], the difference between field-theoretical and statistical-mechanical computations. $x_+$ in Eq. (4.14) is expressed in terms of $T$ by Eq. (4.2). The induced action is determined by Eq. (2.12),

$$\Gamma(x_+,\epsilon,T)=I_L(x_+,\epsilon,T)+NW[K_\epsilon]. \tag{4.15}$$

The action $W[K_\epsilon]$ is the Polyakov action on the annulus $K_\epsilon$:

$$ds^2=\rho^2 d\tau^2+d\rho^2, \quad \rho_-(\epsilon)\leqslant\rho\leqslant\rho_+. \tag{4.16}$$

It is related to the Polyakov action on the cylinder $Q_b$:

$$ds^2=d\tau^2+dy^2, \quad 0\leqslant\tau\leqslant 2\pi, \tag{4.17}$$

$$dy=d\ln\rho, \quad 0\leqslant y\leqslant b, \tag{4.18}$$

by the conformal transformation,

$$W[K_\epsilon]=W[Q_b]-\frac{b}{12}, \tag{4.19}$$

$$b=\ln\frac{\rho_+}{\rho_-}\simeq\ln\frac{x_+T}{\epsilon}. \tag{4.20}$$

Because of the conformal invariance the spectrum of single particle excitations of two-dimensional massless scalars coincides with the spectrum of these fields on related flat ultrastatic space (4.17) (see for the details, e.g., [3]). As a result of this property, the effective action on the cylinder is related to the statistical-mechanical free energy of a single massless constituent in a simple way:

$$TW[Q_b]=F_0(\epsilon,T). \tag{4.21}$$

Equations (4.12), (4.14), (4.15), and (4.21) enable us to find the expression for the vacuum energy

$$\mathcal{E}_v(\epsilon,T)=T\left(I_L(x_+,T)-\frac{N}{12}\ln\frac{\rho_+}{\rho_-}+C\right). \tag{4.22}$$

(Here we took into account that Liouville action on the annulus and that on the disc differ by a constant in the limit $\epsilon\to 0$.) The divergence of the vacuum energy is the result of the divergence of the density of states near the horizon.

The statistical-mechanical entropy of a two-dimensional gas on the interval with the size $b$ can be computed exactly in the limit of large $b$, see, e.g., [27]. If the temperature is $\beta^{-1}$, the free energy, entropy and energy are

$$F^{\mathrm{sm}}(b,\beta)\simeq-\frac{\pi}{6}\frac{b}{\beta^2}-\frac{1}{2\beta}\ln\frac{\beta}{2b}, \tag{4.23}$$

$$S^{\mathrm{sm}}(b,\beta)\simeq\frac{\pi}{3}\frac{b}{\beta}+\frac{1}{2}\ln\frac{\beta}{2b}, \tag{4.24}$$

$$E^{\rm sm}(b,\beta)\simeq\frac{\pi}{6}\frac{b}{\beta^2}-\frac{1}{2\beta}, \tag{4.25}$$

where all constants which are finite at large $b$ are omitted.

The parameter $\beta$ coincides with the periodicity of the Euclidean time $\tau$ in Eq. (4.17). In our case $\beta=2\pi$. By taking into account that the physical temperature is $T$ one can write

$$F_0(\epsilon,T)=2\pi TF^{\rm sm}(b,2\pi). \tag{4.26}$$

This relation in combination with Eqs. (4.12), (4.23)–(4.25) gives the following result:

$$F(\epsilon,T)=E(\epsilon,T)-TS(\epsilon,T), \tag{4.27}$$

$$E(\epsilon,T)=\mathcal{E}_v(\epsilon,T)+2\pi NTE^{\rm sm}(b,2\pi)$$
$$=TI_L(x_+)+CT, \tag{4.28}$$

$$S(\epsilon,T)=NS^{\rm sm}(b,2\pi). \tag{4.29}$$

Note that the total energy $E(\epsilon,T)$ is finite in the limit $\epsilon\to0$ because the divergence of zero-point fluctuations is compensated by the divergence of the thermal excitations of the massless constituents. Moreover, by using Eqs. (4.2) and (4.3) we find that

$$E^{BH}(T)=E(0,T)+CT, \tag{4.30}$$

where $C$ is a numerical constant. By using an arbitrariness in relation (4.14) between the Euclidean action and the free energy one can always make $C$ equal to 0. After this ''normalization'' the statistical-mechanical energy of the induced gravity constituents coincide with black hole energy (4.8).

