Duality invariance of cosmological perturbation spectra

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I show that cosmological perturbation spectra produced from quantum fluctuations in massless or self-interacting scalar fields during an inflationary era remain invariant under a two parameter family of transformations of the homogeneous background fields. This relates slow-roll inflation models to solutions which may be far from the usual slow-roll limit. For example, a scale-invariant spectrum of perturbations in a minimally coupled, massless field can be produced by an exponential expansion with $a \propto e^{Ht}$, or by a collapsing universe with $a \propto (-t)^{2/3}$. [S0556-2821(99)02814-3]

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The spectrum of perturbations on large scales is a key test of any models of the early universe. During an inflationary era quantum fluctuations on small scales become stretched beyond the horizon generating inhomogeneities on super-horizon scales that are otherwise inexplicable in the standard big bang model [1–3]. Conventional models of slow-roll inflation generally predict an almost scale-invariant spectrum of adiabatic density perturbations [4]. As major observational programs are now under way to produce detailed maps of these perturbations, it is important to investigate whether one can uniquely reconstruct the inflationary history of the universe from the spectrum of inhomogeneities. This question has received considerable attention in recent years in the context of slow-roll inflation [5] where it has been realized that there is a degeneracy in the spectrum of adiabatic density perturbations. This could be removed by a detection of the gravitational wave background on the same scale which, in the slow-roll approximation, gives a direct record of the evolution of the scale factor, and hence of the inflaton potential.

By contrast there has been relatively little study of the reconstruction of the evolution in the non-slow-roll case [6]. Recently it has been discovered that the spectra of perturbations produced in so-called pre-big bang models of the early universe [7], based on solutions of the low energy string effective action, are invariant under SL(2,R) symmetry transformations (including S-duality transformations) of the background fields [8]. This raises the interesting question of what is the most general type of cosmological evolution that yields a given perturbation spectrum.

I will consider linear perturbations, $\delta\phi(\eta,x^i)$, about a homogeneous background, $\phi(\eta)$, in a homogeneous cosmology. For a minimally coupled massless field we can neglect any back-reaction upon the spacetime curvature to first order, and so perturbations obey the wave equation

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \nabla^2 \delta\phi = 0, \tag{1}$$

where a dot denotes derivatives with respect to cosmic time t, and $3H$ corresponds to the expansion rate of the homogeneous hypersurfaces. One can decompose the perturbations into independent wavemodes $\delta\phi_k(\eta)Q_k(x^i)$, where $Q_k(x^i)$ is the eigenfunction of the spatial Laplacian ∇^2 with eigenvalue $-k^2$.

The canonically normalized quantum field corresponds to the conformal field perturbation $u = a\,\delta\phi$, where the scale factor $a = \int H dt$. Perturbations obey the wave equation

$$u_k'' + (k^2 + \mu^2) u_k = 0, \tag{2}$$

which corresponds to an oscillator with a time dependent effective mass-squared

$$\mu^2 = -\frac{a''}{a},$$

where a prime denotes a derivative with respect to conformal time $\eta = \int dt/a$. During a conventional inflationary era μ^2 is negative and decreases monotonically, leading to the amplification of vacuum fluctuations. Modes on arbitrarily small scales ($k^2/|\mu^2| \to \infty$) are presumed to occupy the flat spacetime vacuum state ($u_k \to e^{-ik\eta}/\sqrt{2k}$). These vacuum fluctuations eventually lead to a specific spectrum of perturbations on large scales ($k^2/\mu^2 \to 0$), the form of which is determined solely by $\mu^2(\eta)$.

Consider the most commonly studied case of a power-law expansion [9], where the scale factor grows as $a \propto t^p$, which corresponds, in terms of the conformal time, to

$$a = a_0 \left(\frac{\eta}{\eta_0} \right)^{(1-2\nu)/2}, \tag{3}$$

where

$$\nu = \frac{3}{2} + \frac{1}{p-1}. \tag{4}$$

Note that $a \to \infty$ as $t \to \infty$ only coincides with the limit $\eta \to \infty$ for $p < 1$. During an inflationary expansion with $p > 1$ there is an event horizon and $\eta \to 0$ from below as $a \to \infty$.

