Particle production and complex path analysis

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This paper discusses particle production in Schwarzschild-like spacetimes and in a uniform electric field. Both problems are approached using the method of complex path analysis which is used to describe tunnelling processes in semiclassical quantum mechanics. Particle production in Schwarzschild-like spacetimes with a horizon is obtained here by a new and simple semiclassical method based on the method of complex paths. Hawking radiation is obtained in the (t,r) coordinate system of the standard Schwarzschild metric *without* requiring the Kruskal extension. The coordinate singularity present at the horizon manifests itself as a singularity in the expression for the semiclassical propagator for a scalar field. We give a prescription whereby this singularity is regularized with Hawking's result being recovered. The equation satisfied by a scalar field is also reduced to solving a one-dimensional effective Schrödinger equation with a potential $(-1/x^2)$ near the horizon. Constructing the action for a fictitious nonrelativistic particle moving in this potential and applying the above mentioned prescription, one again recovers Hawking radiation. In the case of the electric field, standard quantum field theoretic methods can be used to obtain particle production in a purely time-dependent gauge. In a purely space-dependent gauge, however, the tunnelling interpretation has to be resorted to in order to recover the previous result. We attempt, in this paper, to provide a tunnelling description using the formal method of complex paths for both the time and space dependent gauges. The usefulness of such a common description becomes evident when ''mixed'' gauges, which are functions of both space and time variables, are analyzed. We report, in this paper, certain mixed gauges which have the interesting property that mode functions in these gauges are found to be a combination of *elementary* functions unlike the standard modes which are transcendental parabolic cylinder functions. Finally, we present an attempt to interpret particle production by the electric field as a tunnelling process between the two sectors of the Rindler spacetime.
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I. INTRODUCTION AND SUMMARY

In this paper, we present a critical review of particle production in Schwarzschild-like spacetimes and in a uniform electric field in Minkowski spacetime. We approach both problems by the method of complex paths discussed by Landau in [1] where it is used to describe tunnelling processes in nonrelativistic semiclassical quantum mechanics. This powerful technique will be used as the basis to provide a new method of recovering Hawking radiation in the usual Schwarzschild coordinates without requiring the Kruskal extension. It will also be used to describe particle production in an electric field in different nontrivial gauges and to link particle production in an electric field to tunnelling processes occurring in the Rindler spacetime.

The Schwarzschild-like spacetimes we consider here are the usual black hole, the Rindler and the de Sitter spacetimes. In the standard black-hole spacetime, particle production was obtained by Hartle and Hawking [2] using semiclassical analysis. In this method, the semiclassical propagator for a scalar field propagating in the Schwarzschild spacetime is analytically continued in the time variable t to complex values. This analytic continuation gives the result that the probability of emission of particles from the past horizon is not the same as the probability of absorption into the future horizon. The ratio between these probabilities is of the form

$$P[\text{emission}] = P[\text{absorption}]e^{-\beta E}, \qquad (1.1)$$

where E is the energy of the particles and $\beta = 1/8\pi M$ is the standard Hawking temperature. The above relation is interpreted to be equivalent to a thermal distribution of particles in analogy with that observed in any system interacting with black body radiation. In the latter case, the probability of emission of radiation by the system is related to the probability of absorption by the system by a similar relation as given above. In Hawking's derivation, the Kruskal extension is of vital importance in obtaining the thermal spectrum.

In this paper, we propose an alternate derivation of Hawking radiation *without using the Kruskal extension*. Our motivation for using the standard Schwarzschild coordinates rather than the Kruskal system are the following: (1) The Schwarzschild spacetime is a static spacetime. It contains a global Killing vector ξ^a and the symmetry generated by this vector is respected in the usual Schwarzschild coordinates (t,r). That is, the surfaces $t=\text{const}$ where t is the Schwarzschild ''time'' variable has the same structure for all t since the metric components g_{ab} is independent of t. Though this Killing vector is spacelike in the region interior to the horizon (and timelike in the region exterior to the horizon), it is still a symmetry of the system. (2) The surface area of spheres of constant ''radial'' coordinate r happen to be $4\pi r^2$, which is that of a sphere in flat spacetime, and hence these surfaces can be used to measure r. In contrast, the

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Kruskal system is an explicitly *time-dependent* spacetime since the metric components depend on the Kruskal time coordinate. It does not possess a global Killing vector associated with surfaces $u=\text{const}$ where u is the Kruskal ''time'' coordinate. Indeed, the presence and structure of such a global Killing vector is not immediately apparent from the form of the metric in these coordinates. Surfaces of constant ''radial'' coordinate do not possess the same property that corresponding surfaces in the Schwarzschild spacetime do. Further, the sectors which contain the past horizon and the time-reversed copy of the usual Schwarzschild spacetime are unphysical. In any realistic collapse scenario, these sectors cannot exist. The nonstatic nature of the Kruskal manifold imply that the ''in'' and ''out'' vacua cannot be the same and explicit particle production can take place.

The difficulty, of course, is that the standard Schwarzschild coordinates possess a coordinate singularity at the horizon. As we shall see, this bad behavior of the coordinates appears as a singularity in the expression for the semiclassical propagator near the horizon and we have to provide a specific prescription to bypass it. *This prescription gives the same result as that obtained by Hawking and can be used in all spacetimes with a Schwarzschild-like metric*. Note that the method described above is fundamentally different from the usual method of calculating the Bogoliubov coefficients for an eternal black hole given, for example, in Ref. [3]. In Ref. [3], appropriate linear combinations of the Schwarzschild mode functions that are analytic on the full Kruskal manifold (except at the past and future singularites at $r=0$) are constructed. The scalar field is then expanded in terms of these modes with the vacuum state being the Kruskal vacuum. Such an expansion provides the appropriate connnection between the annihilation and creation operators for the scalar field in the Schwarzschild and Kruskal coordinate systems. Then, for a given time like observer in either the left or right Schwarzschild sector, whose vacuum state is the Schwarzschild vacuum, the number operator is easily calculated and is found to correspond to a thermal spectrum. In our method, however, the action functional is constructed using the Hamilton-Jacobi method in the appropriate coordinates. The singularity in the action caused by the singular behavior of the coordinate system (in the *unextended* Schwarzschild coordinates) at the horizon is handled by the prescription to obtain particle production. We also reduce the problem of the massive scalar field propagating in such a spacetime to an effective Schrödinger equation which has a singular effective potential near the horizon. In this case, the singular behavior of the coordinates manifests itself as a singularity in the potential. The nonrelativistic action for a particle moving under the influence of such a potential is constructed and the same prescription is used to bypass this singularity. We again recover Hawking radiation.

We next consider the problem of a scalar field propagating in flat spacetime in an uniform electric field background. The total particle production due to the presence of the electric field up to the one-loop approximation is correctly calculated by the gauge invariant method proposed by Schwinger [4]. The same problem can be reduced, in a time dependent gauge, to an equivalent Schrödinger equation with an inverted harmonic oscillator potential. Such an equation can be solved exactly using the standard flat spacetime quantum field theoretic methods. Since the problem is explicitly time dependent, the vacuum state at $t\to-\infty$ and at $t\to\infty$ are not the same. The Bogoliubov coefficients between the ''in'' and ''out'' vacua are easily calculated and the total particle production turns out to be the same as that calculated by the Schwinger method. However, if a space dependent gauge is used to describe the same field, the vacuum state of the field remains the same for all time and hence no particle production can take place. To recover the standard result, it is usual to resort to the tunnelling interpretation. This interpretation is useful since it provides a dynamical picture of particle production (see, for example [5]) and is the only way in which the standard gauge invariant result can be recovered in a time independent gauge.

In this paper, we attempt a tunnelling description based on the method of complex paths for both the time-dependent and time-independent gauges. This method is used to calculate the transmission and reflection coefficients (or the tunnelling coefficients) for the equivalent quantum mechanical problem. Then, an interpretation of these coefficients, in order to explicitly obtain the standard gauge invariant result is provided. However, the usefulness of the tunnelling description is seen when simple ''mixed'' gauges, which are functions of both space and time, are considered. In some gauges, the scalar wave equation can be reduced to solving effective Schrödinger equations in suitable new variables which are combinations of the usual spacetime coordinates. By applying the method of complex paths to these Schrödinger equations, the tunnelling coefficients are seen to match those obtained either for the purely time or purely space dependent gauges. The gauge invariant result is now recovered using the appropriate interpretation needed to identify the Bogoliubov coefficients. In certain other gauges, the mathematics becomes simpler and the field equation can be reduced to a *first order* differential equation rather than an effective Schrödinger equation. For such cases, the exact mode functions themselves are used to set up a tunnelling scenario and the resulting tunnelling coefficients are interpreted according to the tunnelling interpretation to recover the gauge invariant result. The mixed gauge functions we report in this paper have the interesting and useful property that the mode functions are combinations of *elementary* functions. This is in contrast to the standard modes which are transcendental due to the presence of the parabolic cylinder functions. These new modes are found to be singular on the lightcone. This property is very similar to the modes of the Schwarzschild-like spacetimes since these too are singular on the horizon.

In classical theory, the action of a uniform electric field on a charge imparts a constant uniform acceleration to it. The spacetime metric in the rest frame of the charge is the Rindler frame. Quantum field theory, on the other hand, predicts particle production arising due to the presence of an electric field in the spacetime. It is therefore of interest to ask if particle production is linked in some way to the presence of a Rindler frame since both are, in a sense, natural for this problem. We attempt to link particle production by a uniform electric field with processes occuring in the Rindler frame by proposing an interpretation of the standard result in terms of tunnelling between the two Rindler sectors. We do this in a heuristic manner and show that this tunnelling process between the Rindler sectors gives rise to the leading exponential factors in the expression for the effective Lagrangian [see Eq. (4.1)] [6].

The layout of the paper is as follows. Section II contains the semiclassical derivation of Hawking radiation without taking recourse to the Kruskal extension. It also contains the reduction to an effective Schrödinger equation and the subsequent recovery of Hawking radiation. Section III discusses particle production in a uniform electric field in a set of new and nontrivial gauges. It relies heavily on the method of complex paths outlined in the Appendix A. In Sec. IV, we attempt to link particle production in the electric field to processes occuring in the Rindler frame.

II. PARTICLE PRODUCTION IN SPACETIMES WITH HORIZON

Hawking's result that a black hole radiates is essentially a semiclassical result with the thermal radiation arising because of the presence of a horizon in the spacetime structure. We will review briefly the conventional derivation of the thermal radiation using path integrals. Consider a patch of spacetime, which in a suitable coordinate system, has one of the following forms (we assume $c=1$):

$$ds^2=B(r)dt^2-B^{-1}(r)dr^2-r^2(d\theta^2+\sin^2(\theta)d\phi^2) \tag{2.1}$$

or

$$ds^2=B(x)dt^2-B^{-1}(x)dx^2-dy^2-dz^2, \tag{2.2}$$

where $B(r)$ and $B(x)$ are functions of r and x respectively. The horizon in the above spacetimes is indicated by the surface $r=r_0$ $(x=x_0)$ at which $B(r)$ $[B(x)]$ vanishes. We further assume that $B'(r)=dB/dr$ $[B'(x)=dB/dx]$ is finite and nonzero at the horizon. Coordinate systems of the form (2.1) can be introduced in parts of the Schwarzschild and de Sitter spacetimes while that of the form (2.2) with the choice $B(x)=1+2gx$ represents a Rindler frame in flat spacetime. Given the coordinate system of Eq. (2.1) say, in some region $\mathcal{R}$, we first verify that there is no physical singularity at the horizon, which in the case of the Schwarzschild black hole, is at the coordinate value $r_0=2M$ where M is the mass of the black hole. Having done that, we extend the geodesics into the past and future and arrive at two further regions of the manifold not originally covered by the coordinate system in Eq. (2.1) (the Kruskal extension). It is now possible to show that the probability for a particle with energy E to be lost from the region $\mathcal{R}$ in relation to the probability for a particle with energy E to be gained by the region $\mathcal{R}$ is given by the relation

$$P_{\rm loss}=P_{\rm gain}e^{-\beta E}, \tag{2.3}$$

where $\beta=8\pi M$. This is equivalent to assuming that the region $\mathcal{R}$ is bathed in radiation at temperature β^{-1}. In the derivation given in the paper by Hartle and Hawking [2], thermal radiation is derived using the semiclassical kernel by an analytic continuation in the time coordinate t to complex values and it was shown that the probability of emission (loss) from the past horizon was related to absorption (gain) into the future horizon by the relation (2.3).

Since all the physics is contained in the (t,r) plane [or the (t,x) plane], we will discuss Hawking radiation in $(1+1)$ dimensions first and show subsequently that the results generalize naturally to $(3+1)$ dimensions without modification. We first derive the semiclassical result in the (t,r) [or (t,x)] plane by applying a certain prescription to bypass the singularity encountered at the horizon. After this, we reduce the problem of the Klein-Gordon field propagating in the Schwarzschild spacetime to an effective Schrödinger problem in $(1+1)$ dimensions and rederive the semiclassical result by using the same prescription.