Let us consider now the entropy of the constituents. Because this quantity corresponds to the fields propagating outside the horizon it can be interpreted as the entanglement entropy. As is well known, the entanglement entropy is divergent when the cutoff is removed. This is its key distinction from the black hole entropy $S^{BH}$. However, as follows from Eqs. (4.7), (4.24), and (4.29), the two entropies are related:[9]

$$S^{BH}(T)=S(\epsilon,T)-S_R(\epsilon), \tag{4.31}$$

$$S_R(\epsilon)\simeq-\frac{2}{\gamma^2}\ln\epsilon-\frac{6}{\gamma^2}\ln|\ln\epsilon|. \tag{4.32}$$

It is not difficult to see that $S_R(\epsilon)$ can be interpreted as the entropy of the constituents on the Rindler space with the same cutoff

$$ds^2=\rho^2d\tau^2+d\rho^2,\quad \epsilon<\rho\leqslant1,\quad 0\leqslant\tau\leqslant2\pi. \tag{4.33}$$

[9]An analogous subtraction formula for two-dimensional black holes was discussed in [27]. A similar computation of the entanglement entropy in two dimensions can be also found in [28]. However, these papers are dealing with quantum corrections to the black hole entropy rather than to the entropy itself.

It is important that if one considers variations of the parameters of the black hole at fixed value of the parameter $\epsilon$, changes of entropy (4.29) coincide with changes of thermodynamical entropy (4.7) of a black hole,

$$\Delta S^{BH}(T)=\Delta S(\epsilon,T). \tag{4.34}$$

Thus, from the point of view of thermodynamics, the two entropies are equivalent. Moreover, the above relation does not depend on the choice of the regularization prescription. Instead of using the cutoff near the horizon one can arrive at Eq. (4.34) using the dimensional or Pauli-Villars regularization schemes which also enable one to eliminate the divergences related to the horizon (see for details, e.g., [3]).

The analysis of this section demonstrates that the thermodynamics of $AdS_2$ black holes has a statistical-mechanical explanation in terms of the constituents of ILG. One can conclude that the constituents are the real degrees of freedom of the black hole. It is interesting to point out the black hole entropy is related to the massless constituents only. It does not mean, however, that the massive fields are irrelevant. As we saw, these constituents provide the finite cosmological constant and give a contribution to the vacuum energy which depends on the black hole parameters.

#### *2. Conformal field theory*

The above result for the entropy can be computed by means of a conformal field theory (CFT) along the lines of computations of the entropy of BTZ black holes [6]. Massless constituents of ILG are described by a CFT with the central charge $c=N$, see Eq. (2.17).

The computation of the entropy is as follows [29,30]. The relation between the Hamiltonian of the system and generators of Virasoro algebra follow from the representation of the metric (4.17) in the form

$$d\overline{s^2}=\left(\frac{b}{\pi}\right)^2(-d\eta^2+dz^2)=\left(\frac{b}{\pi}\right)^2du\ dv, \tag{4.35}$$

$$u=\frac{z+\eta}{2},\quad v=\frac{z-\eta}{2}. \tag{4.36}$$

Consequently,

$$\partial_t=\frac{\pi}{2b}(\partial_u-\partial_v). \tag{4.37}$$

In Eq. (4.35) the coordinate $z$ ranges from 0 to $\pi$. This corresponds to a theory on an interval where the points $z=0$ and $z=\pi$ are independent. In order to carry out the computations it is convenient to pass to a theory where $z$ is a periodic coordinate. This can be done if one considers two equivalent CFT's on the intervals with the length $\pi$ and makes from them a CFT on a circle by gluing together the ends of the intervals. In the obtained theory $z$ has the periodicity $2\pi$.