The effective mass in Eq. (2) is

$$\mu^2 = -\frac{\nu^2 - 1/4}{\eta^2}. \tag{5}$$

Note that for $\nu = \pm 1/2$ the effective mass μ^2 vanishes and there is no particle production. This corresponds to a static universe ($p = 0$) or a spatially flat Friedmann-Robertson-Walker (FRW) radiation dominated universe ($p = 1/2$). During an inflationary expansion with $p>1$ and $\nu>3/2$, $\mu^2\to-\infty$ as $\eta\to 0^-$ which leads to fluctuations on scales larger than the horizon, $|k\eta|>1$. However, it has previously been noted [10] that a collapsing universe could in principle produce large scale perturbations from small scale quantum fluctuations. $|\eta|$ also represents an effective event horizon in a collapsing model with $p<1$ where $\eta\to 0$ from below as $a\to 0$. Any given comoving mode k gets pushed outside horizon as $|\eta|$ decreases and $k\eta\to 0$ as $t\to 0^-$ for $p<1$. For $|\nu|<1/2$, i.e., $0<p<1/2$, the effective mass-squared μ^2 is positive and becomes large as $\eta\to 0^-$ which strongly suppresses fluctuations on large scales ($|k\eta|\ll 1$).

The general solution of the wave equation (2) during a power-law expansion-contraction is [11]

$$u_k(\eta)=\sqrt{|k\eta|}[u_+H^{(1)}_{|\nu|}(|k\eta|)+u_-H^{(2)}_{|\nu|}(|k\eta|)], \qquad (6)$$

where $H^{(i)}_{|\nu|}$ are Hankel functions of order $|\nu|$. Choosing the quantum vacuum state at early times on small scales ($k\eta\to-\infty$) then determines the spectrum of perturbations on large scales ($k\eta\to 0$)

$$\mathcal{P}_u=\frac{C^2(|\nu|)k^2(-k\eta)^{1-2|\nu|}}{(2\pi)^2}, \qquad (7)$$

where the power spectrum is conventionally defined as $\mathcal{P}_u=k^3|u_k^2|/2\pi^2$, and the numerical coefficient

$$C(|\nu|)\equiv\frac{2^{|\nu|}\Gamma(|\nu|)}{2^{3/2}\Gamma(3/2)}. \qquad (8)$$

The spectrum of scalar field perturbations produced on large scales ($|k\eta|\gg 1$) can therefore be written as

$$\mathcal{P}_{\delta\phi}=\left(\frac{C(|\nu|)}{\nu-1/2}\right)^2\left(\frac{H}{2\pi}\right)^2(-k\eta)^{3-2|\nu|}. \qquad (9)$$

In the limit of de Sitter expansion in flat Friedmann-Robertson-Walker (FRW) spacetime, $p\to\infty$ and $\nu\to 3/2$, we recover the famous result $\mathcal{P}_{\delta\phi}=(H/2\pi)^2$ at horizon crossing ($k\eta=-1$) and the spectrum is independent of scale.

Notice, however, that the spectrum $\mathcal{P}_u$ given in Eq. (7) is invariant under the transformation $\nu\to\tilde{\nu}=-\nu$ or, equivalently,

$$p\to\tilde{p}=\frac{1-2p}{2-3p}. \qquad (10)$$

The perturbation spectrum produced during a power-law inflationary expansion with $p>1$ is indistinguishable from the spectrum produced during a power-law collapse $\tilde{p}<1$, where $\tilde{p}$ is given by Eq. (10). There are two fixed points where $\tilde{p}=p$. These occur where $p=1/3$ or $p=1$, which correspond to $\nu=0$ and $|\nu|\to\infty$ respectively.

