Strengths of singularities in spherical symmetry

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Covariant equations characterizing the strength of a singularity in spherical symmetry are derived and several models are investigated. The difference between central and non-central singularities is emphasized. A slight modification to the definition of singularity strength is suggested. The gravitational weakness of shell crossing singularities in collapsing spherical dust is proven for timelike geodesics, closing a gap in the proof. [S0556-2821(99)02214-6]

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I. INTRODUCTION

Over thirty years have passed since the appearance of the first of a series of theorems establishing that under very general circumstances, space-times must develop singularities. This first result, due to Penrose [1], appeared in 1965 and the body of work which grew up around the singularity theorems is contained in the book of Hawking and Ellis [2] which was first published in 1973. However, our understanding of the nature of these singularities remains far from complete. This lack is best exemplified by the absence of a proof, or definitive refutation, of the cosmic censorship conjecture (CCC) [3,4].

An important aspect of a singularity is its gravitational strength [5]. A singularity is termed gravitationally strong, or simply strong, if it destroys by crushing or stretching any object which falls into it. The most familiar example is the singularity at $r=0$ in the Schwarzschild solution. (Throughout this paper, we will refer to a singularity at $r=0$ as a central singularity, and to others as non-central.) A radially infalling object is infinitely stretched in the radial direction and crushed in the tangential directions, with the net result of crushing to zero volume. A singularity is termed weak if *no* object which falls into the singularity is destroyed in this way. The mathematical description of these ideas runs as follows [5,4].

Let $\gamma:[\tau_0,0)\rightarrow M$ be a causal geodesic which approaches a singularity as $\tau\rightarrow 0^-$. Define $J_\nu(\gamma)$ for $\nu\in[\tau_0,0)$ to be the set of maps $Z:[\nu,0)\rightarrow TM$ satisfying the geodesic deviation equation along γ such that $Z(\tau)\in T_{\gamma(\tau)}M$, $g_{ab}k^a(\tau)Z^b(\tau)=0$ where k^a is the tangent to γ and

$$Z(\nu)=0;$$

i.e. $J_\nu(\gamma)$ is the set of Jacobi fields along γ which vanish at $\gamma(\nu)$. Along a timelike geodesic, three independent Jacobi fields define, via the exterior product, a volume element $V(\tau)$ along γ. Along a null geodesic, two such fields define an area element which we also denote $V(\tau)$. A timelike (null) geodesic terminates in a *strong curvature singularity* if for all $\nu\in[\tau_0,0)$ and all independent triads (dyads) in $J_\nu(\gamma)$ we have

$$\lim_{\tau\rightarrow 0^-} V(\tau)=0.$$

Then the singularity itself is said to be *strong* if every causal geodesic which approaches it terminates in a strong singularity. The geodesic terminates in a weak singularity if the limit above is finite and non-zero, and the singularity is weak if every causal geodesic approaching it terminates weakly. We will argue below for a slight modification of this definition whereby the term strong is also attached to a singularity if the norms of the Jacobi fields themselves have zero or infinite limit.

The importance of the notion of the gravitational strength of a singularity for the CCC is that a statement of such possibly need not rule out the occurrence of naked weak singularities. This is based on the belief that one may extend the geometry of space-time through a weak singularity without traumatic effects [6,7]. A general description of this extension does not exist—indeed as far as the author can determine, only two examples of this procedure exist in the literature, one due to Papapetrou and Hamoui [8] and the other due to Clarke and O'Donnell [9]. Both deal with extending through a shell crossing singularity in collapsing spherical dust. However the fact that at a weak singularity one has, along any timelike geodesic, a finite non-degenerate triad of Jacobi fields, from which it may be possible to construct a metric in a canonical way, lends support to the idea.

Our aim here is to give a comprehensive analysis of the strengths of singularities in spherical symmetry. This has been the arena of some of the most interesting developments in general relativity in recent years, and an understanding of what can and cannot occur in spherical symmetry may be a valuable guide for more general situations. Specifically, we study the Jacobi equations for arbitrary radial causal geodesics. This allows us to give covariant equations identifying the geometrical terms which control the strength of the singularity. As one would expect, the results are simpler than those obtained by Clarke and Krolak [10] which apply to general space-times. We study three different models which help illustrate the different situations which occur, and by way of application, demonstrate that (i) a non-central singularity is always weak along null directions and (ii) the shell-crossing singularities in collapsing spherical dust are weak. (This has only been demonstrated previously for radial null

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directions [11]; we complete the proof by showing that it is also true for radial timelike directions.)

II. RADIAL CAUSAL GEODESICS, JACOBI FIELDS AND THE VOLUME ELEMENT

The line element of a spherically symmetric space-time may be written as

$$ds^2 = -2e^{-2f}du dv + r^2 d\omega^2, \tag{2.1}$$

where $f=f(u,v)$, $r=r(u,v)$ and $d\omega^2$ is the line element of the unit 2-sphere. The function r is an invariant of the space-time which we will, quite properly, refer to as the radius. The coordinates u and v are both null, labelling the null hypersurfaces generated by the two families of null geodesics orthogonal to the orbits of the $SO(3)$ symmetry group of the space-time. The form (2.1) is invariant under the transformations $u \to u'(u)$, $v \to v'(v)$.