One has two copies of Virasoro algebra where the elements $L_n$ and $\overline{L_n}$ can be defined in a standard way as the generators of the coordinate transformations, $\delta u=e^{inu}$ and

$\delta v = e^{inv}$, respectively. As a result of relation (4.37), the Hamiltonian of massless constituents which generates transformations along the Killing time $t$ coincides with the operator $\pi(L_0 - \bar{L}_0)/2b$. Similarly, translations of the system along $y$ are generated by the momentum $\pi(L_0 + \bar{L}_0)/2b$. Because the system is at rest the average momentum is zero. On the other hand, the average value of the energy is $NE^{\mathrm{sm}}(b,\beta)$. This fixes the average values $h$ and $\bar{h}$ of $L_0$ and $\bar{L}_0$, respectively. In the leading approximation

$$h = -\bar{h} = \frac{N}{6}\frac{b^2}{\beta^2}. \tag{4.38}$$

In the limit when $b$ is large ($\epsilon$ goes to zero), $h \gg 1$ and one can use Cardy's formula to estimate the degeneracy of $L_0$ and $\bar{L}_0$. In this approximation the total degeneracy $D$ is

$$\ln D = 2\pi\sqrt{\frac{ch}{6}} + 2\pi\sqrt{\frac{c|\bar{h}|}{6}} \tag{4.39}$$

and by taking into account that in our case the central charge $c = N$ we find

$$\ln D = 2N\frac{\pi}{3}\frac{b}{\beta}. \tag{4.40}$$

Finally, we have to remember that $D$ is the number of states of the system with the doubled Hilbert space which results from the trick of imposing periodicity of the coordinate $z$. The real number of states of the system we are interested in is $\sqrt{D}$. Thus, the entropy is

$$\frac{1}{2}\ln D = NS^{\mathrm{sm}} \tag{4.41}$$

and it coincides exactly with the required value in Eq. (4.24) for $N$ fields.

A remark concerning fixing of the Virasoro level is in order. In our calculation the eigenvalues $h$, $\bar{h}$ are not connected with the observable energy (4.8) of a black hole but are determined by the average value of the normally ordered Hamiltonian, i.e., by the thermal energy of the fields.

## V. HYPERBOLIC AND PARABOLIC SOLUTIONS

It is also worth studying thermodynamics of parabolic and hyperbolic instantons of ILG. Below we briefly comment on its features.

The solutions have two boundaries, and the fiducial space $\bar{\mathcal{M}}$ in the both cases [see Eqs. (3.13) and (3.17)] is an annulus. For this reason there are no difficulties related to the horizon as in the case of $AdS_2$ black holes. This automatically provides the agreement between thermodynamics of the solutions and statistical mechanics of the constituents. To begin with it is convenient to replace the fiducial space $\bar{\mathcal{M}}$ in Eq. (2.12) to a cylinder by making coordinate and conformal transformations and changing the Liouville field. Then Eqs. (3.13) and (3.17) can be written as

$$ds^2 = e^{\gamma\phi}\overline{ds^2}, \tag{5.1}$$

$$\overline{ds^2} = d\tau^2 + dy^2, \tag{5.2}$$

$$0 \leq \tau \leq \beta, \quad y_- \leq y \leq y_+. \tag{5.3}$$

In case of the hyperbolic solution,

$$e^{\gamma\phi} = \frac{4}{\mu}\frac{1}{\sin^2 y}, \tag{5.4}$$

and $\beta = 2\pi m$ where $m$ is defined in Eq. (3.13). The boundary coordinates subject to the restrictions $y_- > 0$ and $y_+ < \pi$. For the parabolic solution

$$e^{\gamma\phi} = \frac{4}{\mu}\frac{1}{y^2}, \tag{5.5}$$

where $y_+ < 0$. In this case $\beta$ is an arbitrary parameter. In fact, the parabolic metric does not change when one rescales the coordinates $y$ and $\tau$ with the same coefficient. Thus one can choose any periodicity for $\tau$ by appropriate redefinition of the boundary values $y_\pm$.