Thus one obtains a scale invariant spectrum of perturbations not just for de Sitter inflation in flat FRW spacetime (where $p\to\infty$) but also for $p=2/3$, which corresponds to a collapsing dust-dominated FRW universe. This result is rather surprising at first sight since the scale invariance of the de Sitter spectrum can be understood as being due to the time-invariant nature of this solution, and a collapsing dust universe seems to be far from static. However, there is an important difference between the two cases. For $\nu>0$ (which includes conventional inflation models with $p>1$) the scalar field perturbations become frozen in on large scales as $H^2\propto\eta^{2\nu-3}$ in Eq. (9). Thus $\mathcal{P}_{\delta\phi}$ remains constant for any given mode k as $|k\eta|\to 0$. But for $\nu<0$, the perturbations grow outside the horizon with $\mathcal{P}_{\delta\phi}\propto\eta^{-4|\nu|}$ as $|k\eta|\to 0$. The amplitude of the perturbations as they cross outside the horizon ($|k\eta|=1$) grows as $H^2\propto\eta^{2\nu-3}$, and thus the amplitude of modes already outside the horizon grows at precisely the same rate for $\nu=-3/2$ and at any given time the spectrum is scale-invariant on large scales ($|k\eta|\ll 1$).

If one asks what is the most general cosmological evolution which will lead to an equivalent time-dependent mass for the perturbations and a scale-invariant spectrum of perturbations in massless fields, one obtains the simple solution

$$\tilde{a}(\eta)=C_1\left[\left(\frac{\eta}{\eta_1}\right)^{-1}+\left(\frac{\eta}{\eta_1}\right)^2\right], \qquad (11)$$

which describes a non-singular metric smoothly interpolating between a collapsing dust solution at early times ($\eta\to-\infty$) and an exponentially expanding de Sitter solution at late times ($\eta\to 0$).

One can go on to ask whether given any particular cosmological solution $a(\eta)$ one can write down the most general evolution $\tilde{a}(\eta)$ that would give rise to an equivalent time-dependent mass-squared, μ^2, and hence perturbation spectrum $\mathcal{P}_u$. The answer turns out to be that the same spectrum of perturbations on large scales will be produced by the two parameter family of solutions

$$a(\eta)\to\tilde{a}(\eta)=Ca(\eta)\int_{\eta_*}^{\eta}\frac{d\eta'}{a^2(\eta')}. \qquad (12)$$

The constant C describes an arbitrary rescaling of the whole metric which does not change the essential physics of the solutions, but the constant of integration η_* describes a one parameter family of different solutions.

For example, substituting in the power-law inflationary solutions given in Eq. (3) one obtains

$$\tilde{a}(\eta)=C_1\left(\frac{\eta}{\eta_1}\right)^{1/2}\left[\left(\frac{\eta}{\eta_1}\right)^{\nu}+\left(\frac{\eta}{\eta_1}\right)^{-\nu}\right]. \qquad (13)$$

Gravitational waves (transverse, traceless perturbations of the metric) in Einstein gravity obey the same wave equation as a minimally coupled massless scalar field [1] and hence the graviton spectrum is proportional to $\mathcal{P}_{\delta\phi}$. This is often assumed to give an unambiguous record of the evolution of the cosmological scale factor, $a(\eta)$. However an identical spectrum of gravitational waves will be produced by the two parameter family of solutions given in Eq. (12).

This invariance of cosmological perturbation spectra has already been noted in the context of superstring cosmology where the perturbation spectra of fields in the low energy string effective action may be invariant under symmetries of the action. In the pre big bang scenario [7] the graviton and dilaton fields are minimally coupled in the conformal Einstein frame where the metric evolves as $a\propto t^{1/3}$. $p=1/3$ is a fixed point of the transformation given in Eq. (10) and the graviton and dilaton spectra on large scales remain invariant under *T*-duality or *S*-duality transformations of the background model. However the axion-type fields are minimally coupled in the conformally related axion frames [8,12]. SL$(2,Z)$ *S*-duality transformations of the power-law vacuum solutions lead to a scale factor in the axion frame which evolves as given in Eq. (13) [8]. By constructing explicitly SL$(2,R)$ invariant perturbation variables it was shown that both the axion and dilaton spectra remained invariant under arbitrary SL$(2,R)$ transformations [8]. Equation (12) generalizes this result to arbitrary background solutions for $a(\eta)$, and to theories which may or may not have their origin in superstring theory.