The non-vanishing Ricci tensor components are

$$R_{uu} = -2r^{-1}(r_{uu} + 2r_u f_u) \tag{2.2a}$$

$$R_{vv} = -2r^{-1}(r_{vv} + 2r_v f_v) \tag{2.2b}$$

$$R_{uv} = -2r^{-1}(r_{uv} - r f_{uv}) \tag{2.2c}$$

$$R_{\theta\theta} = \csc^2\theta R_{\phi\phi} = 1 + 2e^{2f}(r_u r_v + r r_{uv}). \tag{2.2d}$$

We use the convention that subscripts attached to lower case letters refer to partial derivatives, but elsewhere refer to tensor components in the associated coordinate basis. The Misner-Sharp energy [12] is

$$E = \frac{r}{2}(1 + 2e^{2f} r_u r_v), \tag{2.3}$$

and the Weyl tensor is completely determined by the Newman-Penrose term Ψ_2, calculated on a null tetrad based on the principal null directions of the space time. Thus Ψ_2 is an invariant of the space-time and is given by

$$\Psi_2 = \frac{e^{2f}}{3r}(r_{uv} + r f_{uv}) - \frac{E}{3r^3}. \tag{2.4}$$

We note the further invariant,

$$e^{2f} f_{uv} = \frac{E}{r^3} + 2\Psi_2 - \frac{R}{12}, \tag{2.5}$$

where R is the Ricci scalar.

The radial geodesic equations are

$$-2e^{-2f}\dot{u}\dot{v} = \epsilon \tag{2.6a}$$

$$\ddot{u} - 2f_u \dot{u}^2 = 0 \tag{2.6b}$$

$$\ddot{v} - 2f_v \dot{v}^2 = 0 \tag{2.6c}$$

where $\epsilon = 0$ for null geodesics and -1 for timelike; the overdot is respectively, differentiation with respect to affine parameter and proper time. Space-like geodesics will not concern us here.

We now look at the Jacobi fields along arbitrary radial causal geodesics, beginning with time-like geodesics.

The unit tangent to an arbitrary time-like curve $\gamma(\tau)$ in the radial 2-space can be written in the form

$$\vec{k} = h\frac{\partial}{\partial u} + \frac{1}{2}e^{2f}h^{-1}\frac{\partial}{\partial v},$$

where $h = h(u,v)$. The condition that γ be geodesic is then

$$h_v + 2h^2 e^{-2f}(h_u - 2f_u h) = 0. \tag{2.7}$$

This follows from the geodesic equations (2.6a)–(2.6c). The variation of any scalar quantity s along this geodesic is given by

$$\dot{s} = k^a \nabla_a s = h s_u + \frac{1}{2}e^{2f}h^{-1}s_v, \tag{2.8}$$

and using Eq. (2.7),

$$\ddot{s} = h^2(s_{uu} + 2f_u s_u) + \frac{1}{4}e^{4f}h^{-2}(s_{vv} + 2f_v s_v) + e^{2f}s_{uv}. \tag{2.9}$$

A Jacobi field Z^a along γ satisfies the geodesic deviation equation

$$\ddot{Z}^a + R^a_{cbd}Z^b k^c k^d = 0, \tag{2.10}$$

which is a linear equation for Z^a and so a basis for the Jacobi fields may be found by obtaining all independent Jacobi fields in the radial 2-space and in the tangential 2-space.

We take

$$\vec{\xi}_{(1)} = x(u,v)\frac{\partial}{\partial\theta}, \quad \vec{\xi}_{(2)} = y(u,v)\csc\theta\frac{\partial}{\partial\phi}$$

as candidates for the Jacobi fields in the tangential 2-space. Note that the norms of $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are $r|x|$ and $r|y|$ respectively. The geodesic deviation equation (2.10) applied to $\xi^a_{(1)}$ yields the following equation for x (the same result applies to y):

$$r(4h^4(x_{uu} + 2f_u x_u) + e^{4f}(x_{vv} + 2f_v x_v) + 4h^2 e^{2f}x_{uv}) + 2(2h^2 x_u + e^{2f}x_v)(2h^2 r_u + e^{2f}r_v) = 0.$$

Using Eqs. (2.8) and (2.9), this assumes the remarkably simple form

$$r\ddot{x} + 2\dot{r}\dot{x} = 0, \tag{2.11}$$

which can be integrated to obtain

$$x(\tau) = x_0 \int_{\tau_1}^{\tau} \frac{d\tau'}{r^2(\tau')}, \tag{2.12a}$$

where x_0 is constant and we have included the initial condition $x(\tau_1)=0$, so that $\vec{\xi}_{(1)}(\tau_1)=0$. The second linearly independent solution is $x(\tau)\equiv 0$. We obtain the same result for $\vec{\xi}_{(2)}$:

$$y(\tau)=y_0\int_{\tau_1}^{\tau}\frac{d\tau'}{r^2(\tau')}. \tag{2.12b}$$

Before dealing with the radial Jacobi fields along the time-like geodesics, we describe the situation for radial null geodesics. It turns out to be remarkably simple. From Eq. (2.6a), either $\dot{u}$ or $\dot{v}$ vanishes along a radial null geodesic. Take it to be the latter. We can then integrate Eq. (2.6b) to obtain $\dot{u}=ce^{2f}$ and so the tangent is

$$\vec{k}=ce^{2f}\frac{\partial}{\partial u}.$$

The variation of a scalar s along the geodesic is $\dot{s}=ce^{2f}s_u$ and

$$\ddot{s}=c^2e^{4f}(s_{uu}+2f_us_u).$$

In the null case, there are only tangential Jacobi fields. Candidates for such are given by $\vec{\xi}_{(1,2)}$ as above, and it turns out that the norms x,y obey the same equation as in the time-like case, and thus the solutions are given by Eq. (2.12a).