A. Hawking radiation in (1+1) dimensions

Consider a certain patch of spacetime in $(1+1)$ dimensions which in a suitable coordinate system has the line element (with $c=1$)

$$ds^2=B(r)dt^2-B^{-1}(r)dr^2, \tag{2.4}$$

where $B(r)$ is an arbitrary function of r. We assume that the function $B(r)$ vanishes at some $r=r_0$ with $B'(r)=dB/dr$ being finite and nonzero at r_0. The point $r=r_0$ indicates the presence of a horizon. It can be easily verified that no physical singularity exists at the horizon since the curvature invariants do not have a singularity on the horizon. Therefore, near the horizon, we expand $B(r)$ as

$$B(r)=B'(r_0)(r-r_0)+\mathcal{O}[(r-r_0)^2]=R(r_0)(r-r_0), \tag{2.5}$$

where it is assumed that $R(r_0)\neq 0$. We now use the equation satisfied by the minimally coupled scalar field Φ with mass m_0 propagating in the spacetime represented by the metric (2.4) to obtain the Hamilton-Jacobi equation satisfied by the action functional S_0. The semiclassical propagator can be constructed using S_0 which will be used to analyze the singularity at the horizon. (We emphasize again that this method is different from that used to compute the Bogoliubov coefficients using appropriate superpositions of mode functions as outlined in [3].)

The equation satisfied by the scalar field is

$$\left(\Box+\frac{m_0^2}{\hbar^2}\right)\Phi=0, \tag{2.6}$$

where the $\Box$ operator is to be evaluated using metric (2.4). Expanding the left-hand side (LHS) of equation (2.6), one obtains

$$\frac{1}{B(r)}\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial}{\partial r}\left(B(r)\frac{\partial\Phi}{\partial r}\right)=-\frac{m_0^2}{\hbar^2}\Phi. \tag{2.7}$$

The semiclassical wave functions satisfying the above equation are obtained by making the standard ansatz for Φ which is

$$\Phi(r,t)=\exp\left[\frac{i}{\hbar}S(r,t)\right], \tag{2.8}$$

where S is a function which will be expanded in powers of $\hbar$. Substituting into the wave equation (2.7), we obtain

$$\left[\frac{1}{B(r)}\left(\frac{\partial S}{\partial t}\right)^2-B(r)\left(\frac{\partial S}{\partial r}\right)^2-m_0^2\right]+\left(\frac{\hbar}{i}\right)\left[\frac{1}{B(r)}\frac{\partial^2 S}{\partial t^2}-B(r)\frac{\partial^2 S}{\partial r^2}-\frac{dB(r)}{dr}\frac{\partial S}{\partial r}\right]=0. \tag{2.9}$$

Expanding S in a power series of $(\hbar/i)$,

$$S(r,t)=S_0(r,t)+\left(\frac{\hbar}{i}\right)S_1(r,t)+\left(\frac{\hbar}{i}\right)^2 S_2(r,t)\ \ldots \tag{2.10}$$

and substituting into Eq. (2.9) and neglecting terms of order $(\hbar/i)$ and greater, we find to the lowest order

$$\frac{1}{B(r)}\left(\frac{\partial S_0}{\partial t}\right)^2-B(r)\left(\frac{\partial S_0}{\partial r}\right)^2-m_0^2=0. \tag{2.11}$$

Equation (2.11) is just the Hamilton-Jacobi equation satisfied by a particle of mass m_0 moving in the spacetime determined by the metric (2.4). The solution to the above equation is

$$S_0(r,t)=-Et\pm\int^r\frac{dr}{B(r)}\sqrt{E^2-m_0^2B(r)}, \tag{2.12}$$

where E is a constant and is identified with the energy. Notice that in the case of $m_0=0$, Eq. (2.11) can be exactly solved with the solution

$$S_0(r,t;m_0=0)=F_1(t-r^*)+F_2(t+r^*), \tag{2.13}$$

where the ''tortoise'' coordinate r^* is defined by

$$r^*=\int\frac{dr}{B(r)}, \tag{2.14}$$

and F_1 and F_2 are arbitrary functions. If F_1 is chosen to be $F_1=-Et+Er^*$ and F_2 chosen to be $F_2=-Et-Er^*$, then it is clear that the solution given in Eq. (2.13) is the same as that in Eq. (2.12) with m_0 set to zero. Therefore, in the case $m_0=0$, the semiclassical ansatz is exact. In the following analysis we will specialize to the case $m_0=0$ for simplicity. The case $m_0\neq 0$ will be considered later. The essential results do not change in any way.

The semiclassical kernel $K(r_2,t_2;r_1,t_1)$ for the particle to propagate from (t_1,r_1) to (t_2,r_2) in the saddle point approximation can be written down immediately as

$$K(r_2,t_2;r_1,t_1)=N\exp\left(\frac{i}{\hbar}S_0(r_2,t_2;r_1,t_1)\right), \tag{2.15}$$

where S_0 is the action functional satisfying the classical Hamilton-Jacobi equation in the massless limit and N is a suitable normalization constant. $S_0(r_2,t_2;r_1,t_1)$ is given by the relation

$$S_0(r_2,t_2;r_1,t_1)=S_0(2,1)=-E(t_2-t_1)\pm E\int_{r_1}^{r_2}\frac{dr}{B(r)}. \tag{2.16}$$

The sign ambiguity (of the square root) is related to the ''outgoing'' ($\partial S_0/\partial r>0$) or ''ingoing'' ($\partial S_0/\partial r<0$) nature of the particle. As long as points 1 and 2, between which the transition amplitude is calculated, are on the same side of the horizon (i.e., both are in the region $r>r_0$ or in the region $r<r_0$), the integral in the action is well defined and real. But if the points are located on opposite sides of the horizon then the integral does not exist due to the divergence of $B^{-1}(r)$ at $r=r_0$.

Therefore, in order to obtain the probability amplitude for crossing the horizon we have to give an extra prescription for evaluating the integral [7]. Since the horizon defined by $B(r_0)=0$ is null we may carry out the calculation in Euclidean space or —equivalently—use an appropriate $i\epsilon$ prescription to specify the complex contour over which the integral has to be performed around $r=r_0$. The prescription we use is that we should take the contour for defining the integral to be an infinitesimal semicircle *above* the pole at $r=r_0$ for outgoing particles on the left of the horizon and ingoing particles on the right. Similarly, for ingoing particles on the left and outgoing particles on the right of the horizon (which corresponds to a time reversed situation of the previous cases) the contour should be an infinitesimal semicircle *below* the pole at $r=r_0$. Equivalently, this amounts to pushing the singularity at $r=r_0$ to $r=r_0\mp i\epsilon$ where the upper sign should be chosen for outgoing particles on the left and ingoing particles on the right while the lower sign should be chosen for ingoing particles on the left and outgoing particles on the right. For the Schwarzschild case, this amounts to adding an imaginary part to the mass since $r_0=2M$.

The prescription outlined above has its origin and basis in the method of complex paths which is outlined in the Appendix (see also [1]). This method is used to compute the transmission and reflection coefficients in standard semiclassical quantum mechanics (and finds wide applicability in the theory of optics) by specifying a suitable complex contour for a given tunnelling scenario. This contour is chosen between two semiclassical regions (where the wave function can be approximated using the semiclassical ansatz with negligible error) such that the semiclassical approximation holds all along the contour. If singularities, which represent distinctive features of the system under consideration, are present in the quantum system and these lie between the semiclassical regions, the appropriate complex contour contains useful information that decides the steady state behavior of the system. In the black hole spacetimes considered in this section, the singularity that appears in the action functional in Eq. (2.16) is directly attributable to the presence of a horizon. Since the semiclassical approximation is applicable on either side of the horizon and arbitrarily close to it, the complex contours needed to bypass the singularity follow from the demand that the semiclassical approximation hold all along the contour. The type of singularity encountered here is similar to that encountered in the one-dimensional Schrödinger system with a potential of the form $(-1/x^2)$. The method of complex paths gives the appropriate contours when dealing with right moving or left moving waves propagating across the singularity at $x=0$. This will be further discussed in Sec. II B [also see Appendix A 1 C where the transmission and reflection coefficients are calculated for the $(-1/x^2)$ potential using the complex paths method].

Consider therefore, an outgoing particle $(\partial S_0/\partial r>0)$ at $r=r_1<r_0$. The modulus square of the amplitude for this particle to cross the horizon gives the probability of emission of the particle. The contribution to S_0 in the ranges $(r_1,r_0-\epsilon)$ and $(r_0+\epsilon,r_2)$ is real. Therefore, choosing the contour to lie in the upper complex plane,

$$S_0[\text{emission}]=-E\lim_{\epsilon\to 0}\int_{r_0-\epsilon}^{r_0+\epsilon}\frac{dr}{B(r)}+(\text{real part})$$
$$=\frac{i\pi E}{R(r_0)}+(\text{real part}), \tag{2.17}$$

where the minus sign in front of the integral corresponds to the initial condition that $\partial S_0/\partial r>0$ at $r=r_1<r_0$. For the sake of definiteness we have assumed $R(r_0)$ in Eq. (2.5) to be positive, so that $B(r)<0$ when $r<r_0$. (For the case when $R<0$, the answer has to be modified by a sign change.) The same result is obtained when an ingoing particle $(\partial S_0/\partial r<0)$ is considered at $r=r_1<r_0$. The contour for this case must be chosen to lie in the lower complex plane. The amplitude for this particle to cross the horizon is the same as that of the outgoing particle due to the time reversal invariance symmetry obeyed by the system.

Consider next, an ingoing particle $(\partial S_0/\partial r<0)$ at $r=r_2>r_0$. The modulus square of the amplitude for this particle to cross the horizon gives the probability of absorption of the particle into the horizon. Choosing the contour to lie in the upper complex plane, we get

$$S_0[\text{absorption}]=-E\lim_{\epsilon\to 0}\int_{r_0+\epsilon}^{r_0-\epsilon}\frac{dr}{B(r)}+(\text{real part})$$
$$=-\frac{i\pi E}{R(r_0)}+(\text{real part}). \tag{2.18}$$

The same result is obtained when an outgoing particle $(\partial S_0/\partial r>0)$ is considered at $r=r_2>r_0$. The contour for this case should be in the lower complex plane and the amplitude for this particle to cross the horizon is the same as that of the ingoing particle due to time reversal invariance.

Taking the modulus square to obtain the probability P, we get

$$P[\text{emission}]\propto\exp\left(-\frac{2\pi E}{\hbar R}\right) \tag{2.19}$$

and

$$P[\text{absorption}]\propto\exp\left(\frac{2\pi E}{\hbar R}\right), \tag{2.20}$$

so that

$$P[\text{emission}]=\exp\left(-\frac{4\pi E}{\hbar R}\right)P[\text{absorption}]. \tag{2.21}$$

Now time reversal invariance implies that the probability for the emission process is the same as that for the absorption process proceeding backwards in time and *vice versa*. Therefore we must interpret the above result as saying that the probability of emission of particles is not the same as the probability of absorption of particles. In other words, if the horizon emits particles at some time with a certain emission probability, the probability of absorption of particles at the same time is different from the emission probability. This result shows that it is more likely for a particular region to gain particles than lose them. Further, the exponential dependence on the energy allows one to give a ''thermal'' interpretation to this result. In a system with a temperature β^{-1} the absorption and emission probabilities are related by

$$P[\text{emission}]=\exp(-\beta E)P[\text{absorption}]. \tag{2.22}$$

Comparing Eq. (2.22) and Eq. (2.21), we identify the temperature of the horizon in terms of $R(r_0)$. Equation (2.21) is based on the assumption that $R>0$. If $R<0$ there will be a change of sign in the equation. Incorporating both the cases, the general formula for the horizon temperature is

$$\beta^{-1}=\frac{\hbar|R|}{4\pi}. \tag{2.23}$$

For the Schwarzschild black hole,

$$B(r)=\left(1-\frac{2M}{r}\right)\approx\frac{1}{2M}(r-2M)+\mathcal{O}[(r-2M)^2] \tag{2.24}$$

giving $R=(2M)^{-1}$, and the temperature $\beta^{-1}=\hbar/8\pi M$. For the de Sitter spacetime,

$$B(r)=(1-H^2r^2)\approx 2H(H^{-1}-r)=-2H(r-H^{-1}) \tag{2.25}$$

giving $\beta^{-1}=\hbar H/2\pi$. Similarly for the Rindler spacetime

$$B(r)=(1+2gr)=2g(r+(2g)^{-1}) \tag{2.26}$$

giving $\beta^{-1}=g\hbar/2\pi$. The formula for the temperature can be used for more complicated metrics as well and gives the same results as obtained by more detailed methods.

