Thirring Model in Terms of Currents: Solution and Light-Cone Expansions

G. F. Dell'Antonio
Istituto di Fisica Teorica dell'Universita, Napoli, Italy

and

Yitzhak Frishman*
Weizmann Institute, Rehovot, Israel

and

Daniel Zwanziger†
New York University, New York, New York 10012
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Exact expansions of operator products, in terms of c-number functions singular on the light cone and regular operators, are exhibited explicitly in the Thirring model. For the products $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$ of fermion fields the expansion reduces to one term only, with the c-number function having a singularity on the light cone which depends on the coupling constant, and the regular operator depending only on the currents, which are free. The resulting formula allows one to calculate all Wightman functions in terms of current matrix elements and thereby provides a simple and complete solution to the Thirring model. The different charge sectors are realized as inequivalent irreducible representation spaces of the canonical current commutation relations, on which the charged field ψ acts as an intertwining operator.

I. INTRODUCTION

Renewed interest in the Thirring model[1] has been occasioned by recent investigations of scale invariance and operator products at short distances.[2] In particular, the question of anomalous asymptotic dimensionality of local operators is nicely answered in that model, where the asymptotic dimensionality depends on the coupling constant, as demonstrated by Wilson[2,3] and Lowenstein[3] using the Johnson[4] and Klaiber[5] solutions, respectively.

The short-distance expansion of Wilson[2] was recently generalized to lightlike distances[6-8] (light-cone expansions). It is a very interesting problem to investigate the structure of possible light-cone expansions of products of operators in the Thirring model. We undertake this task in the present paper.

We show that light-cone expansions for products of fermion field operators exist in the Thirring model. More explicitly, we demonstrate that the products $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$ are equal to a product of a c-number function times a regular operator [see Eqs. (6.4)]. The c-number function has a singularity on the light cone, the strength of which depends on the coupling constant. The regular operator is written in terms of the currents only. Hence the light-cone expansion in the Thirring model for the above-mentioned products is extremely simple, containing one term only with one given singularity. Thus the behavior of all matrix elements of each of the above products are related. The most singular matrix element at short distances is the vacuum expectation value. All matrix elements have the same light-cone singularity.[9]

Our method differs from other treatments of the Thirring model[4] in that we do not start by defining the current in terms of the spinor fields, and we thus avoid all complications connected with separating the points in those fields by an infinitesimal amount. The properties of the currents are entirely determined by their conservation laws and by their commutation relations, namely, the Schwinger term.[10] Since the canonical commutation relations for the spinor field break down in the Thirring model,[4] there is an advantage in avoiding manipulations based on them. The commutation rules between the currents and the fields are those of Johnson.[4]

The structure of the operator products $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$ is determined in two ways. In one we use equations of motion as derived from commutation rules with the Hamiltonian and total momentum operators, given as integrals over a Sugawara-type[11,12] energy-momentum tensor written in terms of currents only. In the other we use a consistency condition, taking the products of four field operators in different orders. In both approaches, we use an irreducibility assumption for the currents, namely, that any operator which commutes with the currents is a function of the charges only.

Our principal result is a formula which expresses the products $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$, for any x and x', as a singular c-number function which multiplies a regular operator which is a function of the currents only. This formula provides a very simple solution of the Thirring model. For by expressing the product of two spinor fields in terms of the currents, the Wightman functions for the spinor fields are reduced to vacuum expectation values of operators depending on currents only. These are trivially obtained because the currents in the Thirring model are free fields. Our derivation shows that the Thirring model is determined by the dynamics of the currents and the commutation relations between the currents and the spinor fields; the equations of motion and commutation relations of the spinor fields are an algebraic consequence of the irreducibility assumption about the currents. The circle of reasoning back to the original formulation of the Thirring model, defined in terms of spinor fields only, is completed when, at the end, the currents are expressed in terms of limits of products of spinor fields.

Finally, we construct the Hilbert space on which the charged field ψ acts as a local field. Each charge sector is realized as the representation space for an irreducible representation of the commutation relations of the currents (which are canonical). Different values of the charges correspond to different inequivalent representations, only the vacuum sector being represented by the Fock representation. The charged field ψ is then constructed as an intertwining operator between the different charge sectors.

The program of the paper is as follows. In Sec. II we set up our equations of motion, using the Sugawara energy-momentum tensor, and all the relevant commutation rules. The Poincaré and dilatation generators are expressed in terms of currents, and the dimensionality of the spinor field is found by commutation with the dilatation generator. In Sec. III we obtain the operator expansion for the products $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$ in the two ways mentioned above. The strength of the singularities in those products agrees with the value of the dimension of the spinor field as calculated in Sec. II. In Sec. IV commutators between spinor fields are calculated and all Wightman functions are reduced to current matrix elements. In addition the singularities of the remaining products $\psi_\alpha(x)\psi_\beta^\dagger(x')$ and $\psi_\alpha^\dagger(x)\psi_\beta(x')$ are determined, in agreement with the results of Ref. 3 for the four-point functions. In Sec. V the Hilbert space is constructed on which j and ψ act as local fields. Finally in Sec. VI we summarize our results and discuss them with reference to short-distance and light-cone expansions.