We now treat the radial Jacobi fields along a time-like geodesic. A space-like vector in the radial 2-space orthogonal to k^a has the form

$$\vec{\xi}=ah\frac{\partial}{\partial u}-\frac{1}{2}e^{2f}ah^{-1}\frac{\partial}{\partial v},$$

where $a=a(u,v)$. This has norm $|a|$. Using Eq. (2.9), we find that the condition for $\vec{\xi}$ to satisfy the geodesic deviation equation along k^a is

$$\ddot{a}+2e^{2f}f_{uv}a=0. \tag{2.13}$$

Recall that according to Eq. (2.5), $e^{2f}f_{uv}$ is an invariant of the space-time. Thus Eqs. (2.12a), (2.12b), and (2.13) provide a *covariant* set of equations which will determine the strength of the singularity. To see how this comes about, we obtain the relationship between $V(\tau)$ and the quantities a,x,y.

Since each of a, x and y satisfy second order linear ordinary differential equations, there are six independent Jacobi fields along a time-like geodesic γ. An arbitrary triad of corresponding 1-forms is given by

$$\mathbf{z}_\alpha=a_\alpha\mathbf{e}+x_\alpha r^2d\theta+y_\alpha\sin\theta d\phi,$$

where $\mathbf{e}=(2h)^{-1}du-e^{-2f}hdv$ and $\alpha=1,2,3$. Then the general volume element along γ has the form

$$V(\tau)=\mathbf{z}_1\wedge\mathbf{z}_2\wedge\mathbf{z}_3$$
$$=6a_{[1}x_2y_{3]}r^4\sin\theta\mathbf{e}\wedge d\theta\wedge d\phi.$$

The norm $\|\mathbf{W}\|$ of a p-form $\mathbf{W}=W_{[i_1\ldots i_p]}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ is given by

$$\|\mathbf{W}\|^2=W_{|i_1\ldots i_p|}W^{i_1\cdots i_p},$$

where the vertical bars indicate summation only over $i_1<i_2<\cdots<i_p$. This gives

$$\|V(\tau)\|=6|a_{[1}x_2y_{3]}|r^2. \tag{2.14}$$

The existence of six independent solutions of the geodesic deviation equation indicates that in a general space-time, the different $V(\tau)$ form a six parameter family. This surfeit of possibilities would produce problems if one wanted to make statements about singularity strengths based on the behavior of all such $V(\tau)$. However the definition of J_{τ_1} reduces the number significantly. Note that $\mathbf{z}_\alpha$ vanishes if and only if each of a_α, x_α and y_α vanish. The general solution for a has the form

$$a=c_+a_++c_-a_-$$

where $c_\pm$ are arbitrary constants and $a_\pm$ are any two independent solutions of Eq. (2.13). A similar result holds for x and y. The initial condition $a(\tau_1)=0$ fixes the ratio c_+/c_-, so that for the problem in hand, there is only *one* choice, up to a constant multiple, for each of a, x and y. Therefore the norm of every relevant volume element has the simple form

$$\|V(\tau)\|=|axy|r^2 \tag{2.15}$$

where a, x and y here represent the *general* solutions of the appropriate equations (2.12a), (2.12b), and (2.13), into which constants may be absorbed.

We thus have a simple and direct way of assessing the strength of a singularity. We determine the limiting behavior of solutions of Eqs. (2.12a), (2.12b), and (2.13) as the singularity is approached and then use Eq. (2.15) to calculate $\lim_{\tau\to 0^-}\|V(\tau)\|$.

III. POSSIBLE SOLUTIONS OF THE MAIN EQUATIONS

We now consider the various possible limiting behaviors which can occur for solutions of the main equations, i.e., Eqs. (2.12a), (2.12b), and (2.13).

From Eq. (2.15), the forms rx and ry arise naturally and occur in the following proposition which applies to both x and y.

Proposition One. *(i) For a non-central singularity,* $\lim_{\tau\to 0^-}rx$ *is finite and non-zero.*

(ii) For a central singularity, if

$$\lim_{\tau\to 0^-}\int_{\tau_1}^{\tau}r^{-2}(\tau')d\tau'<\infty,$$

then $\lim_{\tau\to 0^-}rx=0$. *Otherwise*

$$\lim_{\tau\to 0^-}rx=-\lim_{\tau\to 0^-}\frac{1}{\dot{r}(\tau)}.$$

Here and below, the limit refers to the limit as $\tau\to 0^-$ along a geodesic which approaches the singularity at $\tau=0$. The proofs of part (i) and the first part of part (ii) follow immediately from Eq. (2.12a), and that of the second part from Eq. (2.12a) and l'Hôpital's rule. A consequence of this is that the strength of a non-central singularity is completely determined by the limiting behavior of a at the singularity.