The prescription given for handling the singularity is analogous to the analytic continuation in time proposed by Hawking [2] to derive black hole radiance. If one started out on the left of the horizon and went around the singularity $r=r_0$ by a 2π rotation instead of a rotation by π, it can be easily shown that it has the effect of taking the Kruskal coordinates (v,u) to $(-v,-u)$. A full rotation by 2π around the singularity can be split up into two parts to give the amplitude for emission and subsequent absorption of a particle with energy E. Since the amplitudes for the two processes are not the same in the presence of a horizon, one obtains the usual Hawking radiation given in Eq. (2.21) with the value of $R(r_0)$ being $(2M)^{-1}$. This process is similar to that given in [2] which relates the amplitudes involving the past and future horizons. In Hawking's paper, analytically continuing the time variable t to $t-4Mi\pi$ takes the Kruskal coordinates (v,u) to $(-v,-u)$ and since the path integral kernel is analytic in a strip of $4Mi\pi$ below the real t axis, Hawking radiation is obtained by deforming the contour of integration appropriately. *Note however—in our approach—we did not require the Kruskal extension and worked entirely in the* (t,r) *coordinates*.

When $m_0 \neq 0$, the validity of the semiclassical ansatz must be verified. To do this, consider the perturbative expansion (2.10). Retaining the terms of order $\hbar/i$ and neglecting higher order terms, one finds, upon substituting for S_0 given by the relation (2.12) and solving for S_1,

$$S_1 = -E_1 t \pm E E_1 \int \frac{dr}{B(r)} \frac{1}{\sqrt{E^2 - m_0^2 B(r)}} - \frac{1}{4}\ln(E^2 - m_0^2 B(r)), \tag{2.27}$$

where E_1 is a constant. From the above equation, it is seen that S_1 has a singularity of the same order as S_0 at $r=r_0$. When calculating the amplitude to cross the horizon, the contribution from the singular term just appears as a phase factor multiplying the semiclassical kernel and is inconsequential. The nonsingular finite terms do contribute to the kernel but they contribute the same amount to S[emission] and S[absorption] and they do not affect the relation between the probabilities P[emission] and P[absorption]. Subsequent calculation of the terms S_2, S_3, and so on, show that all these terms have a singularity at the horizon of the same order as that of S_0. Their contribution to the probability amplitude is just a set of terms multiplied by powers of $\hbar$ which can be neglected. From this we can conclude that the semiclassical ansatz, in the perturbative limit, is a valid one.

The generalization to (3+1) dimensions is straightforward. We will work with Eq. (2.1) which is in spherical polar coordinates. The results obtained are extendable to Eq. (2.2) in a straightforward manner. The Klein-Gordon equation, written using the metric (2.1), is

$$\frac{1}{B(r)} \frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 B(r) \frac{\partial \Phi}{\partial r}\right) - \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial \Phi}{\theta}\right) - \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{m_0^2}{\hbar^2}\Phi. \tag{2.28}$$

Since the problem is a spherically symmetric one, one can put $\Phi = \Psi(r,t) Y_l^m(\theta,\phi)$ to obtain

$$\frac{1}{B(r)} \frac{\partial^2 \Psi}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 B(r) \frac{\partial \Psi}{\partial r}\right) + \left(\frac{l(l+1)}{r^2} + \frac{m_0^2}{\hbar^2}\right)\Psi = 0. \tag{2.29}$$

Making the ansatz $\Psi = \exp((i/\hbar)S(r,t))$ and substituting into the above equation, we obtain

$$\left[\frac{1}{B(r)}\left(\frac{\partial S}{\partial t}\right)^2 - B(r)\left(\frac{\partial S}{\partial r}\right)^2 - m_0^2 - \frac{l(l+1)\hbar^2}{r^2}\right] + \frac{\hbar}{i}\left[\frac{1}{B(r)} \frac{\partial^2 S}{\partial t^2} - B(r)\frac{\partial^2 S}{\partial r^2} - \frac{1}{r^2}\frac{d(r^2 B)}{dr}\frac{\partial S}{\partial r}\right] = 0. \tag{2.30}$$

Expanding S in a power series as in Eq. (2.10), we obtain, to the zeroth order in $\hbar/i$,

$$S_0 = -Et \pm \int^r \frac{dr}{B(r)} \sqrt{E^2 - B(r)(m_0^2 + L^2/r^2)}, \tag{2.31}$$

where $L^2 = l(l+1)\hbar^2$ is the angular momentum. It is easy to see from the above equation that near the horizon, the presence of the L^2 term can be neglected since it is multiplied by $B(r)$. Therefore, the semiclassical result of Sec. II A follows even in the case of (3+1) dimensions. The semiclassical ansatz is valid in this case as can be seen by calculating explictly the higher order terms in the expansion for S. All these terms have a singularity at the horizon of the same order as that of S_0 and they contribute to the semiclassical propagator either as phase factors or as terms multiplied by powers of $\hbar$ which are entirely negligible. Expanding the Klein-Gordon equation for Φ using the metric (2.2) gives analogous results and will not be explictly given.

B. Reduction to an effective Schrödinger problem in (1+1) dimensions

Consider the relativistic equation for the wave function Φ in Eq. (2.7). We include the mass m_0 here but we shall see later that it does not appear in the final result. Setting

$$\Psi(r) = \frac{e^{-iEt/\hbar}}{\sqrt{B(r)}} Q(r), \tag{2.32}$$

we get the equation

$$-\frac{d^2 Q(r)}{dr^2} - \left[-\frac{B''(r)}{2B(r)} + \frac{(B'(r))^2}{4B^2(r)} + \frac{E^2}{\hbar^2 B^2(r)} - \frac{m_0^2}{\hbar^2 B(r)}\right] Q(r) = 0, \tag{2.33}$$

where $B' = (dB/dr)$ and $B'' = (d^2B/dr^2)$. Near the horizon, we use the expansion of $B(r)$ given in Eq. (2.5). Neglecting terms of order $1/(r-r_0)$ as compared to terms of order $1/(r-r_0)^2$, we get, in the limit of $\hbar \to 0$,

$$-\frac{d^2 Q(r)}{dr^2} - \frac{g}{(r-r_0)^2} Q(r) = 0, \quad \text{where} \quad g = \frac{E^2}{\hbar^2 R^2}. \tag{2.34}$$

[Notice that m_0 does not appear to the leading order in the above equation. Very close to the horizon, the term contain-ing the mass does not contribute significantly. Equation (2.34) is therefore applicable to both massless and massive scalar particles.] Making the transformation $x=(r-r_0)$, we finally obtain the effective Schrödinger equation for the system with $\hbar=2m=1$ with a potential $(-g/x^2)$:

$$-\frac{d^2Q(x)}{dx^2}-\frac{g}{x^2}Q(x)=0. \qquad (2.35)$$

This potential is symmetric but singular at the origin $x=0$. To make the analogy with the Schrödinger equation, we will replace the right-hand side of Eq. (2.35) by $\tilde{E}Q(x)$ and finally take the limit of $\tilde{E}\to 0$. This ''energy'' $\tilde{E}$ should not be confused with the energy E of the field in the original relativistic system. We therefore consider

$$-\frac{d^2Q(x)}{dx^2}-\frac{g}{x^2}Q(x)=\tilde{E}Q(x). \qquad (2.36)$$

The energy spectrum is continuous for all values of $\tilde{E}$ which, for $\tilde{E}<0$, is peculiar to this potential since, for energies less than the potential energy, the spectrum is usually discrete.

The semiclassical analysis follows closely the method adopted in Sec. II A. The action functional $\mathcal{A}$ for a classical particle moving in a potential $-g/x^2$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial\mathcal{A}}{\partial t}+\left(\frac{\partial\mathcal{A}}{\partial x}\right)^2-\frac{g}{x^2}=0. \qquad (2.37)$$

The solution can be immediately written down as

$$\mathcal{A}=-\tilde{E}t\pm\int^x\frac{dx}{x}\sqrt{\tilde{E}x^2+g}. \qquad (2.38)$$

Equation (2.38) has an integral which is divergent if the action is computed for points lying on the opposite sides of the horizon $x=0$. Since this has a similar form to Eq. (2.12), the prescription used in evaluating S[emission] and S[absorption] can be similarly used to evaluate $\mathcal{A}$[emission] and $\mathcal{A}$[absorption]. The results are

$$\mathcal{A}[\text{emission}]=i\pi\sqrt{g}+(\text{real part}),$$

$$\mathcal{A}[\text{absorption}]=-i\pi\sqrt{g}+(\text{real part}). \qquad (2.39)$$

Constructing the semiclassical propagator as before and taking the modulus square to obtain the probabilities for outgoing and ingoing particles, we get

$$P[\text{outgoing}]=\exp\left[-\frac{4\pi E}{\hbar R}\right]P[\text{ingoing}]. \qquad (2.40)$$

The temperature β^{-1} for the system is the same as that in Eq. (2.23) and one recovers the usual result.

To verify that the semiclassical analysis is valid, one must compute the correction terms and check that these have a singularity of the same order as possessed by $\mathcal{A}$. To do this, consider the effective Schrödinger equation (2.36) with factors of $\hbar$ put in:

$$-\hbar^2\frac{d^2Q(x)}{dx^2}-\frac{g}{x^2}Q(x)=\tilde{E}Q(x). \qquad (2.41)$$

Putting $Q(x)=\exp(iA(x)/\hbar)$, and substituting into Eq. (2.41),

$$-i\hbar\frac{d^2A(x)}{dx^2}+\left(\frac{dA(x)}{dx}\right)^2=\tilde{E}+\frac{g}{x^2}. \qquad (2.42)$$

Expanding A in powers of $\hbar/i$, we get

$$A=\mathcal{A}+\frac{\hbar}{i}A_1+\left(\frac{\hbar}{i}\right)^2A_2+\cdots. \qquad (2.43)$$

Substituting into Eq. (2.42) and proceeding as usual, we find that $\mathcal{A}$ is given by Eq. (2.38). The next term A_1 is given by

$$A_1=g\int\frac{dx}{x}\frac{1}{\tilde{E}x^2+g}. \qquad (2.44)$$

The relation for A_1 also has a singularity at the origin of the same order as $\mathcal{A}$. Explicit calculation of the subsequent terms in the expansion of A reveals that all these terms have a singularity of the same order as that of $\mathcal{A}$ and therefore their net contribution to the kernel is either as phase factors or as the exponential of finite terms multiplied by powers of $\hbar$. Therefore, we conclude as before that the semiclassical ansatz is valid.

We show now that the effective Schrödinger equation in (3+1) dimensions is the same as in Eq. (2.35). We consider here the reduction of the Klein-Gordon equation in spherical polar coordinates obtained in Eq. (2.28) using metric (2.1). Setting

$$\Psi=\exp(-iEt/\hbar)Y_l^m(\theta,\phi)\Psi(r) \qquad (2.45)$$

and substituting into Eq. (2.28), we obtain

$$B(r)\frac{d^2\Psi}{dr^2}+\frac{1}{r^2}\frac{d(r^2B)}{dr}\frac{d\Psi}{dr}+\left(\frac{E^2}{\hbar^2B(r)}-\frac{m^2}{\hbar^2}-\frac{L^2}{\hbar^2r^2}\right)\Psi=0. \qquad (2.46)$$

Making the substitution

$$\Psi=\frac{1}{\sqrt{r^2B(r)}}Q(r), \qquad (2.47)$$

we get the result

$$-\frac{d^2Q}{dr^2}-\left[\frac{1}{B^2}\left(\frac{(B')^2}{4}+\frac{E^2}{\hbar^2}\right)-\frac{1}{B}\left(\frac{B''}{2}+\frac{B'}{r}+\frac{m_0^2}{\hbar^2}+\frac{L^2}{\hbar^2r^2}\right)\right]Q=0, \qquad (2.48)$$

where B' and B'' are the first and second derivatives of $B(r)$ respectively. Near the horizon $r=r_0$, using the expansion for

$B(r)$ given in Eq. (2.5), it is easy to see that the $1/B^2$ term in the above equation dominates over the $1/B$ term which can therefore be neglected. The resulting Schrödinger equation is the same as in Eq. (2.35) in the limit of $\hbar \to 0$. It can easily be proved that the effective Schrödinger equation for the Cartesian metric (2.2) gives exactly the same result.

III. PARTICLE PRODUCTION IN A UNIFORM ELECTRIC FIELD

We now move on to the discussion of particle production in a uniform electric field. We study a system consisting of a minimally coupled scalar field Φ propagating in flat spacetime in a uniform electric field background. The two standard gauges, namely the purely time dependent and the purely space dependent gauges, are first considered and it is shown how the method of complex paths outlined in Appendix A 1 a can be used to construct a viable tunnelling interpretation in each case. The standard quantum field theoretic result will not be rederived here since it is well known (though its results are used to construct the tunnelling interpretation). The tunnelling description is then applied to a few nontrivial but simple ''mixed'' gauges which are functions of both space and time.