II. EQUATIONS OF MOTION AND COMMUTATION RELATIONS

We have, in the Thirring model,[1] a vector current j_μ and an axial-vector current $j_{5\mu}$, the components of which are related by

$$j_{5\mu}=\epsilon_{\mu\nu}j^\nu, \tag{2.1}$$

with $\epsilon_{10}=-\epsilon_{01}=1$ and $\epsilon_{00}=\epsilon_{11}=0$. Both currents are conserved. The equation of motion for the spinor field ψ is

$$i\partial\!\!\!/\psi(x)=g j\!\!\!/(x)\psi(x), \tag{2.2}$$

where g is the coupling constant and $a\!\!\!/\equiv a^\mu\gamma_\mu$, with γ_μ the Dirac matrices.

The conservation of the vector and axial-vector currents follow from the definitions $j_\mu(x)=\overline{\psi}(x)\gamma_\mu\psi(x)$ and $j_{5\mu}(x)=\overline{\psi}(x)\gamma_\mu\gamma_5\psi(x)$ and Eq. (2.2). The relation (2.1) follows directly from this definition. However, for the purpose of further use of the equation of motion (2.2) Johnson had to separate the points of the two fields in the current such that $j_\mu(x)=\overline{\psi}(x+\epsilon)\gamma_\mu\psi(x-\epsilon)$, in order to be able to calculate Green's functions. The advantage of our method is that we do not look into the structure of the current in terms of the fields. The commutation rules containing the Schwinger term turn out to be sufficient to determine the structure of the operator products mentioned in the Introduction.

The commutation rules are

$$[j_0(x,t),j_0(y,t)]=0, \tag{2.3a}$$

$$[j_0(x,t),j_1(y,t)]=ic\delta'(x-y), \tag{2.3b}$$

$$[j_1(x,t),j_1(y,t)]=0. \tag{2.3c}$$

c is a positive number,[10] which fixes the normalization of j.

At this stage it is useful to introduce the following variables:

$$u=t+x, \tag{2.4a}$$

$$v=t-x, \tag{2.4b}$$

and

$$j_+(u,v)=j_0(u,v)+j_1(u,v), \tag{2.5a}$$

$$j_-(u,v)=j_0(u,v)-j_1(u,v). \tag{2.5b}$$

Hence

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v},$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}-\frac{\partial}{\partial v},$$

and thus

$$\partial^\mu j_\mu=\tfrac{1}{2}[\partial_u j_-(u,v)+\partial_v j_+(u,v)]=0, \tag{2.6a}$$

$$\epsilon^{\mu\nu}\partial_\mu j_\nu=\tfrac{1}{2}[-\partial_u j_-(u,v)+\partial_v j_+(u,v)]=0. \tag{2.6b}$$

We conclude, therefore,

$$j_+\equiv j_+(u), \tag{2.7a}$$

$$j_-\equiv j_-(v). \tag{2.7b}$$

Combining this result with the commutation rules Eqs. (2.3a)–(2.3c), one obtains

$$[j_+(u),j_+(u')]=2ic\delta'(u-u'), \tag{2.8a}$$

$$[j_-(v),j_-(v')]=2ic\delta'(v-v'), \tag{2.8b}$$

$$[j_+(u),j_-(v)]=0. \tag{2.8c}$$

Following Johnson,[4] we postulate the following commutation rules between the current and the spinor field:

$$[j_0(x,t),\psi(y,t)]=-a\psi(y,t)\delta(x-y), \tag{2.9a}$$

$$[j_{50}(x,t),\psi(y,t)]=[j_1(x,t),\psi(y,t)]=-\bar{a}\gamma_5\psi(y,t)\delta(x-y), \tag{2.9b}$$

where a and $\bar{a}$ are yet to be determined. For the free spinor field with $g=0$, we know that $a=\bar{a}=1$. Combining these commutation relations with Eq. (2.7a)–(2.7b), one immediately gets

$$[j_+(u),\psi(u'v')]=-(a+\bar{a}\gamma_5)\psi(u'v')\delta(u-u'), \tag{2.10a}$$

$$[j_-(v),\psi(u'v')]=-(a-\bar{a}\gamma_5)\psi(u'v')\delta(v-v'). \tag{2.10b}$$

Note that the current j_μ satisfies the Klein-Gordon equation with zero mass, $\Box j_\mu=0$ [this is immediate from Eqs. (2.7a)–(2.7b), since $\Box=4\partial_u\partial_v$]. Hence we can decompose $j_\pm$ into positive and negative frequencies in an invariant manner. We can thus define normal ordering for products of currents. We also define a normal ordering for a product of a current with a spinor field by

$$:j_\mu(x)\psi(x):=j_\mu^{(+)}(x)\psi(x)+\psi(x)j_\mu^{(-)}(x), \tag{2.11}$$

where $j_\mu^{(+)}(x)$ is the part that contains the creation operators (namely, positive frequencies) and $j_\mu^{(-)}(x)$ that which contains the annihilation operators (negative frequencies). Following Klaiber,[5] we redefine Eq. (2.2) to read

$$i\partial\!\!\!/\psi(x)=g:j\!\!\!/(x)\psi(x):. \tag{2.12}$$

As we shall see later, the last equation is well defined, singular factors being taken care of by the normal ordering.