Next we summarize the possible behavior of a in the appropriate limit. Define $F(\tau)=2e^{2f}f_{uv}(\tau)$ and $G(\tau)=\tau^2F(\tau)$. Then Eq. (2.13) is equivalent to

$$\tau^2\ddot{a}+G(\tau)a=0, \tag{3.1}$$

and we wish to determine the behavior of a at $\tau=0$ which is a singular point of Eq. (3.1). We quote the following results from Bender and Orszag [13], which should also be found in any text on linear differential equations. All the asymptotic relations below hold as $\tau\to 0^-$ and $c,c_\pm$ are arbitrary constants.

The equation is said to have a *regular singular point* at $\tau=0$ if $G(\tau)$ is analytic in a neighborhood of $\tau=0$. Otherwise, $\tau=0$ is called an *irregular singular point*. For a regular singular point, we define $G_0=G(0)$. In fact the results below apply more generally, namely if $G(\tau)=O(1)$ as $\tau\to 0^-$ with a very low degree of differentiability; $G\in C^1(-\tau_0,0]$ is sufficient. The method of Frobenius applies. The roots of the indicial equation are

$$v_{1,2}=\frac{1}{2}\pm\frac{1}{2}(1-4G_0)^{1/2}.$$

The following possibilities arise.

(RSP1) $v_1-v_2\notin Z$. Then $a(\tau)\sim c_+|\tau|^{v_1}+c_-|\tau|^{v_2}$. Three subcases arise depending on the value of G_0.

(RSP1a) $1/4<G_0$. Then $a(\tau)\sim c|\tau|^{1/2}$, so that $\lim_{\tau\to 0^-}a(\tau)=0$..

(RSP1b) $0<G_0<1/4$. Then $v_{1,2}$ are both positive so that $\lim_{\tau\to 0^-}a(\tau)=0$.

(RSP1c) $G_0<0$. Then $v_2<0$ so that $\lim_{\tau\to 0^-}a(\tau)=\infty$.

(RSP2) $v_1=v_2\Leftrightarrow G_0=1/4$ Then $a(\tau)\sim c_+|\tau|^{1/2}+c_-|\tau|^{1/2}\ln|\tau|$. Again, $\lim_{\tau\to 0^-}a(\tau)=0$.

(RSP3) $v_1-v_2\in N-0\Leftrightarrow G_0=(1-k^2)/4$, $k\in N^+$. Then $a(\tau)\sim c_+|\tau|^{v_1}+c_-(|\tau|^{v_2}+d|\tau|^{v_1}\ln|\tau|)$ where d is a *fixed* constant. We mention under this last heading one special case of particular importance, that for which $G_0=0$. This includes singularities whereat $F(\tau)$ is finite and, typically, space-times with weak non-central singularities.

(RSP3a) $G_0=0$. Then $a(\tau)\sim c_-+c_+|\tau|+c_-d|\tau|\ln|\tau|)$, and so $\lim_{\tau\to 0^-}a(\tau)$ is finite and non-zero.

The second class of possibilities arises when Eq. (3.1) has an irregular singular point at $\tau=0$. If $\lim_{\tau\to 0^-}|G(\tau)|=\infty$, then the WKB approximation holds. This gives

$$a(\tau)\sim c(F(\tau))^{-1/4}\exp\left\{\pm\int_{\tau_1}^{\tau}(F(\tau'))^{1/2}d\tau'\right\}.$$

There are two possibilities here.

(ISP1) $\lim_{\tau\to 0^-}F(\tau)=+\infty$. Then $\lim_{\tau\to 0^-}a(\tau)=+\infty$.

(ISP2) $\lim_{\tau\to 0^-}F(\tau)=-\infty$. Then $\lim_{\tau\to 0^-}a(\tau)=0$.

This does not cover all possibilities since there are irregular singular points for which the limit $\lim_{\tau\to 0^-}|G(\tau)|$ does not exist. Typically, this would occur if $G(\tau)$ is oscillatory in a neighborhood of $\tau=0$, e.g.

$$G_p(\tau)=k|\tau|^{-1}\sin(|\tau|^{-p}),$$

where $p>0$. Taking this form for $G(\tau)$ and defining $x=|\tau|^{-1}$, $b=x^{(1+p^{-1})/2}a$ (3.1) becomes

$$b''+\left[\frac{k}{p^2}\frac{\sin x}{x^{2-p^{-1}}}+\frac{1-p^{-2}}{4x^2}\right]b=0.$$

The dominant coefficient of b is the decaying oscillatory term, and this determines the asymptotic behavior of the solutions. There are three different cases, depending on the value of p. We quote the result for the simplest case, which is $p>1$. The asymptotic behavior in this case is given by [14]

$$a_1\sim x^{(1-p^{-1})/2},\quad a_2\sim x^{-(1+p^{-1})/2}\quad (x\to\infty).$$

Thus the singularity is strong and stretching for $p>1$. It turns out that the singularity is strong and crushing for $1/2<p\leq 1$, see [14] for details. On the other hand, if we take $G(\tau)=k\sin(|\tau|^{-1}|)$, the same procedure leads to

$$b''+\frac{k}{p^2}\frac{\sin x}{x^2}b=0,$$

the asymptotic solutions of which lead to [14]

$$a_1\sim 1,\quad a_2\sim x^{-1}\quad (x\to\infty)$$

and so in this case the singularity is weak. Notice that we have in this case $G(\tau)=O(1)$, but G is not differentiable at $\tau=0$.