The wave equation (2) for the perturbation u may be derived from an effective action

$$S=\frac{1}{2}\int d\eta\int d^3x\{u'^2-u_{,i}u_{,i}-\mu^2u^2\}, \qquad (14)$$

with the corresponding Hamiltonian

$$\mathcal{H}=\frac{1}{2}\int d^3x\{\pi_u^2+u_{,i}u_{,i}+\mu^2u^2\}, \qquad (15)$$

where the momentum canonically conjugate to u is $\pi_u=u'$. The action S and Hamiltonian $\mathcal{H}$ both remain invariant under the transformation given in Eq. (12) which leaves $u(\eta)$ and $\mu^2(\eta)$ invariant. It is interesting to compare this with a different invariance which has also recently been noted in the context of superstring cosmology [13], and applied to generalized cosmological perturbations [14]. This is an invariance of the effective action

$$\hat{S}=\frac{1}{2}\int d\eta\int d^3x a^2\{\delta\phi'^2-\delta\phi_{,i}\delta\phi_{,i}\}, \qquad (16)$$

and corresponding Hamiltonian

$$\hat{\mathcal{H}}=\frac{1}{2}\int d^3x\{a^{-2}\pi_{\delta\phi}^2+a^2\delta\phi_{,i}\delta\phi_{,i}\}, \qquad (17)$$

written in terms of the field perturbation $\delta\phi$ and its conjugate momentum $\pi_{\delta\phi}=a^2\delta\phi'$. The Lagrangian in Eq. (16) differs from that in Eq. (14) by a total derivative

$$\hat{S}=S-\frac{1}{2}\int d\eta\int d^3x\frac{d}{d\eta}\left(\frac{a'}{a}u^2\right), \qquad (18)$$

which does not affect the equation of motion, Eq. (1), but does change the Hamiltonian

$$\hat{\mathcal{H}}=\mathcal{H}+\int d^3x\left\{\frac{a''}{a}u^2-\frac{1}{2}\left(\frac{a'}{a}u^2\right)'\right\}. \qquad (19)$$

There is a duality invariance of the action $\hat{S}$ and Hamiltonian $\hat{\mathcal{H}}$ under which the ''pump field'' $a^2\to\tilde{a}^2=a^{-2}$ is inverted and the field perturbation $\delta\phi$ is exchanged with its canonical momentum $\pi_{\delta\phi}$ [14]. The Hamiltonian $\hat{\mathcal{H}}$ does not remain invariant under the transformation in Eq. (12), but neither does the Hamiltonian $\mathcal{H}$ remain invariant under the duality transformation in Ref. [14]. The effective action is only defined up to boundary terms and due to the explicit time-dependence of $a(\eta)$, the Hamiltonian is not uniquely defined. Both transformations, however, represent symmetries of the equation of motion.

The most cosmologically significant perturbation spectrum produced during an inflationary era in the early universe is likely to be the primordial spectrum of adiabatic density perturbations on large scales induced by the perturbations in the scalar field which drives inflation. To study the evolution of this field requires us to include the self-interaction potential of the field and the back-reaction of metric fluctuations. Fortunately Mukhanov [15] has shown that the wave equation for the gauge invariant field perturbation

$$u=a\left[\delta\phi+\dot{\phi}\frac{\psi}{H}\right], \qquad (20)$$

where ψ is the gauge-dependent curvature perturbation [16], can still be written in the form given in Eq. (2) but with a time-dependent mass-squared

$$\mu^2=-\frac{z''}{z}, \qquad (21)$$

where $z=a\dot{\phi}/H$. Quite generally we can write

$$z=a\sqrt{\frac{3\gamma}{8\pi G}}, \qquad (22)$$

where the effective barotropic index $\gamma\equiv\dot{\phi}^2/(V+\dot{\phi}^2/2)$. In the special case of power-law inflation driven by a scalar field with exponential potential, $\dot{\phi}\propto H$ and hence γ is a constant and we have $z\propto a$.

