Reply to ''Comment on 'Hara's theorem in the constituent quark model' ''

V. Dmitrašinović*
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

(Received 27 April 1998; published 25 May 1999)

In the preceding Comment, it is alleged that a ''hidden loophole'' in the proof of Hara's theorem has been found, which purportedly invalidates the conclusions of the paper commented upon. I show that there is no such loophole in the constituent quark model, and that the ''counterexample'' presented in the Comment is not gauge invariant. [S0556-2821(99)05301-1]

PACS number(s): 11.30.Hv, 13.30.−a, 14.20.Jn

In the previous Comment [1] on my paper [2], it is pointed out that the ''hidden assumption'' of sufficiently localized current is tacitly made in my argument and then claimed that this assumption is not valid in the constituent quark model.

In the following, I show that: (a) this assumption is well known, (b) the mathematical formulation of this assumption as used in the Comment [1] is incorrect, (c) the correct form of this assumption is satisfied by the constituent quark model, and (d) the electromagnetic (EM) current conservation is violated in the constituent quark model calculation of Kamal and Riazuddin (KR) [3] which current is used as a ''counterexample'' in Ref. [1].

(a) That ''a sufficiently localized current'' is a well-known condition for the existence of the multipole expansion, can be seen in textbooks, see for example p. 54 of Ref. [4], [between Eqs. (7.23) and (7.24)], where it emerges from the demand that all surface terms vanish in integrations by parts leading to the multipole expansion of the EM Hamiltonian matrix element (ME)

$$H_{fi}=H_{fi}(\mathbf{q})=-\int d\mathbf{R}\;\hat{\boldsymbol{\varepsilon}}_M\cdot\mathbf{J}_{fi}(\mathbf{R})\;\exp(i\mathbf{q}\cdot\mathbf{R})$$
$$=\sum_{J=1}^{\infty}\sqrt{2\pi(2J+1)}\,i^J[\lambda\langle\Psi_f|\hat{T}^{\mathrm{mag}}_{JM}|\Psi_i\rangle+\langle\Psi_f|\hat{T}^{\mathrm{el}}_{JM}|\Psi_i\rangle],\tag{1}$$

where $M=\pm$ and $q=|\mathbf{q}|$. Equivalently, the assumption is that the EM Hamiltonian ME $H_{fi}(\mathbf{q})$ itself be a well-defined (proper) integral.

(b) The mathematical formulation of the ''implicit assumption'' that was offered in Ref. [1] reads

$$\mathbf{J}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+<R^{-3},\quad\text{as}\quad R=|\mathbf{R}|\to\infty.\tag{2}$$

Strictly speaking, this inequality is meaningless, since it compares an operator valued (q-number) left-hand side with an ordinary (c-number) function right-hand side. If we accept this inequality as a statement about matrix elements, then we can show that the conclusions drawn in Ref. [1] do *not* follow from it. Specifically, this inequality, although violated by the ''counterexample'' transverse current in Eq. (7) of Ref. [1], is actually *not* the source of the unusual threshold behavior in $\hat{T}^{\mathrm{el}}_{1M}$, Eqs. (8), (12) in Ref. [1]. Rather, the real ''culprit'' is the singular behavior of $\mathbf{J}(R)\cdot\hat{\boldsymbol{\varepsilon}}_+$ as $R\to 0$.

To prove this assertion, note that the EM Hamiltonian $H_{fi}(\mathbf{q})$, Eq. (1), with the ''counterexample'' EM current defined by Eq. (4) in Ref. [1], is infinite as its stands. That fact alone should have been enough to suggest that the subsequent conclusions would be questionable. To better define this integral, a ''regularization'' procedure was introduced into Eq. (4) of Ref. [1] in the form of a Gaussian [as a function of the new parameter ε] multiplying the integrand. But, rather than keeping the Gaussian regularization until the end of the calculation, it was removed too soon. This procedure led to the erroneous conclusions drawn in Ref. [1]. Specifically, the $\varepsilon\to 0$ limit is taken under the integral sign in Eq. (11) in Ref. [1], the relevant part being

$$\alpha=\lim_{\varepsilon\to 0}q\int_0^{\infty}dr\,j_1(qr)\,\mathrm{erf}\left(\frac{r}{2\sqrt{\varepsilon}}\right)=\int_0^{\infty}dz\,j_1(z).\tag{3}$$