Our main point here is that both strong and weak singularities may occur in this class and the analysis to determine which case obtains may be quite difficult.

(ISP3) $\lim_{\tau\to 0^-}G(\tau)$ does not exist. The singularity may be either strong or weak.

Keep in mind that the behavior described here is characteristic of a particular radial timelike geodesic which runs into the singularity, and not of the singularity itself. We will therefore refer to, for example, a *type (RSP1a) geodesic*, and to a *type (RSP1a) singularity* only if *all* the radial timelike geodesics terminating there are type (RSP1a).

In this language, the central singularity of Schwarzschild space-time is type (RSP1c), with $a(\tau)\sim c_+|\tau|^{4/3}+c_-|\tau|^{-1/3}$. Also, $rx(\tau)\sim x_0|\tau|^{1/3}$, $ry(\tau)\sim y_0|\tau|^{1/3}$, so that overall, $\|V(\tau)\|\sim d|\tau|^{1/3}$, giving a singularity which is strong along timelike approaches. Suppose instead the behavior was $rx(\tau)\sim x_0|\tau|^{1/6}$, $ry(\tau)\sim y_0|\tau|^{1/6}$. Then $\|V(\tau)\|\sim d$ (constant), so by the current definition, the singularity is weak along timelike approaches. It would be of very little comfort to an observer jumping into such a singularity to realize, as he watched his legs elongate and disintegrate, that such volume forms were preserved on his journey. The possibility of the existence of such a singularity was noted by Tipler [5].

We give an example of such below. This motivates the following addendum to the definition of a strong singularity.

We will say that a causal geodesic $\gamma:[\tau_0,0)\to M$ approaching a singularity as $\tau\to 0^-$ terminates in a strong singularity if for all $\tau_1\in[\tau_0,0)$, except some suitably small set (finite, countable, zero-measure), the general element of $J_{\tau_1}(\gamma)$ is degenerate or infinite in the limit $\tau\to 0^-$. We will say that γ terminates in a weak singularity if the general element of $J_{\tau_1}(\gamma)$ is finite and non-degenerate in the limit. The terms will be applied to the singularity itself if *all* causal geodesics approaching the singularity behave in one of the two ways.

By degenerate, we mean that both of the independent Jacobi fields in some particular direction (or mutually orthogonal directions) orthogonal to k^a shrink to zero magnitude. A non-central type (RSP1b) singularity would be an example of such.

We now gather the results above into some general statements.

Proposition Two. *For a non-central singularity and for a central singularity for which $\dot{r}$ has a finite and non-zero limit along every causal geodesic approaching the singularity, the strength of the singularity is determined by Eq. (2.13). If the singularity is of type (RSP3a), then it is weak. The only other type which may be weak is (ISP3); singularities of the other types are strong. A central singularity for which $\dot{r}$ has zero or infinite limit along every causal geodesic approaching the singularity is strong.*

The proof follows from Proposition One and from the definitions above; essentially it amounts to some useful book-keeping. A great many singularities will have different behaviors along different geodesics approaching the singularity, and so will not be covered by this result. There remains the problem of determining the behavior of $a(\tau)$ and $\dot{r}(\tau)$ in the limit as the singularity is approached. However we have identified which elements of the geometry determine the strength of a singularity and listed the various possibilities.

We now give some applications of the results laid out above.

IV. EXAMPLES

In this section, we study the strengths of some singularities in four different (classes) of space-times. The first three, two toy models and Roberts' space-time [15], are used to illustrate the types of singularities which may arise and some of the points made above. The fourth is the marginally bound case of a Lemaître-Tolman-Bondi (LTB) collapsing dust sphere [16]. We use the theory above to demonstrate conclusively the weakness of shell-crossing singularities in this space-time.

A. A toy model

We consider the space-time with line element

$$ds^2=-du\,dv+\left(\frac{v-u}{2}\right)^{2\alpha}d\omega^2,$$

i.e. $2e^{-2f}=1$ and $r=((v-u)/2)^\alpha$. We take $\alpha\geqslant 1$. The case $\alpha=1$ is flat space-time. The Ricci scalar is

$$R=\frac{2}{r^2}(1-(3\alpha^2-2\alpha)r^{2-2/\alpha}),$$

and so there is a scalar curvature singularity at $r=0$. Since f is constant, the radial Jacobi field orthogonal to an arbitrary timelike geodesic will satisfy, according to Eq. (2.13), $\ddot{a}=0$, with general solution $a=c_++c_-\tau$. Thus the strength of this central singularity will be determined by the behavior of the tangential Jacobi fields.

Along a radial null geodesic, we have (without loss of generality) $v=$constant and $u=c\tau+d$, where c,d are constants and τ is an affine parameter. Thus after a reparametrization of τ, we have

$$r=c|\tau|^\alpha.$$

The same result holds for all radial timelike geodesics. From Eqs. (2.12) and (2.15), we find

$$\|V(\tau)\|\propto(c_++c_-|\tau|)|\tau|^{2-2\alpha}.$$

Thus the singularity is strong. Notice that $\|V(\tau)\|\to\infty$ as the singularity is approached along any radial causal geodesic. The deformation results from infinite tangential stretching.