Starting from any known solution $z(\eta)$ we obtain the identical spectrum of perturbations $\mathcal{P}_u$ from the two parameter family of solutions

$$z(\eta)\to\tilde{z}(\eta)=Cz(\eta)\int_{\eta_*}^{\eta}\frac{d\eta'}{z^2(\eta')}, \qquad (23)$$

which leaves μ^2 given in Eq. (21) invariant.

The gauge-invariant curvature perturbation ζ [3,16,4] is related to the field perturbation u by

$$\zeta=\psi+\frac{H}{\dot{\phi}}\delta\phi=\frac{u}{z}. \qquad (24)$$

This is usually evaluated in terms of the quantities at horizon crossing. This is because ζ becomes constant on super-horizon scales for adiabatic perturbations. In this case the z

acquires an implicit scale dependence due to the different times at which different scales are evaluated. However one can also evaluate ζ at a fixed time, such as the end of inflation, in which case the scale dependence of ζ is due solely to the scale dependence of u, and z contributes a scale independent factor. Thus under the transformation given by Eq. (23) the curvature perturbation is rescaled by an overall factor $z/\tilde{z}$, but the spectral index

$$n \equiv 1 + \frac{d \ln \mathcal{P}_\zeta}{d \ln k}, \tag{25}$$

remains invariant.

For instance, it is well known that the extreme slow-roll limit of inflation corresponding de Sitter expansion driven by a massless scalar field, where $z \propto \eta^{-1}$, leads to a scale-invariant Harrison-Zel'dovich ($n=1$) spectrum of curvature perturbations. However substituting this familiar form for $z(\eta)$ into Eq. (23) yields the most general evolution which gives a scale-invariant spectrum as

$$\tilde{z}(\eta) = C_1 \left[\left(\frac{\eta}{\eta_1} \right)^{-1} + \left(\frac{\eta}{\eta_1} \right)^{2} \right]. \tag{26}$$

This shows that it is in fact possible to produce a scale-invariant spectrum of curvature perturbations from inflation that is far from the usual slow-roll limit.

Unfortunately it is not possible to uniquely determine the form of the self-interaction potential $V(\phi)$ for a given $z(\eta)$, such as that given in Eq. (26). For example, both power-law inflation [9] driven by an exponential potential, and natural inflation [17] where the potential energy remains effectively constant, can give rise to a power-law spectrum of curvature perturbations with $n=$constant [18]. However it is possible to test the consistency of the slow-roll approximation for a given $z(\eta)$. The slow-roll approximation requires that the effective barotropic index, γ in Eq. (22), is small and slowly varying, so that to zeroth order in the slow-roll parameters [5], the evolution of z is determined by the growth of the scale factor $a \sim \eta^{-1}$. This implies that $z''z/z'^2 \approx 2$. For the general form of $\tilde{z}(\eta)$ which yields a scale-invariant spectrum of curvature perturbations, given in Eq. (26), this slow-roll condition is badly broken at early times for $|\eta/\eta_1| \gg 1$.

Even in the slow-roll limit, the spectrum of curvature perturbations is sufficient only to determine the inflation potential up to a one parameter class of solutions [5]. The amplitude of the gravitational wave perturbations is then required to fix the actual amplitude of the inflation potential. In this paper I have demonstrated that if one allows behavior which may be far from the slow-roll limit there is a degeneracy even in the spectrum of gravitational wave perturbations. The general solution which yields an almost scale-invariant spectrum of gravitational waves interpolates between an initially collapsing universe and a quasi–de Sitter expansion. However, the asymptotic behavior at late times reproduces the usual slow-roll result, so in practice this need not be a serious limitation for reconstructing the evolution in the context of conventional inflation models [5].

On the other hand the transformation presented here suggests that it might be possible to use slow-roll techniques to analyze perturbations in models far from the usual slow-roll limit if they can be related to slow-roll models. An example of this is provided by solutions to the low energy string effective action where perturbation spectra in general axion-dilaton cosmologies can be related to much simpler dilaton-vacuum solutions by a duality transformation [8].

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