The method of complex paths is a useful tool used to calculate the transmission and reflection coefficients in semiclassical quantum mechanics in one space dimension [1] and we briefly summarize it here, leaving the details to the Appendix. (Readers unfamiliar with this approach should read the Appendix at this juncture since those results will be used extensively.) First two disjoint regions are identified where the semiclassical wave functions can be written down. [The Schrödinger potential that will be encountered most often will be the inverted harmonic oscillator potential $(-x^2)$ which has the semiclassical, disjoint, regions located at $x \to \pm\infty$.] Then a tunnelling scenario is set up by imposing appropriate boundary conditions on the semiclassical wave functions in these regions. One region is assumed to contain the transmitted wave while the other contains the incident and reflected waves. To obtain the tunnelling coefficients, the solution is analytically continued to the complex plane and the behavior of the transmitted wave is studied along a complex contour (with the space variable now considered complex) joining the two regions. The contour is chosen such that the semiclassical condition is satisfied all along the contour. Rotation along the contour transforms the transmitted wave either to the incident wave or to the reflected wave thus relating the transmission amplitude to either the incident or the reflected wave amplitude. Using the normalization condition satisfied by the tunnelling coefficients, both coefficients can be determined.

The complex contour should be chosen so that singularities (real and complex) in the potential, where the semiclassical ansatz is invalid, are avoided. If such singularites exist, they contribute to the determination of the tunnelling coefficients. Notice that this method works for the exact mode functions too (as it must). An appropriate tunnelling scenario has to be set up and the complex path method can be applied. Such situations will be encountered below in certain gauges where the explicit modes are known but the problem cannot be reduced to an effective Schrödinger equation.

A. Time dependent gauge

The four vector potential A^μ giving rise to a constant electric field in the x direction is assumed to be of the form

$$A^\mu = (0, -E_0 t, 0, 0). \tag{3.1}$$

The electric field is $\mathbf{E} = E_0 \hat{\mathbf{x}}$. The minimally coupled scalar field Φ propagating in flat spacetime, satisfies the Klein-Gordon equation

$$[(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2]\Phi = 0, \tag{3.2}$$

where m is the mass and q is the charge of the field. The mode functions of Φ can be expressed in the form $\Phi(t,\mathbf{x}) = f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$ where $f_{\mathbf{k}}(t)$ satisfies the equation

$$\frac{d^2}{dt^2} f_{\mathbf{k}} + [m^2 + k_\perp^2 + (k_x + qE_0 t)^2] f_{\mathbf{k}} = 0; \quad \mathbf{k}_\perp = (k_y, k_z). \tag{3.3}$$

Introducing the variables,

$$\tau = \sqrt{qE_0}\, t + \frac{\omega}{\sqrt{qE_0}}, \quad \lambda = \frac{k_\perp^2 + m^2}{qE_0}, \tag{3.4}$$

we obtain the equation

$$-\frac{d^2}{d\tau^2} f_{\mathbf{k}} - \tau^2 f_{\mathbf{k}} = \lambda f_{\mathbf{k}}. \tag{3.5}$$

The above equation is essentially a Schrödinger equation in an inverted oscillator potential with a positive ''energy'' λ. Since the energy is positive, the problem is essentially an *over the barrier reflection* problem. Using the results of Sec. A 1 b, we can calculate the reflection and transmission coefficients exactly as

$$R = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}, \quad T = \frac{1}{1 + e^{-\pi\lambda}}, \tag{3.6}$$

where we have put $\hbar = 2m = g_1 = 1$ and set $E_1 = \lambda$ in Eqs. (A24),(A25). To identify the Bogoliubov coefficients α_λ and β_λ, we recast the normalization condition $R + T = 1$ in the form

$$\frac{1}{T} - \frac{R}{T} = 1 \tag{3.7}$$

and then identify $|\beta_\lambda|^2$ with R/T and $|\alpha_\lambda|^2$ with $1/T$. Therefore, the Bogoliubov coefficients are given by

$$|\beta_\lambda|^2 = e^{-\pi\lambda} = \exp\left(-\frac{\pi(k_\perp^2 + m^2)}{qE_0} \right),$$

$$|\alpha_\lambda|^2 = e^{-\pi\lambda}+1 = \exp\left(-\frac{\pi(k_\perp^2+m^2)}{qE_0}\right)+1. \tag{3.8}$$

The transmission and reflection coefficients are time reversal invariant and are dependent only on the energy (magnitude and sign). They are also independent of the direction in which the boundary conditions are applied. To obtain a dynamical picture of particle production, we have to interpret these quantities suitably. In the present case, the following interpretation seems adequate. A purely positive frequency wave with amplitude square T in the infinite past, $t\to-\infty$, evolves into a combination of positive and negative frequency waves in the infinite future $t\to\infty$ with the negative frequency waves having an amplitude square R and the positive frequency waves having an amplitude unity. The quantity R/T determines the overlap between the negative frequency modes in the distant future and the positive frequency modes in the distant past (the notation here differs from the treatment given in [7], [5]). This is identified with the modulus square of the Bogoliubov coefficient β_λ which is the particle production per mode λ. Using the normalization condition satisfied by the Bogoliubov coefficients, $|\alpha_\lambda|^2-|\beta_\lambda|^2=1$, $|\alpha_\lambda|^2$ can be calculated to be $1/T$. Once the Bogoliubov coefficients have been identified, the effective Lagrangian can be easily calculated. This derivation will not be repeated here. We refer the reader to Ref. [5] and Ref. [7] for the explicit calculation. Note that the particular interpretation given in this case is due to its similarity with the more rigourous calculation by quantum field theoretic methods. In the next section, in which we discuss the space dependent gauge, we will be forced to adopt a different interpretation in order to identify particle production.

B. Space dependent gauge

The four vector potential A^μ giving rise to a constant electric field in the x direction is now assumed to be of the form

$$A^\mu = (-E_0 x, 0,0,0). \tag{3.9}$$

The electric field is $\mathbf{E}=E_0\hat{\mathbf{x}}$ as before. The field Φ satisfies Eq. (3.2) as before. Substituting for the potential A^μ from Eq. (3.9) into Eq. (3.2), we obtain

$$(\partial_t^2-\nabla^2-2iqE_0x\partial_t-q^2E_0^2x^2+m^2)\Phi=0. \tag{3.10}$$

We write Φ in the form

$$\Phi = e^{-i\omega t}e^{ik_y y+ik_z z}\phi(x) \tag{3.11}$$

and obtain the differential equation satisfied by ϕ as

$$\frac{d^2\phi}{dx^2}+[(\omega+qE_0x)^2-k_\perp^2-m^2]\phi=0, \tag{3.12}$$

where we have used the notation $k_\perp^2=k_y^2+k_z^2$. Making the following change of variables:

$$\rho=\sqrt{qE_0}x+\frac{\omega}{\sqrt{qE_0}}, \quad \lambda=\frac{k_\perp^2+m^2}{qE_0}, \tag{3.13}$$

into the differential equation for ϕ, it reduces to the form

$$-\frac{d^2\phi}{d\rho^2}-\rho^2\phi=-\lambda\phi. \tag{3.14}$$

In this form, we see that the above differential equation has the form of an effective Schrödinger equation with an inverted harmonic oscillator potential and an negative energy $-\lambda$. If we apply the results of Sec. A 1 b, we obtain the result for *tunnelling through the barrier*. Following the treatment in Ref. [7] and using the results of Sec. A 1 b we can calculate the reflection and transmission coefficients exactly as

$$R=\frac{e^{\pi\lambda}}{1+e^{\pi\lambda}}, \quad T=\frac{1}{1+e^{\pi\lambda}}, \tag{3.15}$$

where we have put $\hbar=2m=g_1=1$ and set $E_1=-\lambda$ in Eqs. (A24),(A25). We cast the renormalization condition $R+T=1$ in the form

$$\frac{1}{R}-\frac{T}{R}=1 \tag{3.16}$$

and then identify the rate of particle production per mode with T/R. The interpretation of particle production using the tunnelling picture now proceeds as follows. A right moving travelling wave of amplitude square $1/R$ is incident on the potential. A fraction T/R is transmitted through it and a wave of unit amplitude is scattered back. The tunnelling probability, which is T/R, is interpreted as the rate at which particles are being produced by the background electric field. This matches exactly with the expression for $|\beta_\lambda|^2$ given in Eq. (3.8). With this interpretation, we recover the usual gauge independent result.

C. Mixed gauges

We shall now study the problem in a new set of gauges which prove to be useful and instructive. (As far as the authors know these gauges have not been studied in the literature before.) When parameters which specify these gauges are varied, the problem can be mapped either onto a ''tunnelling through the barrier'' or ''over the barrier reflection'' system. Then, using the tunnelling description developed previously for each of these systems, the gauge invariant result is obtained.

For certain ranges of these parameters, the scalar wave equation can be reduced to solving a second order equation which can be converted to an effective Schrödinger equation. This equation is studied using the complex path method as in Sec. III A and Sec. III B. The solutions to these effective Schrödinger equations are usually transcendental in form.

For other parameter ranges, however, the scalar field equation reduces to a *first* order differential equation whose solution is a combination of elementary functions. To recover the gauge invariant result, the solution and its complex conjugate are used to set up a tunnelling scenario from which the tunnelling coefficients are obtained. Then, using the tunnelling interpretation, the gauge invariant result of particle production is recovered.

1. Gauge type 1

The first gauge type we consider is a simple generalization of the space-dependent gauge [in Eq. (3.9)] of the form

$$A^{\mu}=(-E_0x+D_1t,0,0,0), \tag{3.17}$$

where E_0 is as usual the magnitude of the electric field and D_1 is a gauge parameter which can be positive or negative. It is easily verified that the above gauge type gives the same electric field as the pure time and space gauges did. The differential equation for the scalar field Φ is

$$[\partial_t^2-\nabla^2+2iq(-E_0x+D_1t)\partial_t-q^2(-E_0x+D_1t)^2+iqD_1+m^2]\Phi=0. \tag{3.18}$$

We make a judicious choice of variables of the form

$$u=-E_0x+D_1t,\quad v=E_0x+D_1t. \tag{3.19}$$

Notice that the coefficients in Eq. (3.18) are dependent only on the variable u and not on v, y or z and so the above equation is separable in the variables (u,v,y,z). Expressing the derivatives (∂_t,∂_x) in terms of (∂_u,∂_v) and writing Φ in the form

$$\Phi=e^{ik_yy+ik_zz}e^{-i\gamma v}\phi(u), \tag{3.20}$$

one obtains

$$(D_1^2-E_0^2)\frac{d^2\phi}{du^2}+2i(qD_1u-\gamma(D_1^2+E_0^2))\frac{d\phi}{du}+(m^2+k_\perp^2+iqD_1-\gamma^2(D_1^2-E_0^2)-q^2u^2+2qD_1\gamma u)\times\phi(u)=0. \tag{3.21}$$

It is easy to see from the above equation that two distinct cases can be identified here, namely, $|D_1|\neq|E_0|$ and $D_1=\pm E_0$. In the first case, the differential equation is a second order one and the effective Schrödinger equation can be obtained by eliminating the first derivative. In the second case, however, the resulting equation is a *first order* differential equation whose solution is an elementary function.

Consider first the case $|D_1|\neq|E_0|$. Writing $\phi(u)$ in the form

$$\phi(u)=Q(u)\exp-i\left[\frac{qD_1}{2(D_1^2-E_0^2)}u^2-\frac{\gamma(D_1^2+E_0^2)}{(D_1^2-E_0^2)}u\right] \tag{3.22}$$

and defining a new variable $\rho=u-(2\gamma D_1/q)$, one obtains the effective Schrödinger equation as

$$-\frac{d^2Q}{d\rho^2}-\frac{q^2E_0^2\rho^2}{(D_1^2-E_0^2)^2}Q(\rho)=\frac{(m^2+k_\perp^2)}{(D_1^2-E_0^2)}Q(\rho). \tag{3.23}$$

The effective potential is clearly seen to be that of an inverted oscillator. Notice that, depending on whether $D_1^2<E_0^2$ or $D_1^2>E_0^2$, the problem reduces to a ''tunnelling through the barrier'' problem or an ''over the barrier reflection'' problem respectively. Using the results of Sec. A 1 b and especially Eq. (A21), the value of the quantity $2\varepsilon_1$ is found to be (with $\hbar=2m=1$)

$$2\varepsilon_1=\sqrt{\frac{1}{g_1}}E_1=\frac{k_\perp^2+m^2}{qE_0}\,\mathrm{sgn}(D_1^2-E_0^2)=\lambda\ \mathrm{sgn}(D_1^2-E_0^2), \tag{3.24}$$

where sgn(x) is the sign function which is positive for $x>0$ and negative for $x<0$ and λ is defined in Eq. (3.5). Thus, the tunnelling coefficients are given by either Eq. (3.6) or Eq. (3.15) depending on $\mathrm{sgn}(D_1^2-E_0^2)$ but are nevertheless independent of D_1 as expected. The tunnelling interpretation required to recover the standard result proceeds accordingly. Note that only the $Q(u)$ part of the full solution to Φ contributes to the transmission and reflection coefficients. The solutions to $Q(u)$ are the usual parabolic cylinder functions.