The induced action (2.12) for hyperbolic and parabolic solutions has a very simple form because the Polyakov action on the cylinder (5.2) is determined by the statistical-mechanical free energy (4.23) at the temperature $\beta^{-1}$ on the interval of the length $b = y_+ - y_-$,

$$W(y_+ - y_-, \beta) = \beta F^{\mathrm{sm}}(y_+ - y_-, \beta). \tag{5.6}$$

Note, as we explained in the previous section, $F^{\mathrm{sm}}(y_+ - y_-, \beta)$ coincides with the free energy of a single massless scalar constituent of ILG. One can represent the induced action (2.12) in the form

$$\Gamma(y_+, y_-, \beta) = \beta[f(y_+) - f(y_-) + NF^{\mathrm{sm}}(y_+ - y_-, \beta)]. \tag{5.7}$$

The function $f(y)$ can be easily found by calculating Liouville action (2.16) for the given solutions. One has

$$f(y) = \frac{1}{\pi\gamma^2}\left(4\cot y + 2y - \frac{\gamma}{2}\phi(y)\right) \tag{5.8}$$

for the hyperbolic solution, and

$$f(y) = \frac{1}{\pi\gamma^2}\left(\frac{4}{y} - \frac{\gamma}{2}\phi(y)\right) \tag{5.9}$$

for the parabolic one. As follows from these expressions, the parabolic action is a limiting form of the hyperbolic functional at the small values of $y_\pm$.

The thermodynamics of these solutions is the thermodynamics of a system in a finite volume. The corresponding thermodynamical state is fixed by the boundary values $y_\pm$ which characterize the size of the system and the temperature $T$ measured at one of the boundaries. The free energy is determined by the induced action (5.7) as usually, $F(T,y_+,y_-)=T\Gamma(\beta,y_+,y_-)$. The inverse temperature $T^{-1}$ is the circumference length of the boundary and is proportional to $\beta$. Thus, one immediately concludes from Eq. (5.7) that the thermodynamical entropy coincides with the statistical entropy of the massless constituents of ILG:

$$S(T,y_+,y_-)=NS^{\rm sm}(\beta,y_+-y_-), \tag{5.10}$$

where

$$S(T,y_+,y_-)=-\left(\frac{\partial F(T,y_+,y_-)}{\partial T}\right)_{y_+,y_-}, \tag{5.11}$$

and $S^{\rm sm}$ is defined in Eq. (4.24). The equality (5.10) is more strong than relation (4.34) for the elliptic solutions.

To conclude this section we should note that the first law for the hyperbolic and the parabolic solutions has a more general form

$$\delta E=T\delta S-p_+\delta y_+-p_-\delta y_-, \tag{5.12}$$

where

$$E=F+TS \tag{5.13}$$

is the energy of the system and the quantities

$$p_\pm=-\left(\frac{\partial F(T,y_+,y_-)}{\partial y_\pm}\right)_{T,y_\mp} \tag{5.14}$$

can be interpreted as pressures at the boundaries. They can be computed with the help of Eqs. (5.7)–(5.9).

## VI. ILG DUAL TO BTZ BLACK HOLES

Our discussion includes as a particular example induced Liouville gravity which is dual to the BTZ black hole sector of the three-dimensional gravity with the negative cosmological constant ($AdS_3$ gravity). The Euclidean action of this theory [31],

$$I^{(3)}=-\frac{1}{16\pi G}\left[\int d^3x\sqrt{g}\left(R+\frac{2}{l^2}\right)+\int_\infty d^2x\sqrt{h}K\right], \tag{6.1}$$

where $K$ is the extrinsic curvature of the asymptotic boundary, corresponds to a canonical free energy of the system. Let us consider for simplicity a static BTZ black hole with the mass $M$. Functional (6.1) taken on the corresponding instanton has the canonical form

$$I^{(3)}=\beta M^{BTZ}-S^{BTZ}, \tag{6.2}$$

where $\beta$ and $S^{BTZ}$ are the inverse temperature and the entropy of the black hole, respectively. The parameters of the black hole are determined by the radius $r_+$ of the horizon