The purpose of examining this model is to gave an explicit example where the behavior at the singularity is clearly pathological and destructive, but which would not previously have been described as a strong singularity. We note that

$$T_{ab}\left(\frac{\partial}{\partial u}\right)^a\left(\frac{\partial}{\partial u}\right)^b=-\frac{\alpha}{8\pi}(\alpha-1)r^{-2/\alpha},$$

so that the weak energy condition is violated for the values of α of interest here.

B. Roberts' solution

Roberts' solution has been used in studies of cosmic censorship [15] and critical collapse [17]. The line element is

$$ds^2=-du\,dv+\frac{1}{4}(u^2-2uv+(1-p^2)v^2)d\omega^2,$$

where p is constant. $p=0$ gives flat space-time. The Ricci scalar is

$$R=\frac{1}{2r^4}p^2uv,$$

and so there is a central scalar curvature singularity. As above, the strength of the singularity is determined by the tangential Jacobi fields. In this case we find that $r=c|\tau|$ along any radial causal geodesic terminating at $r=0$ at parameter value $\tau=0$. We use Eqs. (2.12a), (2.12b), and (2.15) to obtain

$$\|V(\tau)\|\sim k(c_++c_-|\tau|)$$

with $rx, ry \sim$ constant as $\tau \to 0^-$, and so this central singularity is weak. Thus the examples where this singularity is naked may not be genuine counterexamples to cosmic censorship. See also [3] for related comments.

C. Another toy model

A model with slightly more complicated dynamics and which illustrates well some of the points made above is that with the line element

$$ds^2 = -\left(\frac{v-u}{2}\right)^{2\alpha} du dv + \left(\frac{v-u}{2}\right)^2 d\omega^2.$$

We take $\alpha \geq 0$; $\alpha = 0$ is flat space-time. We find that

$$2e^{2f} f_{uv} = -4\alpha(2\alpha+1) r^{-4\alpha-2}, \tag{4.1}$$

and so there is a scalar curvature singularity at $r=0$ (recall that this term is an invariant).

For an arbitrary radial null geodesic, we make take $u = u_0 =$constant. Then we find

$$v - u = v - u_0 = (c\tau + d)^{1/(2\alpha+1)},$$

so that

$$r = k|\tau|^{1/(2\alpha+1)}$$

after an appropriate shift in the affine parameter τ. Applying the second part of Proposition One, we see that all radial null geodesics approaching $r=0$ terminate in a strong curvature singularity with the area element obeying $\|V(\tau)\| \to 0$ in every case.

To solve for the radial timelike geodesics, we make the change of variables $r = (v-u)/2$, $t = (v+u)/2$. Then the line element takes the form

$$ds^2 = r^{2\alpha}(-dt^2 + dr^2) + r^2 d\omega^2.$$

The geodesic equations for radial infall yield

$$\dot{r} = -r^{-2\alpha}(c^2 - r^{2\alpha})^{1/2}.$$

According to Eqs. (2.13) and (3.1), we need to determine the behavior of r as proper time $\tau \to 0$. [As usual, we fix the origin of proper time so that $r(0)=0$.] The previous equation may be solved asymptotically by expanding the right hand side and then inverting the resulting integral with the result

$$r = c|\tau|^{1/(2\alpha+1)} + O(|\tau|^{\beta}) \tag{4.2}$$

where $\beta > 1/(2\alpha+1)$.

Then the tangential Jacobi fields have the asymptotic behavior

$$rx \sim c|\tau|^{2\alpha/(2\alpha+1)} \quad (\alpha \neq 1/2),$$

$$rx \sim c|\tau|^{1/2} \ln|\tau| \quad (\alpha = 1/2).$$

The behavior of the radial Jacobi fields is dictated by Eq. (4.1) which from the above has the behavior

$$F(\tau) \sim c_1 |\tau|^{-2}$$

where c_1 is a *negative* constant. Then in the notation used above, G_0 is a negative constant, and so this is a type (RSP1c) singularity. The asymptotic behavior of a is

$$a(\tau) \sim c_+ |\tau|^{v_1} + c_- |\tau|^{v_2}$$

where $v_{1,2} = (1 \pm (1-4G_0)^{1/2})/2$, and so

$$\|V(\tau)\| \sim V_0 |\tau|^{v_2 + 4\alpha/(2\alpha+1)}.$$

Therefore, for any value of α, there will exist radial timelike geodesics along which $\|V(\tau)\|$ diverges, has zero limit and has finite limit as the singularity is approached. These different possibilities arise from the different choices available for c in Eq. (4.2) which give the value $G_0 = -4\alpha(2\alpha+1)c^{-4\alpha-2}$. Starting from some fixed value $r = r_*$ at $\tau = \tau_* < 0$, we see that c is essentially a measure of the initial velocity of an observer falling radially inwards from r_*. By tuning this velocity, an observer could in principle ensure that his $\|V(\tau)\|$ is finite in the approach to the singularity. However, in practice this would be of little help to him since as pointed out above, the observer experiences infinite tangential crushing and radial stretching in the infall. Furthermore, his initial velocity would have to be tuned with infinite precision to obtain $0 \neq \lim_{\tau \to 0^-} \|V(\tau)\| < \infty$. According to the definition above, this is a strong singularity.