Now, consider the more interesting case $D_1=+E_0$ with $u=E_0(t-x)$. The differential equation in Eq. (3.21) for ϕ reduces to

$$2i(qE_0u-2\gamma E_0^2)\frac{d\phi}{du}+(m^2+k_\perp^2+iqE_0-q^2u^2+2qE_0\gamma u)\phi(u)=0. \tag{3.25}$$

The solution is easily obtained to be

$$\phi(u)=\left[\sqrt{\frac{q}{E_0}}u-2\gamma\sqrt{\frac{E_0}{q}}\right]^{i\lambda/2-1/2}e^{-iqu^2/4E_0}, \tag{3.26}$$

with λ being defined in Eq. (3.13). This solution resembles the asymptotic forms for some of the parabolic cylinder functions except that it is *exact* and is clearly a combination of elementary functions. (Notice that the solution is singular on the surface $t-x=2\gamma/q$ which is reminiscent of the behavior of black hole modes near the horizon. The implications of this will be discussed in a future publication.) To recover the gauge invariant result in this case, notice that the complex conjugate $\phi^*(u)$ is also a solution. With this pair of independent modes, one can apply the theory given in Sec. A 1 a to set up a tunnelling problem with the appropriate boundary conditions at $u=\pm\infty$. This is most conveniently done by defining a new dimensionless variable

$$s=\sqrt{\frac{q}{E_0}}u. \tag{3.27}$$

The mode function ϕ now becomes

$$\phi(s)=(s-2\gamma\sqrt{E_0/q})^{i\lambda/2-1/2}e^{-is^2/4}, \tag{3.28}$$

with an analogous expression for $\phi^*(s)$. We now assume that the wave function is a right moving travelling wave ϕ_R as $s\to+\infty$ while as $s\to-\infty$, it is a superposition of an incident wave of unit amplitude and a reflected wave given by ϕ_L. Therefore, we have

$$\phi_R=C_3 s^{-i\lambda/2-1/2}e^{is^2/4} \quad (s\to+\infty),$$

$$\phi_L=(-s)^{i\lambda/2-1/2}e^{-is^2/4} + C_2(-s)^{-i\lambda/2-1/2}e^{is^2/4} \quad (s\to-\infty), \tag{3.29}$$

where C_3 and C_2 are the transmission and reflection amplitudes respectively. We have made the approximation $s-2\gamma\sqrt{E_0/q}\approx s$ which holds in the limit $s\to\pm\infty$. Applying the method of complex paths (see Appendix A) to rotate ϕ_R about the *upper* complex contour (in accordance with the semiclassical ansatz), one obtains

$$C_2=-iC_3\exp\left(\frac{\pi\lambda}{2}\right). \tag{3.30}$$

Using the normalization condition $|C_2|^2+|C_3|^2=1$, we get

$$R=|C_2|^2=\frac{e^{\pi\lambda}}{1+e^{\pi\lambda}}, \quad T=|C_3|^2=\frac{1}{1+e^{\pi\lambda}}, \tag{3.31}$$

which are the usual tunnelling coefficients given in Eq. (3.15). It is therefore seen that the system corresponds to a tunnelling through the barrier problem. The interpretation, of course, follows that given in Sec. III B.

For the case $D_1=-E_0$, the modes are given by

$$\phi(u)=\left[\sqrt{\frac{q}{E_0}}u+2\gamma\sqrt{\frac{E_0}{q}}\right]^{-i\lambda/2-1/2}e^{iqu^2/4E_0}. \tag{3.32}$$

An analysis similar to the case $D_1=+E_0$ shows that the tunnelling coefficients are the same as in Eq. (3.31). This system also corresponds to tunnelling through the barrier with the corresponding interpretation given in Sec. III B required in order to recover particle production. Therefore, it is seen that, for the case $|D_1|\neq|E_0|$, the magnitude of D_1 decides the appropriate tunnelling interpretation even though it does not appear in the final expressions for the tunnelling coefficients. In contrast, for the cases $D_1=\pm E_0$, the system reduces to a tunnelling through the barrier with the sign of D_1 not playing any role in deciding the appropriate interpretation.

The above considerations also work for an analogous generalization of the time-dependent gauge of the form

$$A^\mu=(0,-E_0t+D_1x,0,0,0) \tag{3.33}$$

in an obvious manner and we will not repeat the discussion.

2. Gauge type 2

The second gauge type we consider is of the form

$$A^\mu=(D_1t-D_2x,D_2x-D_1t,0,0), \tag{3.34}$$

where D_1 and D_2 are arbitrary constants such that $E_0=D_1+D_2$. The magnitude and direction of the uniform electric field is seen to be the same. It is in this gauge that, for $D_1=D_2$, the mode functions have the simplest form possible. Writing out the differential equation for the scalar field Φ and defining new variables

$$u=D_1t-D_2x, \quad v=D_1t+D_2x \tag{3.35}$$

and setting $\Phi=e^{ik_yy+ik_zz}e^{-i\gamma v}\phi(u)$ as before, one obtains

$$(D_1^2-D_2^2)\frac{d^2\phi}{du^2}+2i[q(D_1+D_2)u-\gamma(D_1^2+D_2^2)]\frac{d\phi}{du} +[m^2+k_\perp^2+iq(D_1+D_2)-\gamma^2(D_1^2-D_2^2) +2\gamma q(D_1-D_2)u]\phi(u)=0. \tag{3.36}$$

Here too, two distinct cases can be distinguished, namely, $D_1\neq D_2$ and $D_1=D_2$ (we will discuss the case $D_2=-D_1$, which corresponds to a ‘‘pure gauge’’ with zero electric field a little later).

Consider first the case $D_1\neq D_2$. Writing ϕ in the form

$$\phi(u)=\exp-i\left(\frac{q}{2(D_1-D_2)}u^2-\frac{\gamma(D_1^2+D_2^2)}{(D_1^2-D_2^2)}u\right)Q(u) \tag{3.37}$$

and introducing a new variable $\rho=u-(2\gamma D_1D_2/qE_0)$, one obtains an effective Schrödinger equation as

$$-\frac{d^2Q}{d\rho^2}-\frac{q^2E_0^2\rho^2}{(D_1^2-D_2^2)^2}Q(\rho)=\frac{(m^2+k_\perp^2)}{(D_1^2-D_2^2)}Q(\rho), \tag{3.38}$$

where we have used $E_0=D_1+D_2$. This equation has the same form as that in Eq. (3.23). Depending on whether $D_1^2<D_2^2$ or $D_1^2>D_2^2$, the problem reduces to a ‘‘tunnelling through the barrier’’ problem or an ‘‘over the barrier reflection’’ problem respectively. Using the results of Sec. A 1 b and especially Eq. (A21), one finds the value of the quantity $2\varepsilon_1$ to be (with $\hbar=2m=1$)

$$2\varepsilon_1=\sqrt{\frac{1}{g_1}}E_1=\frac{k_\perp^2+m^2}{qE_0}\mathrm{sgn}(D_1^2-D_2^2)=\lambda\ \mathrm{sgn}(D_1^2-D_2^2). \tag{3.39}$$

Thus, the tunnelling coefficients are given by either Eq. (3.6) or Eq. (3.15) depending on $\mathrm{sgn}(D_1^2-D_2^2)$ and are dependent only on E_0 as expected. The tunnelling interpretation required to recover the standard result proceeds accordingly.

However, when $D_1=D_2$, the differential equation for $\phi(u)$ reduces to

$$4i(qD_1u-\gamma D_1^2)\frac{d\phi}{du}+(m^2+k_\perp^2+2iqD_1)\phi(u)=0 \tag{3.40}$$

with the solution

$$\phi(u)=\left(\sqrt{\frac{2q}{E_0}}u-\gamma\sqrt{\frac{E_0}{2q}}\right)^{i\lambda/2-1/2}. \tag{3.41}$$

This is the simplest mode function possible for any gauge of the electric field. This solution is also singular just like the modes in Eq. (3.26) but on the surface $t-x=\gamma/q$. The form of the function $\phi(u)$ clearly indicates the factor responsible for the nonzero tunnelling coefficients. One can, in a manner similar to that outlined for the modes (3.26), use the independent mode functions ϕ and ϕ^* to set up a tunnelling problem in the limit $u\rightarrow\pm\infty$. Rotating the transmitted wave in the *upper* complex plane (this is the only contour possible in accordance with the semiclassical ansatz) the tunnelling coefficients are found to be those in Eq. (3.6). The system, in this case, corresponds to over the barrier reflection and the interpretation required to obtain particle production follows that in Sec. III A.

Finally, we look at the case $D_2=-D_1$. The electric field, in this pure gauge, is identically zero. Solving Eq. (3.36) for this case, the solution for ϕ is found to be

$$\phi(u)=e^{-i\lambda u/2\gamma D_1}e^{-iqu^2/2D_1}. \tag{3.42}$$

It is clear from the above form of ϕ that the transmission coefficient is unity and the reflection coefficient is zero. No particle production takes place. Explicit calculations, of course, verify this. The tunnelling interpretation gives a null result in this case.

The mixed gauge types in Eq. (3.17) and Eq. (3.34) represent the same electric field. Hence, these gauges must be related to the pure space and time gauges in Eqs. (3.1),(3.9) by gauge transformations of the form $\tilde{A}^\alpha=A^\alpha+\partial^\alpha f$ where f is a suitable gauge function. Choosing the base gauge A^α as the space-dependent gauge in Eq. (3.9), one obtains the following gauge functions for the two mixed gauge types:

$$A^\alpha\rightarrow\tilde{A}^\alpha=(-E_0x+D_1t,0,0,0),$$

$$f=\frac{1}{2}D_1t^2,$$

$$A^\alpha\rightarrow\tilde{A}^\alpha=(D_1t-D_2x,D_2x-D_1t,0,0),$$

$$f=D_1tx+\frac{1}{2}(D_1t^2-D_2x^2). \tag{3.43}$$

The gauge transformed scalar fields in the $\tilde{A}^\alpha$ gauge ought to be related to those in the A^α gauge by a phase factor e^{if}. But the solutions give in Eq. (3.23) (with $D_1=E_0$) and Eq. (3.41) (with $D_1=D_2=E_0/2$) are clearly not gauge transformations of the mode functions of gauge (3.9) which contain parabolic cylinder functions. These modes are intrinsic to the gauges themselves and cannot be obtained by a simple gauge transformation of the modes of the space-dependent gauge. This is reminiscent of the situation in the Rindler and Minkowski frames. The Rindler plane wave modes of the form $e^{-i\omega\tau}f(\xi)$ (where τ and ξ are the time and space coordinates in the Rindler frame) are *not* obtained from the Minkowski plane wave modes of the form $e^{-i\omega t+kx}$ (where t and x are the Minkowski time and space coordinates) by a coordinate transformation.

The solutions given in Eq. (3.23) and Eq. (3.41) can be obtained in a more general and elegant fashion as follows. Consider the Minkowski metric expressed in the (u,v,y,z) coordinate system where $u=t-x$ and $v=t+x$:

$$ds^2=du\,dv-dy^2-dz^2. \tag{3.44}$$

Assume a vector potential of the form

$$A^\mu=[g(u),h(u),0,0], \tag{3.45}$$

where $g(u)$ and $h(u)$ are arbitrary functions with $A^\mu=(A^u,A^v,A^y,A^z)$ being expressed in the (u,v,y,z) coordinate system and *not* in the (t,x,y,z) system. The electric field is given by $\mathbf{E}=(\partial g(u)/\partial u)\hat{\mathbf{x}}$. Notice that the v component of the vector potential, $A^v=h(u)$, does not contribute to the electric field and is a ''pure'' gauge function. Writing down the scalar wave equation in the (u,v,y,z) system and setting $\Phi=e^{ik_yy+ik_zz}e^{-i\gamma v}\phi(u)$, one obtains the general solution for $\phi(u)$ as

$$\ln(\phi(u))=-\frac{k_\perp^2+m^2}{2i}\int\frac{du}{qg(u)-2\gamma}-\frac{1}{2}\ln(qg(u)-2\gamma)+\frac{i}{2}\int du\,h(u). \tag{3.46}$$

For the uniform electric field, $g(u)=E_0u+C_0$ with C_0 being a constant. Both the gauges in Eq. (3.17) (with $D_1=E_0$) and Eq. (3.34) (with $D_1=D_2=E_0/2$) have a similar form for $g(u)$, but differ in the form of $h(u)$. To set up a tunnelling scenario the explicit form of $h(u)$ must be known as this will determine whether the system is a tunnelling through the barrier or an over the barrier reflection system. For more general forms for $g(u)$, it is clear that the gauge invariant quantity $E^2-B^2\neq0$ which implies nonzero particle production. This is related to the pole structure of the first term in Eq. (3.46) and will be explored further in a future publication.