$$M^{BTZ}=\frac{r_+}{8Gl^2}, \tag{6.3}$$

$$\beta=\frac{2\pi l^2}{r_+}, \tag{6.4}$$

$$S^{BTZ}=\frac{2\pi r_+}{4G}. \tag{6.5}$$

It is instructive to represent the free energy, entropy, and mass of the black hole in terms of the inverse temperature

$$F^{BTZ}(l,\beta)=\beta^{-1}I^{(3)}=-\frac{\pi^2l^2}{2G\beta^2}, \tag{6.6}$$

$$S^{BTZ}(l,\beta)=\frac{\pi^2l^2}{G\beta}, \tag{6.7}$$

$$M^{BTZ}(l,\beta)=\frac{\pi^2l^2}{2G\beta^2}. \tag{6.8}$$

Let us now demonstrate that this black hole is thermodynamically equivalent to a certain type of ILG. We begin with Euclidean ILG with a zero cosmological constant ($\mu=0$) which is a particular case of ILG models. The boundary of a Euclidean BTZ black hole is a torus, see, e.g., [32]. This leads us to consider ILG where the background $\mathcal{M}$ has the topology of a torus. The induced action of this theory is [see Eq. (2.12)]

$$\Gamma[g]=\bar{I}_L[\phi]+NW[\bar{\mathcal{M}}], \tag{6.9}$$

$$\bar{I}_L[\phi]=-\frac{1}{2\pi}\int_{\bar{\mathcal{M}}}d^2y(\bar{\nabla}\phi)^2. \tag{6.10}$$

The flat space $\bar{\mathcal{M}}$ is a torus

$$d\bar{s}^2=d\tau^2+dx^2. \tag{6.11}$$

We assume that

$$0<\tau\leqslant\beta,\quad 0<x\leqslant b. \tag{6.12}$$

The solutions to this theory are constant $\phi$ and on the solutions the induced action coincides with $NW[\bar{\mathcal{M}}]$. Thus, the on-shell Euclidean action of ILG corresponds to the free energy of massless gas on a circle of length $b$ at the temperature $\beta^{-1}$. In the limit when size $b$ is large one can find the action on the torus from the leading order expression (4.23) for the free energy on the interval of size $b$. We thus obtain

$$F(b,\beta)\simeq -N\frac{\pi}{6}\frac{b}{\beta^2}. \tag{6.13}$$

This free energy corresponds to the CFT with the integer central charge $c=N$. Consider now the BTZ black hole with the same central charge $c$[10]

[10]The conformal Virasoro algebra with an integer central charge has a number of interesting features which were discussed in [33].

$$c_{BTZ}=\frac{3l}{2G}=N. \tag{6.14}$$

This charge corresponds to the group of diffeomorphisms at the asymptotic infinity of the BTZ black hole. By now replacing $N$ in Eq. (6.13) by $c_{BTZ}$ and putting $b=2\pi l$ we come to the identity between the ILG and BTZ free energy (6.6),

$$F^{BTZ}(l,\beta)=F(2\pi l,\beta)|_{N=c_{BTZ}}. \tag{6.15}$$

The corresponding identities can be established for the entropy (6.7) and energy (6.8) of the black hole. Therefore, the static BTZ black hole is *thermodynamically* equivalent to 1 +1 ILG having $c_{BTZ}$ massless fields and given on the circle of the radius $l$ [$l$ is related to the curvature radius of the corresponding $AdS_3$ geometry, see Eq. (6.1)].