A more careful calculation of the integral (3) leads to

$$\alpha=\left(\frac{2}{q}\right)\int_0^{\infty}dz\,j_0(z)\,\delta\left(\frac{z}{q}\right).\tag{4}$$

Note that the integral (4) receives its whole value from only one point—the lower integration bound $r=0$—and not from $r\to\infty$, as implied by inequality (2) and claimed in Ref. [1].

Manifestly, an object with a Dirac delta function singularity is localized. [This is not a proof that the physical EM current hyperon matrix element (ME) is localized—that will be checked in the next section]. So, currents of the type of Eq. (4) in Ref. [1], if they exist, are a new and (very) short-distance phenomenon. They must come from some high-energy extension of the standard model, since they do not exist in the Salam-Weinberg (SW) model.[1] For single-quark current operators, this can be seen from the relevant SW Feynman rules. It is alleged in Ref. [1] that the effective two-quark EM current operators (induced by the $W^{\pm}$,Z^0 exchange graphs) form such an ''abnormal'' current. We shall show in point (d) below that this claim is incorrect because

*Email address: dmitra@tor.physics.sc.edu

[1]The SW model determines the form of the electroweak currents in the constituent quark model.

the said current is *not* conserved. But, first we shall show that the EM current hyperon matrix element in the constituent quark model is sufficiently localized to have a normal threshold behavior.

(c) As already stated in point (a) above, the sufficient condition for localizability is the existence of the Fourier transform (FT) in Eq. (1), i.e., at least absolute-integrability $(\alpha=1)^2$ [5] of $\mathbf{J}_{fi}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+$:

$$\int d\mathbf{R}|\mathbf{J}_{fi}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+|^\alpha=\int d\mathbf{R}|\langle\Psi_f|\mathbf{J}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+|\Psi_i\rangle|^\alpha<\infty;$$

$$\alpha=1,2. \tag{5}$$

Hence we see that the hyperon wave functions $|\Psi_{i,f}\rangle$ play an important role in deciding localizability. The complete hyperon wave function $|\Psi\rangle$ factors into the center-of-mass (c.m.) plane wave and the internal (quark) wave function (WF) $|\Phi\rangle$

$$|\Psi_i(\mathbf{P}_i)\rangle\simeq\exp(i(\mathbf{P}_i\cdot\mathbf{X}-E_it))|\Phi(\boldsymbol{\rho},\boldsymbol{\lambda})\rangle, \tag{6}$$

where $(E_i,\mathbf{P}_i)$ is the initial-state hyperon four-momentum and $\boldsymbol{\rho},\boldsymbol{\lambda}$ are the three-body Jacobi coordinates describing the motion of the three constituent quarks in the hyperon relative to its c.m. co-ordinate $\mathbf{X}$. The internal wave functions $|\Phi_{i,f}\rangle$ are bound-state ones and therefore normalizable, whereas the c.m. plane waves produce a (nonsquare-integrable) momentum-conserving Dirac delta function in all momentum space matrix elements,

$$\begin{aligned}-H_{fi}(\mathbf{q})&=\int d\mathbf{R}\;\hat{\boldsymbol{\varepsilon}}_+\cdot\mathbf{J}_{fi}(R)\;\exp(i\mathbf{q}\cdot\mathbf{R})\\&=(2\pi)^3\delta(\mathbf{P}'-\mathbf{q}-\mathbf{P})\int d\mathbf{R}\;\langle\Phi_f|\mathbf{J}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+|\Phi_i\rangle\\&\quad\times\exp(i\mathbf{q}\cdot\mathbf{R}),\end{aligned} \tag{7}$$