In summary, it is seen that by a judicious choice of interpretation of the transmission and reflection coefficients in each of the two pure gauges, the standard gauge invariant result can be obtained. This tunnelling interpretation works in the case of the mixed gauges too with the gauge parameters deciding whether the system is an ''over the barrier reflection'' one or a ''tunnelling through the barrier'' one. The effective Schrödinger equation that is analyzed in these gauges is expressed in suitable variables that are combinations of the usual spacetime variables. In cases where there is no explicit reduction to an effective Schrödinger form, the exact modes themselves are used to set up a tunnelling scenario with the recovery of the tunnelling coefficients. The two mixed gauges that were considered lend themselves to easy solutions but more complicated ones may be considered in a similar manner. The mode functions in Eq. (3.26) and Eq. (3.41) are a combination of elementary functions unlike the previously known modes. The tunnelling coefficients arise solely due to the presence of a factor $s^{\pm i\lambda/2}$ in the semiclassical wave functions and in the exact mode functions. It is interesting to note that, in the case of black hole spacetimes too, it is precisely a factor of the form $s^{\pm i\epsilon}$ near the horizon that gives rise to the nontrivial result of Hawking radiation.

In the last section, we go on to interpret particle production by an electric field in terms of tunnelling between the two sectors of the Rindler spacetime.

IV. COMPLEX PATHS INTERPRETATION OF PARTICLE PRODUCTION IN ELECTRIC FIELD

In Sec. III, we noted how particle production can be calculated using the tunnelling interpretation. This interpretation gives the same result in both the space as well as the time dependent gauges. The spectrum of particles produced by the electric field is not thermal in contrast to the spectrum seen by a Rindler observer. We use the formal tunnelling method to show, in a heuristic manner, how this particle production can be obtained by tunnelling between the two sectors of the Rindler spacetime.

The imaginary part of the effective Lagrangian Im L_{eff} for the scalar field in a constant electric field, which is related to the probability of the system to remain in the vacuum state for all time, is given by [7]

$$\mathrm{Im}\,L_{eff}=\sum_{n=1}^{\infty}\frac{1}{2}\frac{(qE_0)^2}{(2\pi)^3}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{\pi m^2}{qE_0}n\right), \tag{4.1}$$

where m is the mass, q is the charge and E_0 is the magnitude of the electric field. We will derive the above expression for Im L_{eff} using the general arguments given in Sec. A 1.

Consider the Hamilton-Jacobi equation for the motion of a particle in an electromagnetic field in $(1+1)$ dimensions:

$$\frac{1}{2}\left(\frac{\partial F}{\partial t}+qA^t\right)^2-\frac{1}{2}\left(\frac{\partial F}{\partial x}-qA^x\right)^2-\frac{1}{2}m^2=0, \tag{4.2}$$

where F is the action and $A^\mu=(A^t,A^x,0,0)$ is the four vector potential. We have neglected the dependence of F on the y and z coordinates since it does not change the results. In a time-dependent gauge, given by Eq. (3.1), the action F can be easily solved for by using the ansatz $F=p_x x+f(t)$ to give

$$F(t_1,x_1;t_0,x_0)=p_x(x_1-x_0)\pm\int_{t_0}^{t_1}dt\sqrt{(p_x+qE_0t)^2+m^2}, \tag{4.3}$$

where p_x is the momentum of the particle in the x direction. The trajectory of the particle in the (t,x) plane is the usual hyperbolic trajectory given by

$$(t-t_i)^2-(x-x_i)^2=-\left(\frac{m}{qE_0}\right)^2, \tag{4.4}$$

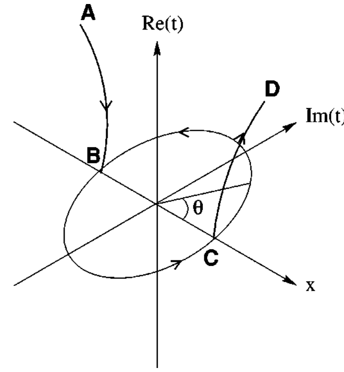

FIG. 1. AB and CD are the hyperbolic branches representing motion in the two disjoint Rindler sectors for a fixed t. The particle tunnels from B to C and back in a circular path along the imaginary time-space plane. If the particle makes n-loops around the circle, the imaginary part of the action will be n times the value for a single loop.

where t_i, x_i are constants. For any fixed position t, there are thus two disjoint trajectories corresponding to motion in the two Rindler wedges.

Let us consider the tunnelling of a particle from one branch of the hyperbola to the other branch *and back* in the imaginary time coordinate t (see Fig. 1). This means that the particle comes back to the same spacetime point as it started from. Choosing the positive sign in Eq. (4.3) (this choice gives a tunnelling probability that is exponentially damped), we have

$$\begin{aligned}F(t_0,x_0;t_0,x_0)&=\oint dt\sqrt{(p_x+qE_0t)^2+m^2}\\&=\frac{m^2}{qE_0}\oint du\sqrt{1+u^2}\\&=i\frac{m^2}{qE_0}\oint d\tau\sqrt{1-\tau^2}\\&=i\frac{m^2}{qE_0}\int_0^{2\pi}d\theta\cos^2(\theta)\\&=\frac{i\pi m^2}{qE_0},\end{aligned} \tag{4.5}$$

where we have made the following changes of variable $(p_x+qE_0t)/m=u=i\tau$ and $\tau=\sin(\theta)$. The expression for $\exp(iF)$ from the above equation is seen to be exactly the same as the exponential term in Eq. (4.1) for $n=1$. The same argument can be repeated for the particle tunnelling n times to and fro to give

$$F_n(t_0,x_0;t_0,x_0)=i\frac{m^2}{qE_0}\int_0^{2n\pi}d\theta\cos^2(\theta)=\frac{i\pi m^2}{qE_0}n. \tag{4.6}$$

Again, the quantity $\exp(iF_n)$ is seen to match with the exponential part of the nth term in Eq. (4.1). Therefore, the imaginary part of the total effective Lagrangian can be written down immediately as

$$\text{Im L}_{\text{eff}}=\sum_{n=1}^{\infty}(\text{prefactor})\exp\left(-\frac{\pi m^2}{qE_0}n\right), \tag{4.7}$$

where the prefactor can only be calculated using the exact kernel. However, the dependence of the prefactor on n and the phase factor $(-1)^{n+1}$ present in Eq. (4.1) can be deduced using the following arguments.

The formal expression of the path integral kernel for the above electric field problem, in the time dependent gauge, is given by [7]

$$K(a,b;s)=\langle a|e^{ish}|b\rangle, \tag{4.8}$$

where $K(a,b;s)$ is the kernel for the particle to propagate between the spacetime points $a=(x^0,\mathbf{x})$ and $b=(y^0,\mathbf{y})$ in a proper time s and h is the Hamiltonian given by

$$h=\frac{1}{2}(i\partial_i-qA_i)(i\partial^i-qA^i)-\frac{1}{2}m^2, \tag{4.9}$$

where A^i is the four vector potential given in Eq. (3.1) and q and m are the charge and mass of the particle respectively. Going over to momentum coordinates and considering the coincidence limit $\mathbf{x}=\mathbf{y}$, the kernel can be written in the form

$$K(x^0,y^0;\mathbf{x},\mathbf{x};s)=-\frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}\frac{dp_x}{s}\mathcal{G}(x^0,y^0;s), \tag{4.10}$$

where $\mathcal{G}(x^0,y^0;s)$ is given by

$$\mathcal{G}(x^0,y^0;s)=\langle x^0|e^{isH}|y^0\rangle \tag{4.11}$$

and H is the Hamiltonian

$$\begin{aligned}H&=-\frac{1}{2}\left(\frac{\partial^2}{\partial t^2}+(p_x+qE_0t)^2+m^2\right)\\&=-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+q^2E_0^2\rho^2+m^2\right),\end{aligned} \tag{4.12}$$

with $\rho=(t+p_x/qE_0)$. In the expression for the kernel in momentum coordinates, we have integrated over the transverse momentum variables p_y and p_z. The above Hamiltonian is that of an inverted harmonic oscillator. [Since all references to p_y and p_z have disappeared in H, the dependence of F on the y and z coordinates was neglected when writing down the expression for the Hamilton-Jacobi equation in Eq. (4.2).] The expression for the effective Lagrangian is then given by

$$\begin{aligned}L_{\text{eff}}&=-i\int_0^{\infty}\frac{ds}{s}K(x^0,x^0;\mathbf{x},\mathbf{x};s)\\&=-\frac{1}{2(2\pi)^2}\int_0^{\infty}\frac{ds}{s^2}\int_{-\infty}^{\infty}dp_x\mathcal{G}(x^0,x^0;s).\end{aligned} \tag{4.13}$$

We would like to evaluate the propagator $\mathcal{G}(x^0,x^0;s)$ for a tunnelling situation where the particle tunnels from the point x^0 and back in loops. Since the path integral is not well defined for closed loops, it will have to be evaluated in some approximate limiting procedure which is outlined below.

Since the tunnelling potential is that of an inverted oscillator, we can use all the results of Sec. A 1 b with the semiclassical wave functions given in Eqs. (A19),(A20). We would like to first account for the factor $(-1)^n$ that arises when a particle tunnels from one side of the barrier to the other and back. Consider an incident wave to the right of the barrier and impinging on it. Using the method of complex paths, we rotate this wave in the lower complex plane (this is the only route possible for the same reason as that given when rotating a right moving travelling wave in the upper complex plane) to obtain a wave again incident on the barrier with a (energy independent) phase factor $\exp(i\pi/2)$ being picked up (other factors dependent on the energy are also picked up but these are not important here). Since this wave is moving in the wrong direction, we assume that the particle that has tunnelled through has the same amplitude as the rotated wave but is moving away from the barrier. This just involves changing the sign of the argument of the exponential in the expression for the rotated wave. Rotating this left moving wave again in the upper complex plane now, the final wave obtained is a right moving wave with another extra phase factor of $\exp(i\pi/2)$ being picked up. The total phase change with respect to the incident wave is thus $\exp(i\pi)$. Since this phase factor is independent of the energy, the propagator for the tunnelling process too, after one such rotation, will pick up a phase of $\exp(i\pi)$. Similarly, for n rotations, n taking the values 1,2,3, . . . , the phase acquired will be $\exp(in\pi)=(-1)^n$.

Therefore, the propagator $\mathcal{G}$ for n loops, $\mathcal{G}_n(x^0,x^0;s)$, can be written as

$$\mathcal{G}_n(x^0,x^0;s)=N(p_x,m,E)e^{in\pi}e^{iS_n(x^0,x^0;s)}, \tag{4.14}$$

where $S_n(x^0,x^0;s)$ is the classical action for the tunnelling process and N is the prefactor that arises after evaluating the "sum over paths." This prefactor is not expected to depend on the proper time s since the tunnelling process takes place instantaneously or on the number of rotations n. So the only quantities it may depend on are p_x, m and E. Though the form of N cannot be determined, we can obtain the constraint on N so as to give the correct result thereby showing the existence of such a factor. The action for the tunnelling process can be determined by solving the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial t}+qA^t\right)^2-\frac{1}{2}\left(\frac{\partial S}{\partial x}-qA^x\right)^2-\frac{1}{2}m^2+\frac{\partial S}{\partial s}=0. \tag{4.15}$$

The solution to the above equation is

$$S=-Es+p_x x\pm\int dt\sqrt{(p_x+qE_0t)^2+(m^2+2E)}. \quad (4.16)$$

Choosing the positive sign and setting $p_x+qE_0t=i\sqrt{m^2+2E}\sin(\theta)$, we obtain

$$S=-Es+p_x x\pm i\frac{m^2+2E}{2qE_0}\int d\theta(1+\cos(2\theta)). \quad (4.17)$$

For *closed* paths, with θ taking the values from 0 to $2n\pi$, the above action can be written as

$$S_n=-Es\pm i\frac{m^2+2E}{2qE_0}\theta. \quad (4.18)$$

We have thrown away the $p_x x$ term while retaining the $-Es$ term since the dependence on the x coordinate is really irrelevant. Defining a new variable $\overline{\theta}=i\theta/2qE_0$ and rescaling $s=\alpha s'$, one obtains

$$S_n=-E\alpha s'+(m^2+2E)\overline{\theta}. \quad (4.19)$$

Choosing α appropriately, S_n can be cast into a form that matches the action for a fictitious free particle in $(1+1)$ dimensions with ‘‘energy’’ αE and ‘‘momentum’’ (m^2+2E) satisfying the classical energy-momentum relation

$$\alpha E=\frac{1}{2}(m^2+2E)^2, \quad (4.20)$$

where the particle’s ‘‘mass’’ is set to unity for convenience. The above equation determines the quantity α and S_n can be written in the form

$$S_n(\overline{\theta}_2,\overline{\theta}_1;s)=\frac{(\overline{\theta}_2-\overline{\theta}_1)^2}{2s'}=\frac{\alpha(\overline{\theta}_2-\overline{\theta}_1)^2}{2s}, \quad (4.21)$$

where $\overline{\theta}_1$ and $\overline{\theta}_2$ are the initial and final states of the free particle with s' being the proper time taken. [Note that $(\overline{\theta}_2-\overline{\theta}_1)=2in\pi/2qE_0$.] Substituting this into the expression for $\mathcal{G}_n(x^0,x^0;s)$ in Eq. (4.14) and evaluating only the integral over s in the expression for the effective Lagrangian in Eq. (4.13) *without* taking the limits, we obtain

$$\begin{aligned}\int\frac{ds}{s^2}\mathcal{G}_n(x^0,x^0;s)&=N(p_x,m,E)e^{in\pi}\int\frac{ds}{s^2}\exp\left(\frac{i\alpha(\overline{\theta}_2-\overline{\theta}_1)^2}{2s}\right)\\&=-N(p_x,m,E)e^{in\pi}\frac{2}{i\alpha(\overline{\theta}_2-\overline{\theta}_1)^2}\exp(iS_n(\overline{\theta}_2,\overline{\theta}_1;s)).\end{aligned} \quad (4.22)$$