D. Marginally bound spherical dust

The marginally bound LTB space-time (spherically symmetric inhomogeneous dust) has line element

$$ds^2 = -dt^2 + (r')^2 d\eta^2 + r^2 d\omega^2,$$

where the prime indicates differentiation with respect to the coordinate η. For the collapsing case,

$$r^3(\eta, t) = \frac{9}{2} m(\eta)(t_0(\eta) - t)^2,$$

where m, t_0 are arbitrary functions of η. See [16] for details. The energy density ρ of the dust, which is proportional to the Ricci scalar, is given by

$$4\pi\rho = \frac{m'}{r^2 r'}.$$

Thus as well as the central singularity at $r=0$ [occurring when $t = t_0(\eta)$], there are so-called shell-crossing singularities occurring when $r' = 0$ [18]. These generally occur before the central singularity, at non-zero radius and so are non-central. It has long been believed that these scalar curvature singularities are weak. However, it seems that this has only been properly established for null geodesics approaching the singularity [6,11]. As we have seen above, this weakness is completely independent of the structure and nature of the singularity apart from the fact that it is non-central. We fill this gap by proving that all radial timelike geodesics terminate in a weak singularity.

According to Proposition Two, the strength of a shell-crossing singularity is governed by (2.13). Using Eq. (2.5), we find that

$$F=2e^{2f}f_{uv}=\frac{m'}{r^2r'}-2\frac{m}{r^3}.$$

The terms m'/r^2 and m/r^3 will both be finite in general in the approach to the singularity, the former being positive, assuming positive energy density. Thus the behavior is governed by $F=1/r'$. We will show that

$$\lim_{\tau\to 0^-}\frac{\tau^2}{r'}=0 \tag{4.3}$$

along any radial timelike geodesic approaching the singularity. This shows that the singularity is type (RSP3a), and is therefore weak by Proposition Two.

The radial timelike geodesic equations are

$$-\dot{t}^2+(r')^2\dot{\eta}^2=-1 \tag{4.4a}$$

$$r'\ddot{\eta}+2r_t'\dot{t}\dot{\eta}+r''\dot{\eta}^2=0 \tag{4.4b}$$

$$\ddot{t}+r'r_t'\dot{\eta}^2=0 \tag{4.4c}$$

where the overdot indicates differentiation with respect to proper time along the geodesic and the subscript is differentiation with respect to the global time coordinate t. Along each geodesic approaching the singularity, we choose the origin of proper time so that the singularity is at $\tau=0$.

We find that

$$r'=\frac{r}{3}\left(\frac{m'}{m}+\frac{2t_0'}{t_0-t}\right),$$

so that at a shell-crossing singularity,

$$\frac{m'}{m}=-\frac{2t_0'}{t_0-t}.$$

The following terms will enter into our analysis:

$$r_t'=\frac{2}{3}\frac{rt_0'}{(t_0-t)^2} \tag{4.5}$$

$$r''=\frac{(r')^2}{r}+\frac{r}{3}\left(\frac{m''}{m}-\left(\frac{m'}{m}\right)^2+\frac{2t_0''}{t_0-t}-\frac{2t_0'}{(t_0-t)^2}\right). \tag{4.6}$$

Then

$$r''(0)=\frac{r}{3m^2t_0'}\left(mm''t_0'-\frac{3}{2}(m')^2t_0'-mm't_0''\right), \tag{4.7}$$

where here and subsequently, evaluation at zero means in the limit $\tau\to 0^-$ along a geodesic.

Generically, $r_t'(0)$ and $r''(0)$ will be non-zero. If this were not the case, there would be extra conditions imposed on m and t_0 for all values of η, which would result in a loss of generality. For example, if $r_t'(0)=0$, then $t_0'(\eta)\equiv 0$ for all η. In this case, the space-time is homogeneous and isotropic. The condition $r''(0)=0$ imposes less severe but nonetheless significant restrictions. So we assume henceforth that $r_t'(0)$ and $r''(0)$ are non-zero.

We also need to track the evolution of r and r' along the geodesics. We have

$$\dot{r}=r_t\dot{t}+r'\dot{\eta}=\sqrt{\frac{2m}{r}}\dot{t}+r'\dot{\eta} \tag{4.8}$$

$$(\dot{r}')=\frac{2}{3}\frac{rt_0'}{(t_0-t)^2}+r''\dot{\eta}. \tag{4.9}$$

We now prove Eq. (4.3), which demonstrates the weakness of the singularity.

Case One: $\lim_{\tau\to 0^-}|\dot{\eta}|<\infty|$.

By Eq. (4.4a), $\dot{t}(0)=1$. The sign comes from the assumption that the geodesic is future directed and the fact that t is a global time coordinate. The past directed case proceeds in an identical manner. By Eq. (4.9), $(\dot{r}')$ will be finite in the limit $\tau\to 0^-$. If this limit is non-zero, we can apply l'Hôpital's rule to τ^2/r' to prove Eq. (4.3). The other possibility is that $(\dot{r}')(0)=0$ So now assume this to be the case.

Suppose further that $|\ddot{\eta}(0)|<\infty$. Then $(r'\ddot{\eta})(0)=0$, and so taking the limit of Eq. (4.4b), we have

$$0=\lim_{\tau\to 0^-}(2r_t'\dot{t}+r''\dot{\eta})$$

$$=\lim_{\tau\to 0^-}(r_t'\dot{t}+(\dot{r}')),$$

which gives $r_t'(0)=0$, in contradiction of one of our assumptions. So if $(\dot{r}')(0)=0$, then we must have $|\ddot{\eta}(0)|=\infty$.