This result can be also generalized to rotating BTZ black holes. If the angular velocity $\Omega$ of the black hole is not zero, the free energy (6.6) is replaced to [32]

$$F^{BTZ}(\Omega,l,\beta)=-\frac{\pi^2 l^2}{2G\beta^2(1-l^2\Omega^2)}. \tag{6.16}$$

The corresponding two-dimensional system equivalent to this black hole is a rotating quantum gas. To be more specific, consider a massless gas where half of the quanta are rotating clockwise with angular velocity $\Omega$ and the other half are moving in the opposite direction. The corresponding free energies of these quanta denoted by $F_\pm$ are determined by the ''boosted'' partition functions

$$e^{-\beta F_\pm}=\mathrm{Tr}\, e^{-\beta(\hat{H}\pm\Omega\hat{M})}, \tag{6.17}$$

where $\hat{H}$ is the Hamiltonian and $\hat{M}$ is the angular momentum related to the momentum $\hat{P}$ along the circle as

$$\hat{M}=\frac{b}{2\pi}\hat{P}. \tag{6.18}$$

[$b/(2\pi)$ is the radius of the circle.] The energy of a relativistic massless particle equals the modulus of its momentum. Therefore

$$e^{-\beta F_\pm}=\mathrm{Tr}\, e^{-\beta(1\pm\bar{\Omega})\hat{H}}, \tag{6.19}$$

$$\bar{\Omega}=\frac{b\Omega}{2\pi}. \tag{6.20}$$

Thus, when $b$ is large one can find $F_\pm$ with the help of Eq. (4.23),

$$F_\pm(\Omega,b,\beta)\simeq-\frac{\pi b}{6\beta^2(1\pm\bar{\Omega})}. \tag{6.21}$$

The total free energy of the system is[11]

$$F(\Omega,b,\beta)=\frac{N}{2}F_+(\Omega,b,\beta)+\frac{N}{2}F_-(\Omega,b,\beta)$$
$$\simeq -N\frac{\pi}{6}\frac{b}{\beta^2(1-\bar{\Omega}^2)}. \tag{6.22}$$

Now by taking into account Eq. (6.20) and that $b=2\pi l$ we can generalize the relation (6.15):

$$F^{BTZ}(\Omega,l,\beta)=F(\Omega,2\pi l,\beta)|_{N=c_{BTZ}}. \tag{6.23}$$

Once one has the thermodynamic relation (6.23) between the BTZ black hole and a two-dimensional rotating gas, the statistical-mechanical explanation of the two-dimensional entropy can be considered as an explanation of the black hole entropy [6]. This, however, cannot be taken as a satisfactory explanation of the entropy, because the degrees of freedom of the two-dimensional theory have nothing to do with the degrees of freedom of the black hole.

## VII. DISCUSSION

Recent interesting computations of the entropy of extremal [4,5] and BTZ [6] black holes leave open the essential question about the real degrees of freedom of the black hole. The difficulty is that the computations concern not black holes themselves but dual systems in flat space-times. Some help in resolving this difficulty may come from studying models of Sakharov's induced gravity [16–18]. In this theory the degrees of freedom responsible for the black hole entropy are the constituents which induce the Einstein gravity in the low-energy limit.

The Liouville induced gravity considered in this paper is a suitable ''firing range'' to study this issue. We have shown that the entropy of induced $AdS_2$ black holes is equivalent to the entanglement entropy of the massless constituents. In two dimensions the divergence usually encountered in the definition of the entanglement entropy does not depend on the thermodynamical parameters of the black hole and, hence, is not observable in physical processes. Therefore, going to two-dimensional induced gravity leads to an essential simplification. In higher-dimensional models the relation between the entanglement and black hole entropies is more complicated. It always requires subtraction a nontrivial Noether charge related to nonminimal couplings of the constituents [17]. Another distinction is that in higher dimensions the

[11]A similar computation can be found in [34].

main contribution to the entanglement entropy is determined only by the constituents with the Planck mass localized in the vicinity of the horizon. In ILG the entropy is related to the fields which are localized in the entire black hole exterior.

Induced Liouville gravity has solutions of different types which may find different applications. The important example are the solutions with a zero cosmological constant on a torus. On the level of thermodynamics they are equivalent to BTZ black holes. It would be interesting to see whether there is a similar relation of the found solutions, including $AdS_2$ black holes, and thermodynamics of the objects in the three-dimensional AdS space. Another interesting problem is to analyze along the lines of Sec. VI the theory [12,13] which appears in the near horizon limit of generic black holes. We leave these issues for future publications.

## ACKNOWLEDGMENT

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada. The work of D.F. was supported in part by RFBR grant N99-02-18146.

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