Notice that the prefactor to the exponential term has no dependence on the proper time s. Now, we use the form for the action given by Eq. (4.18) and taking the limits for s from 0 to ∞, the effective Lagrangian for n loops, $L_{\rm eff}(n)$, can be written in the form

$$\begin{aligned}L_{\rm eff}(n)=&\frac{i}{2}\frac{(qE_0)^2}{(2\pi)^3}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{\pi m^2}{qE_0}n\right)\frac{8}{\alpha}\\&\times\exp\left(-\frac{2\pi E}{qE_0}n\right)\int_{-\infty}^{\infty}\frac{dp_x}{2\pi}N(p_x,m,E),\end{aligned} \quad (4.23)$$

where we have set $(\overline{\theta}_2-\overline{\theta}_1)=2in\pi/2qE_0$. Taking the limit $E\to 0$, so as to obtain the correct result, and using the expression for α in Eq. (4.20), we find that N must satisfy the relation

$$\lim_{E\to 0}\frac{16E}{(m^2+2E)^2}\int_{-\infty}^{\infty}\frac{dp_x}{2\pi}N(p_x,m,E)=1 \quad (4.24)$$

so that the imaginary part of the effective Lagrangian for n loops, $L_{\rm eff}(n)$, matches the nth term in Eq. (4.1). Therefore, in this manner, the contributions to the imaginary part of the effective Lagrangian for the uniform electric field can be thought of as arising from the tunnelling of particles between the two Rindler sectors.

V. CONCLUSIONS

In conclusion, we see that particle production can be obtained in Schwarzchild-like spacetimes in the standard coordinate systems without requiring the maximally extended manifold. The method of complex paths used in ordinary quantum mechanics is modified appropriately to produce a prescription that regularizes the singularity in the action functional and Hawking radiation is recovered as a consequence. In the case of the electric field, particle production in different gauges has been described using the tunnelling description which gives a correspondence between the transmission and reflection coefficients and the standard Bogoliubov coefficients. The interesting feature of the mixed gauges that were considered was that the mode functions could be combinations of elementary functions for certain values of the gauge parameters. The method of complex paths also gives a simple interpretation of particle production in an electric field as arising due to tunnelling between the two disjoint sectors of the Rindler spacetime. Though we have only given a heuristic argument in this paper, we will explore this issue further in a future publication.

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APPENDIX: FACETS OF TUNNELLING

In this section we briefly review the basic concepts of semiclassical quantum mechanics in one dimension and formally describe the tunnelling process. We then apply the formalism to two potentials, namely, $V_1(x) = -x^2$ and $V_2(x) = -1/x^2$, and calculate the transmission and reflection coefficients for both.

1. Semiclassical limit of quantum mechanics

Consider a simple one dimensional quantum mechanical system with an arbitrary potential $V(x)$ where x denotes the space variable (see Ref. [1] for details). To describe the transition of the system from one state to another, we first solve the corresponding classical equations of motion and determine the path of transition. This path is, in general, complex since many processes like tunneling through a potential barrier cannot occur classically. Therefore, the transition point q_0 where the system formally makes the transition is a complex number determined by the classical conservation laws. Then, the action S for the transition from some initial state x_i to a final state x_f given by

$$S(x_f, x_i) = S(x_f, q_0) + S(q_0, x_i) \tag{A1}$$

is calculated. Here, $S(q_0, x_i)$ is the action for the system to move from the initial state x_i to the transition point q_0 while $S(x_f, q_0)$ is that to move from q_0 to x_f. The probability P for the transition to occur is given by the formula

$$P \sim \exp\left(-\frac{2}{\hbar}\mathrm{Im}[S(x_f, q_0) + S(q_0, x_i)]\right). \tag{A2}$$

The above formula is valid only when the argument of the exponential is large. Further, if the potential energy has singular points, these must also be considered as possible values for q_0. If the position of the transition point is not unique, then it must be chosen so that the exponent in Eq. (A2) has the smallest absolute value but still must be large enough so that the above formula be valid.

If the transition point q_0 is real, but lies in the classically inaccessible region, then the above formula gives the transmission coefficient for penetration through a potential barrier, while if the transition point is complex, it solves the problem finding the over the barrier reflection coefficient. The $\sim$ sign in the above formula is used since the coefficient in front of the exponential is not determined. This can be determined by calculating the exact semiclassical wave functions. Generally, it is desirable to find the ratios of two different transitions so that this coefficient does not matter.

The physics of the tunnelling and the ''over the barrier'' reflection processes are very different. In the tunnelling process, the semiclassical analysis gives a transmission coefficient that is an exponentially small quantity with the corresponding reflection coefficient being unity. In contrast, in the ''over the barrier'' reflection process, just the reverse is obtained. The transmission coefficient is unity while the reflection coefficient is an exponentially small quantity. Both these processes will be encountered when the electric field is studied in different gauges.

We will now review the method of calculating the transmission and reflection coefficients for a typical quantum mechanical problem using the method of complex paths for a general potential $V(x)$.

a. Description of the method of complex paths

Consider the motion of a particle of mass m in a region characterized by the presence of a potential $V(x)$ in one space dimension. The problem is to calculate the transmission and reflection coefficients between two asymptotic regions labeled L and R where the semiclassical approximation to the exact wave function is valid. After identifying these regions and writing down the semiclassical wave functions, definite boundary conditions are imposed. The usual boundary conditions considered are that in one region, say L, the wave function is a superposition of an incident wave and a reflected wave while in the second region R, the wave function is just a transmitted wave. Then, a complex path (in the plane of the now complex variable x) is identified from R to L such that (a) all along the path the semiclassical ansatz is valid and (b) the reflected wave is exponentially greater than the incident wave at least in the latter part of the path near the region L. The transmitted wave is then moved along the path to obtain the reflected wave and thus the amplitude of reflection is identified in terms of the transmission amplitude. Having done this, the normalization condition is used, i.e., the sum of the modulus square of the transmission and reflection amplitudes should equal unity, to determine the exact values of the transmission and reflection coefficients.

For a given potential, the turning points q_0 (or transition points) are given by solving the equation

$$p(q_0) = \sqrt{2m(E - V(q_0))} = 0, \tag{A3}$$

where $p(x)$ is the classical momentum of the particle and E is the energy of the particle. In general, q_0 is complex. At these points, the semiclassical ansatz is not valid since the momentum is zero. Further, the potential can possess singularities. At these points too, the semiclassical approximation is invalid. Therefore the contour between the two regions should be chosen to be far away from such points. In general the contour will enclose them. Therefore, the relation between the transmission and reflection amplitudes is determined by taking into account the turning points and the singularities of the potential.

The Schrödinger equation to determine the wave function ψ of the particle is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V(x))\psi. \tag{A4}$$

Referring to [1], the semiclassical wave function, in the classically allowed region where $E > V(x)$, is given by the formula

$$\psi = C_1 p^{-1/2}\exp\left(\frac{i}{\hbar}\int p(x)dx\right) + C_2 p^{-1/2}\exp\left(-\frac{i}{\hbar}\int p(x)dx\right), \tag{A5}$$

while in the classically inaccessible regions of space where $E < V(x)$, the function $p(x)$ is purely imaginary and the wave function is now given by the relation

$$\psi = C_1 |p|^{-1/2} \exp\left(-\frac{1}{\hbar} \int |p(x)| dx \right) + C_2 |p|^{-1/2} \exp\left(\frac{1}{\hbar} \int |p(x)| dx \right). \quad \text{(A6)}$$

The condition on the potential for semiclassicality of the wave function to be valid is

$$\left| \frac{d}{dx} \left(\frac{\hbar}{p(x)} \right) \right| \ll 1, \quad \text{(A7)}$$

or, in another form,

$$\frac{m\hbar |F|}{|p|^3} \ll 1, \quad F = -\frac{dV}{dx}. \quad \text{(A8)}$$

It ought to be noted that the accuracy of the semiclassical approximation is not such as to allow the superposition of exponentially small terms over exponentially large ones. Therefore, it is inapplicable in general to retain both terms in Eq. (A6). We will consider a few cases of interest in this paper and refer the reader to [1] for an exhaustive discussion along with suitable illustrative examples.

Consider the case in which the semiclassical condition (A8) holds in the regions $x \to \pm\infty$. As $x \to -\infty$, the wave function is assumed to be a superposition of incident and reflected waves and is written in the form

$$\psi = p^{-1/2} \exp\left(\frac{i}{\hbar} \int p(x) dx \right) + C_2 p^{-1/2} \exp\left(-\frac{i}{\hbar} \int p(x) dx \right), \quad \text{(A9)}$$

where the incident wave has unit amplitude while the reflected wave has amplitude given by C_2. As $x \to +\infty$, the wave function is assumed to be a right moving travelling wave

$$\psi = \frac{C_3}{\sqrt{p}} \exp\left(\frac{i}{\hbar} \int p\, dx \right). \quad \text{(A10)}$$

The method of complex paths can now be applied on the function (A10). The contour is chosen either in the upper or lower complex plane such that the reflected wave is always exponentially greater than the incident wave along that part of the path near the region $x \to -\infty$. If this is satisfied along one of the contours then C_2 is determined in terms of C_3. To carry out the above procedure however, the exact semiclassical wave functions as $x \to \pm\infty$ have to be determined. This will be done explicitly later for the relevant cases.

A different case arises when the semiclassical ansatz holds in the vicinity of the origin $x = 0$ rather than at $x = \pm\infty$. The boundary conditions assumed in this case are the same as above with the condition $x \to \infty$ replaced by $x \to 0^+$ and $x \to -\infty$ by $x \to 0^-$. Here, the required contour is about the origin and is chosen to be small. But it must still be large enough so that the reflected wave is much larger than the incident wave along the latter part of the contour near the region $x < 0$.

In the above cases the method of complex paths gives the exact transmission and reflection amplitudes. But, in certain cases it is enough to assume that the transmission and incident amplitudes are equal to unity while the reflection amplitude is exponentially damped and consequently very small. Here, the ''over the barrier'' reflection coefficient for energies large enough so that the reflection coefficient is exponentially small, has to be determined. In this case, the condition $E > V(x)$ is always satisfied. Therefore, the transition point q_0 at which the particle reverses its direction is the complex root of the equation $V(q_0) = E$. Let q_0 lie in the upper complex plane for definiteness. Now, the amplitudes of the incident and transmitted waves are equal (both are set to unity within exponential accuracy). To calculate the reflection coefficient, the relation between the wave functions far to the right of the barrier and far to the left of the barrier must be determined. The transmitted wave can be written in the form

$$\psi_T = \frac{1}{\sqrt{p}} \exp\left(\frac{i}{\hbar} \int_{x_1}^{x} p\, dx \right), \quad \text{(A11)}$$

where x_1 is any point on the real axis. We follow the variation of ψ_T along a path C in the upper complex plane which encloses the turning point q_0. The latter part of this path must lie far enough to the left of q_0 so that the error in the semiclassical incident wave is less than the required small reflected wave. Passage around q_0 only causes a change in the sign of the root $\sqrt{E - V(x)}$ and after returning to the real axis, the function ψ_T becomes the reflected wave ψ_R. Going around a complex path in the lower complex plane converts ψ_T into the incident wave. Since the amplitudes of the incident and transmitted waves may be regarded as equal, the required reflection coefficient is given by

$$R = \left| \frac{\psi_R}{\psi_T} \right|^2 = \exp\left(-\frac{2}{\hbar} \mathrm{Im} \int_C p\, dx \right). \quad \text{(A12)}$$

Now we can deform the contour in any way provided it still encloses the point q_0. In particular, the contour can be deformed to go from x_1 to q_0 and back. This gives

$$R = \exp\left(-\frac{4}{\hbar} \mathrm{Im} \int_{x_1}^{q_0} p\, dx \right). \quad \text{(A13)}$$

Since $p(x)$ is real everywhere, the choice of x_1 on the real axis is immaterial. The above formula determines the above the barrier reflection coefficient. It must be emphasized that to apply the above formula the exponent must be large so that $1 - R$ is very nearly equal to unity.