Using l'Hôpital's rule twice, we have in this case

$$\lim_{\tau\to 0^-}\frac{\tau^2}{r'}=\lim_{\tau\to 0^-}\frac{2}{(\ddot{r}')}.$$

We calculate

$$(\ddot{r}')=r_{tt}'\dot{t}^2+r_t'\ddot{t}+2r_t''\dot{t}\dot{\eta}+r''\ddot{\eta}+r'''\dot{\eta}^2.$$

From Eq. (4.4c), $\ddot{t}(0)=0$ and the terms r_{tt}', r_t'' and r''' will be finite in the appropriate limit. Thus the dominant term is $r''\ddot{\eta}$, giving $\lim_{\tau\to 0^-}|\ddot{r}'|=\infty$, proving Eq. (4.3).

Case Two: $\lim_{\tau\to 0^-}|\dot{\eta}|=\infty$.

Suppose that $|r'\dot{\eta}|(0)<\infty$. Then by Eq. (4.4a), $\dot{t}(0)$ is finite and so Eq. (4.9) gives $|(\dot{r}')|(0)=\infty$. We then use l'Hôpital's rule to prove Eq. (4.3).

Finally, suppose that $|r'\dot{\eta}|(0)=\infty$. Then by Eq. (4.4a),

$$\lim_{\tau\to 0^-}\dot{t}=\lim_{\tau\to 0^-}\left(|r'\dot{\eta}|\left(1+\frac{1}{|r'\dot{\eta}|}\right)^{1/2}\right)$$

$$=\lim_{\tau\to 0^-}|r'\dot{\eta}|.$$

Then by Eq. (4.9),

$$\lim_{\tau\to 0^-}|(\dot{r}')|=|r''\pm r' r_t'|(0)\lim_{\tau\to 0^-}|\dot{\eta}|.$$

This can be finite only if $(r''\pm r' r_t')(0)=0$. But this limit is generically equal to $r''(0)$, which is non-zero, and so we have $|(\dot{r'})|(0)=\infty$. Again, l'Hôpital's rule is used to prove Eq. (4.3).

This completes the proof of Eq. (4.3) for all radial time-like geodesics and thus demonstrates the weakness of the singularity.

V. CONCLUSIONS

The central results here are contained in Eqs. (2.12a), (2.12b), (2.13) and (2.15). These provide a set of covariant equations, the asymptotic solutions to which (which require information about causal geodesics) determine the strengths of singularities in spherically symmetric space-times. The notion of ''strength'' is in a slightly modified form to Tipler's original definition [5]; the modification is clearly motivated and is illustrated by the examples in Sec. III.

Proposition One demonstrates the important point that the behavior of null geodesics tells us nothing about the strength of a non-central singularity. Also, a null geodesic approaching a central singularity terminates in a strong singularity unless $\dot{r}$ has a finite, non-zero limit at the singularity. Proposition Two lists the possible ways in which strong or weak singularities may occur.

In addition to studying the toy models, we were able to demonstrate conclusively the weakness of the naked singularity in Roberts' space-time and the shell-crossing singularities in collapsing spherical dust. This latter proof shows that while detailed qualitative information about causal geodesics is required, we do not need the full solution of the geodesic equations. Therefore there is good hope that the results above may be successfully applied to other situations.

One of these is the case where the singularity occurs at a point where the metric is continuous and non-degenerate ($\det(g_{ab})\neq 0$) (we will refer to such as a continuous non-degenerate singularity). It seems plausible that in such a situation, the singularity must necessarily be weak. The argument goes roughly as follows. Solutions of the time-like geodesics of Sec. II typically behave as $\dot{u}=O(1)$, $\dot{v}=O(1)$ as $\tau\to 0$. Then to obtain a strong curvature singularity, the Riemann tensor components must diverge faster than $O(\tau^{-2})$; integrating twice cannot yield a finite metric. However this argument might not hold for an (ISP3) singularity, and perhaps not for other cases. A careful analysis of Eqs. (2.6a)–(2.6c) and (2.13) should be able to yield either a theorem stating that a continuous non-degenerate singularity is indeed weak, or produce examples to the contrary. The statement that a continuous non-degenerate singularity is necessarily weak has been made, or the conclusion been used, on several occasions in the literature in connection with studies of the Cauchy horizon singularity in black holes and singularities in plane wave space-times. This has usually been accompanied by separate calculations verifying that the singularity is indeed weak [19,20], but this has not always been the case [21–23]. Thus it appears to be of importance to determine exactly when one can conclude weakness for a continuous non-degenerate singularity.

Clarke and Krolak [10] have given necessary and sufficient conditions, in arbitrary space-times, for a singularity to be strong, the conditions involving integrals of certain curvature terms along geodesics. An advantage of our work is that it deals with the full set of Jacobi fields J_τ rather than the volume element $V(\tau)$. As the toy model of Sec. III C shows, this can be important. Also, the decisive term here $2e^{2f}f_{uv}$ is slightly simpler than the decisive terms in [10]. It may be possible to use the results here to investigate the connection between Tipler's definition of strengths of singularities and Krolak's limiting focusing conditions [24].

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