which we systematically dropped from the displayed equations in our previous publications. Thus we see that the original integral in the criterion Eq. (5) is singular due to this trivial c.m. motion. It is the remaining (form) factor $\int d\mathbf{R}\;\langle\Phi_f|\mathbf{J}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+|\Phi_i\rangle\;\exp(i\mathbf{q}\cdot\mathbf{R})$, with ‘‘the c.m. motion taken out’’ that must be a well-defined Fourier transform:

$$\int d\mathbf{R}|\langle\Phi_f|\mathbf{J}(\mathbf{R})\cdot\hat{\boldsymbol{\varepsilon}}_+|\Phi_i\rangle|<\infty, \tag{8}$$

which is the final form of the localizablity criterion. The physical meaning of this requirement is clear: the hyperon probability distribution weighted by the EM current must be sufficiently close to its c.m. to yield a finite expectation value integral. This condition is satisfied by the hyperon in the constituent quark model because of the normalizability of its internal (quark) wave functions $|\Phi_{i,f}\rangle$; confinement makes the quark probability density only more localized and the integral faster converging. The EM current operator is at most a polynomial in the momenta (gradients) and spin operators which cannot overcome the exponential decay of the internal wave functions. Consequently, the hyperon EM current ME is localized in the constituent quark model, as advertised.

(d) Finally, we turn to the question of EM current conservation in the constituent quark model calculation of Kamal and Riazuddin [3], which is used as a ‘‘counterexample’’ in Ref. [1]. First note that there is full agreement between the KR paper [3] and Refs. [2,6] on the question of gauge invariance of the *covariant* amplitude described by the Feynman diagrams in Figs. 1, 2 of Ref. [3] (cf. Figs. 1, 2 in Ref. [6]). However, *this does not mean that the result of its nonrelativistic reduction is also gauge invariant*. The nonrelativistic reduction of the parity violating part of the Feynman amplitude leads KR to the current

$$\mathbf{J}\simeq(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2), \tag{9}$$

shown in Eq. (13) of Ref. [3]. The EM current (9) is only the two-body part of the complete EM current. By itself it is *not* conserved, see Eq. (15) in Ref. [2], or Eqs. (3.6)–(3.9) in Ref. [6]. To convince oneself explicitly of this fact, compare the EM current Eq. (9) with the corresponding manifestly conserved current,

$$\mathbf{J}\simeq\hat{\mathbf{q}}\times((\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\times\hat{\mathbf{q}})=(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)-\hat{\mathbf{q}}(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\hat{\mathbf{q}}. \tag{10}$$

One can see that the current (9) is just the first term on the right-hand side of Eq. (10)—there is no term proportional to the three-momentum transfer $\hat{\mathbf{q}}$. Consequently, the current (9) is *not* transverse to $\mathbf{q}$ in momentum space, i.e., *it is not conserved*. Since the two-body term constitutes the whole EM current in the KR calculation, we have proven our last contention: that the calculation in Ref. [3] is *not* gauge invariant.

Thus, we have shown that the objections raised in Ref. [1] are invalid.

The author would like to thank F. Myhrer and K. Kubodera for reading the manuscript and for valuable comments.

[2] Absolute integrability ensures the existence of the (direct) Fourier transform, but not that of the inverse one. To ensure the existence of both, one needs square-integrability $(\alpha=2)$.

[1] P. Żenczykowski, preceding comment, Phys. Rev. D **60**, 018901 (1999).
[2] V. Dmitrašinović, Phys. Rev. D **54**, 5899 (1996).
[3] A. N. Kamal and Riazuddin, Phys. Rev. D **28**, 2317 (1983).
[4] J. D. Walecka, *Theoretical Nuclear and Subnuclear Physics* (Oxford University Press, New York, 1995).
[5] A. Messiah, *Quantum Mechanics* (Wiley, New York, 1959), Appendix A.
[6] V. Dmitrašinović, Nucl. Phys. **A537**, 551 (1992).