Finally consider a situation where the amplitudes of the reflection and incident wave are equal. The transmission coefficient is now an exponentially small quantity. This case corresponds to the standard tunnelling process. The problem is characterized by the presence of real turning points between which lies the classically forbidden region where the energy $E < V(x)$. For definiteness, let there be two real turning points labeled q_- and q_+. The potential $V(x)$, in the immediate vicinity of the turning points of the barrier, is assumed to be of the form

$$E-V(x)\approx F_0(x-q_\pm), \quad F_0=-\left.\frac{dV}{dx}\right|_{x=q_\pm}. \tag{A14}$$

This assumption is equivalent to saying that the particle, near the turning points, moves in a homogeneous field. With this assumption, the amplitude of transmission C_3 is given by (refer to Ref. [1], p. 181)

$$C_3=\exp\left(-\frac{1}{\hbar}\int_{q_-}^{q_+}|p(x)|dx\right). \tag{A15}$$

The transmission coefficient is then given by $|C_3|^2$. The above formula holds only when the exponent is large. In the derivation above, we have assumed that the semiclassical condition holds across the entire barrier except in the immediate vicinity of the turning points. In general, however, the semiclassical condition need not hold over the entire extent of the barrier. The potential, for example, could drop steeply enough so that Eq. (A14) is not valid. In these cases, the exact semiclassical equations have to be determined before applying the method of complex paths. The cases encountered in this paper all satisfy Eq. (A14).

We now apply the above results to two potentials. The first is the well known inverted harmonic oscillator potential $V_1(x)=-g_1x^2$ with $g_1>0$ while the other is $V_2(x)=-g_2/x^2$ with $g_2>0$. The first potential arises when the propagation of a scalar field in a constant electric field background is studied. The second potential arises when the propagation of a scalar field in Schwarzschild-like spacetimes is considered in the vicinity of the horizon.

b. Application to the potential $V_1(x)=-g_1x^2$

Consider the potential given by

$$V_1(x)=-g_1x^2, \tag{A16}$$

where $g_1>0$ is a constant. This potential is the inverted harmonic oscillator potential and is discussed extensively in many places (see for example [1,7,8,9]). We will follow the semiclassical treatment given in Ref. [1] and briefly review the calculation of the reflection and transmission coefficients for both the tunneling and over the barrier reflection cases.

The semiclassicality condition (A8) for the above potential is

$$\left|\frac{\hbar g_1 x}{\sqrt{2m}[E_1+g_1x^2]^{3/2}}\right|\ll 1, \tag{A17}$$

where m is the mass of the particle and E_1 is its energy. The above condition definitely holds for large enough $|x|$ and for any value of E_1, either positive or negative. Therefore the motion of a particle moving under such a potential is semiclassical for large enough $|x|$ and hence holds as $x\to\pm\infty$.

Since the motion is semiclassical for large $|x|$, we can expand the momentum $p(x)$ as

$$p(x)=\sqrt{2m(E_1+g_1x^2)}\approx\sqrt{2mg_1}\left(x+\frac{E_1}{2g_1x}\right). \tag{A18}$$

Using Eq. (A18), the semiclassical wave functions can be written as follows with the following boundary conditions. As $x\to\infty$, we assume that the wave function is a right moving travelling wave ψ_R while as $x\to-\infty$, it is a superposition of an incident wave of unit amplitude and a reflected wave given by ψ_L. Therefore, we have

$$\psi_R=C_3\xi^{i\varepsilon_1-1/2}e^{i\xi^2/2} \quad (\xi\to+\infty), \tag{A19}$$

$$\psi_L=(-\xi)^{-i\varepsilon_1-1/2}e^{-i\xi^2/2}+C_2(-\xi)^{i\varepsilon_1-1/2}e^{i\xi^2/2} \quad (\xi\to-\infty), \tag{A20}$$

where we have made the definitions

$$\xi=\left(\frac{2mg_1}{\hbar^2}\right)^{1/4}x, \quad \varepsilon_1=\frac{1}{\hbar}\sqrt{\frac{m}{2g_1}}E_1. \tag{A21}$$

Following the variation of Eq. (A19) around a semicircle of large radius ρ in the *upper* half plane of the now complex variable ξ, we obtain

$$C_2=-iC_3\exp(-\pi\varepsilon_1). \tag{A22}$$

The conservation of particles is expressed by the condition that

$$|C_3|^2+|C_2|^2=1. \tag{A23}$$

From Eq. (A22) and Eq. (A23), the transmission coefficient is

$$T=|C_3|^2=\frac{1}{1+e^{-2\pi\varepsilon_1}}=\frac{1}{1+e^{-(1/\hbar)\pi\sqrt{2m/g_1}E_1}}, \tag{A24}$$

while the reflection coefficient is

$$R=|C_2|^2=\frac{e^{-(1/\hbar)\pi\sqrt{2m/g_1}E_1}}{1+e^{-(1/\hbar)\pi\sqrt{2m/g_1}E_1}}. \tag{A25}$$

Note that the passage through the *lower* half complex plane to determine C_2 is unsuitable since on the part of the path $-\pi<\phi<-\pi/2$, where ϕ is the phase of the complex variable ξ, the incident wave [first term in Eq. (A20)] is exponentially large compared with the reflected wave. The above formula holds for all energies E_1. This is because, even for negative energies, the semiclassical wave functions given in Eqs. (A19),(A20) are exactly the same with the boundary conditions being fully satisfied.

If E_1 is large and negative, Eq. (A24) gives $T\approx e^{-\pi\sqrt{2m/g_1}|E_1|/\hbar}$ and thus $R\sim 1$. This is in accordance with the formula in Eq. (A15). To apply Eq. (A15) it is necessary to calculate the turning points first. The real turning points are $q_0=-\sqrt{|E_1|/g_1}$ and $q_1=\sqrt{|E_1|/g_1}$. Therefore,

$$C_3=\exp\left(-\frac{1}{\hbar}\int_{q_0}^{q_1}|p(x)|dx\right)$$

$$=\exp\left(-\frac{1}{\hbar}\sqrt{2mg_1}\int_{q_0}^{q_1}\left|\sqrt{x^2-\frac{|E_1|}{g_1}}\right|dx\right)$$

$$=\exp\left(-\frac{1}{2\hbar}\pi\sqrt{2m/g_1}|E_1|\right). \tag{A26}$$

This gives the same answer.

We can calculate the over the barrier reflection coefficient using Eq. (A13) for E_1 large and *positive*. The turning points now are given by solving the equation $p(q_0)=0$ with the condition that $E_1>V_1(x)$ always. Since $E_1>0$, the turning points are $q_0=\pm i\sqrt{E_1/g_1}$. Choosing the positive sign for q_0 and setting $x_1=0$, the integral in Eq. (A13) is evaluated as follows:

$$\int_0^{q_0}p(x)dx=\sqrt{2mg_1}\int_0^{q_0}\sqrt{E_1/g_1+x^2}$$

$$=i\sqrt{2m/g_1}E_1\int_0^1\sqrt{1-y^2}$$

$$=\frac{1}{4}i\pi\sqrt{2m/g_1}E_1. \tag{A27}$$

Therefore,

$$R=\exp\left(-\frac{1}{\hbar}\pi\sqrt{2m/g_1}E_1\right). \tag{A28}$$

The above formula can also be obtained directly from Eq. (A25) by neglecting the exponential term compared to unity which means that the energy has to be large.

c. Application to the potential $V_2(x)=-g_2/x^2$

Consider the potential given by

$$V_1(x)=-\frac{g_2}{x^2}, \tag{A29}$$

where g_2 is a positive constant. The potential has a singularity at the origin. This potential arises when the effective Schrödinger equation is calculated for Schwarzschild-like spacetimes.

The semiclassical condition (A8) for this potential takes the form

$$\left|\frac{\hbar g_2}{\sqrt{2m}}\frac{1}{[E_2x^2+g_2]^{3/2}}\right|\ll 1, \tag{A30}$$

where E_2 is the energy. It is clear that the above relation holds for large $|x|$. It also holds for small $|x|$ if $\sqrt{2mg_2}\gg\hbar$. Notice that the quasiclassicality condition for small $|x|$ is independent of the sign and magnitude of the energy E_2. For this potential, we will be concerned only with the small $|x|$ behavior in contrast with the potential V_1. Since the motion is semiclassical for small $|x|$, we expand the momentum $p(x)$ as

$$p(x)=\sqrt{2m\left(E_2+\frac{g_2}{x^2}\right)}\approx\sqrt{\frac{2mg_2}{x}}+\sqrt{\frac{m}{2g_2}}E_2x. \tag{A31}$$

Notice the similarity between Eq. (A18) and Eq. (A31).

We will calculate the over the barrier reflection coefficient with $E_2>0$ and small, which will be of interest later. Using the expansion in Eq. (A31), the semiclassical wave functions with the following boundary conditions, namely, that for $x>0$ the wave function is a right moving travelling wave while it is a superposition of an incident wave of unit amplitude and reflected wave for $x<0$, are

$$\psi_R=C_3\xi^{i\varepsilon_2+1/2}e^{i\xi^2/2}\quad(\xi>0), \tag{A32}$$

$$\psi_L=(-\xi)^{-i\varepsilon_2+1/2}e^{-i\xi^2/2}+C_2(-\xi)^{i\varepsilon_2+1/2}e^{i\xi^2/2}\quad(\xi<0), \tag{A33}$$

where we have made the definitions

$$\xi=\left(\frac{mE_2^2}{2g_2}\right)^{1/4}x,\quad \varepsilon_2=\sqrt{\frac{2mg_2}{\hbar}}. \tag{A34}$$

Following the variation of Eq. (A32) around a small semicircle of radius $\rho<\sqrt{g_2/|E_2|}$ (in contrast to the potential V_1 where the radius ρ was large) in the *upper* half complex plane, we obtain

$$C_2=C_3\exp\left(-\pi\varepsilon_2+\frac{i\pi}{2}\right). \tag{A35}$$

Setting $T=|C_3|^2=1$ and $R=|C_2|^2$, we finally obtain

$$R=Te^{-2\pi\varepsilon_2}=Te^{-(1/\hbar)2\pi\sqrt{2mg_2}}. \tag{A36}$$

Using the normalization condition $R+T=1$, we obtain

$$T=\frac{1}{1+e^{-(1/\hbar)2\pi\sqrt{2mg_2}}}$$

and

$$R=\frac{e^{-(1/\hbar)2\pi\sqrt{2mg_2}}}{1+e^{-(1/\hbar)2\pi\sqrt{2mg_2}}}. \tag{A37}$$

Notice that the above result is independent of the energy E_2 and hence holds for $E_2<0$ too. For small $|x|$, the lack of dependence on E_2 is not too surprising since the contour is such that it is not too close to the real turning points $q_0=\pm\sqrt{g_2/|E_2|}$. Anyway, when $E_2\sim 0^+$, ρ is "large" and therefore the contour is chosen to lie in the upper complex plane for the same reason as given in the analysis of the potential V_1 in the previous section.

We will derive the result in Eq. (A36) using Eq. (A12). The complex turning points q_0 are the roots of the equation $E_2=-g_2/q_0^2$ where $E_2>0$ and therefore, the turning points are $q_0=\pm i\sqrt{g_2/E_2}=\pm ip_0$. Hence, we have to evaluate the integral

$$\int_C p\,dx = \sqrt{2mE_2}\int_C \sqrt{1+\frac{p_0^2}{x^2}}dx, \tag{A38}$$

where the contour C encircles the point $x = ip_0$ in the upper half complex plane. However, since there is a singularity at the origin, we cannot deform the contour as was done when deriving Eq. (A13). Therefore, as a means of regularization, we modify the potential to

$$V_{\rm mod}(x) = -\frac{g_2}{x^2+\epsilon^2}, \tag{A39}$$

where the limit $\epsilon \to 0$ must be taken at the end of the calculation. The turning points for the modified potential are $x_{\rm mod} = \pm i\sqrt{\epsilon^2 + g_2/E_2}$ while the poles of the modified potential are at $x = \pm i\epsilon < x_{\rm mod}$. Even in this case, there is a singularity on the path of integration which contributes to the integral rather than the turning point. Therefore, integrating up to $+i\epsilon$ using the modified potential and back, we obtain

$$\begin{aligned}\int_C p\,dx &= \lim_{\epsilon\to 0} 2\sqrt{2mE_2}\int_0^{i\epsilon}\sqrt{1+\frac{p_0^2}{x^2+\epsilon^2}}dx \\ &= \lim_{\epsilon\to 0} 2i\sqrt{2mE_2}\,\epsilon\int_0^1 dy\sqrt{1+\frac{p_0^2/\epsilon^2}{1-y^2}} \\ &\approx 2i\sqrt{2mE_2}\,p_0\int_0^1\frac{dy}{\sqrt{1-y^2}} \\ &= i\pi\sqrt{2mE_2}\,p_0 = i\pi\sqrt{2mg_2}. \end{aligned} \tag{A40}$$

We therefore recover the result given in Eq. (A36). From the above calculation it is clear that, due to the singularity at the origin, the reflection coefficient has no contribution from the turning point at all.

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