Let us now choose the following set of γ matrices:

$$\gamma_0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma_5=\gamma_0\gamma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

This is the set in which γ_5 is diagonal. In this basis the conserved charges

$$Q_+=\int j_+(u)du,\quad Q_-=\int j_-(v)dv$$

are shifted by the spinor field ψ according to

$$Q_\pm\psi_1=\psi_1[Q_\pm-(a\pm\bar{a})], \tag{2.13a}$$

$$Q_\pm\psi_2=\psi_2[Q_\pm-(a\mp\bar{a})], \tag{2.13b}$$

and Eq. (2.12) reads

$$\partial_u\psi_2(u,v)=-\tfrac{1}{2}ig:j_+(u)\psi_2(u,v):, \tag{2.14a}$$

$$\partial_v\psi_1(u,v)=-\tfrac{1}{2}ig:j_-(v)\psi_1(u,v):. \tag{2.14b}$$

Note that under a Lorentz transformation given by a "boost" angle θ such that $\tanh\theta=\beta$, the velocity, we have

$$u\to e^{-\theta}u,\quad v\to e^{\theta}v, \tag{2.15}$$

and

$$j_+(u)\to e^{\theta}j_+(e^{\theta}u),\quad j_-(v)\to e^{-\theta}j_-(e^{-\theta}v), \tag{2.16}$$

and also

$$\psi(u,v)\to e^{+\gamma_5\theta/2}\psi(e^{\theta}u,e^{-\theta}v). \tag{2.17}$$

Thus the quantities $j_+(u)$, $j_-(v)$, $\psi_1(u,v)$, and $\psi_2(u,v)$ transform into themselves under a Lorentz transformation. For ψ_1 and ψ_2 this is achieved in the basis in which γ_5 is diagonal.

We should emphasize that the equations of motion (2.14a) and (2.14b) admit any transformation law for ψ, even that of a scalar field [a mass term would determine the law to be (2.17)]. In the present work we adopt the transformation law (2.17), which is the usual one for a spinor field. We also know that the anticommutation relations determine the transformation law. However, we are not going to assume any canonical rules, since these are eventually going to be broken down. Also, we do not start by defining the current as a bilinear product in the Fermi fields with all the complications of point separations needed to get the Schwinger term[10] and for solving for the Green's functions.[4] This is all taken care of by the commutation rules (2.8) and (2.10) and the normal ordering in the equations of motion (2.12). The expressions for the currents as limits of bilinear products of spinor fields are in fact obtained by us in Sec. IV and result from the operator expansion.

The equations of motion (2.6) and the commutation relations (2.3) or (2.8) define a quantum dynamical system for the j's. For this system we introduce Poincaré generators of time translation H, space translation P, Lorentz boost M, and the dilatation generator D, with nonzero commutators

$$i[M,H]=P,\quad i[M,P]=H, \tag{2.18a}$$

$$i[D,H]=H,\quad i[D,P]=P, \tag{2.18b}$$

and defined by

$$i[H,j_+(u)]=i[P,j_+(u)]=\partial_u j_+(u), \tag{2.19a}$$

$$i[H,j_-(v)]=-i[P,j_-(v)]=\partial_v j_-(v), \tag{2.19b}$$

$$i[D,j_+(u)]=+i[M,j_+(u)]=(u\partial_u+1)j_+(u), \tag{2.19c}$$

$$i[D,j_-(v)]=-i[M,j_-(v)]=(v\partial_v+1)j_-(v). \tag{2.19d}$$

These are the commutators for a vector field of canonical dimension. The generators may be obtained from an energy-momentum tensor of Sugawara form,[11,12] built out of currents only,

$$\Theta_{\mu\nu}=\frac{1}{2c}:2j_\mu j_\nu-g_{\mu\nu}j_\alpha j^\alpha:, \tag{2.20}$$

which is divergenceless, symmetric, and traceless, so

$$\Theta_{00}+\Theta_{01}=\Theta_{11}+\Theta_{10}=\frac{1}{2c}:j_+{}^2(u):, \tag{2.21a}$$

$$\Theta_{00}-\Theta_{01}=\Theta_{11}-\Theta_{10}=\frac{1}{2c}:j_-{}^2(v):. \tag{2.21b}$$

For one may verify that with

$$H+P=\frac{1}{2c}\int :j_+{}^2(u):du, \tag{2.22a}$$

$$H-P=\frac{1}{2c}\int :j_-{}^2(v):dv, \tag{2.22b}$$

$$D+M=\frac{1}{2c}\int u:j_+{}^2(u):du, \tag{2.22c}$$

$$D-M=\frac{1}{2c}\int v:j_-{}^2(v):dv, \tag{2.22d}$$

the defining commutation relations (2.19) are satisfied. Because $u=t+x$ and $v=t-x$, these integrals may, of course, be evaluated at fixed time and are time-independent. In Sec. V it will be verified that these operators do exist on the Hilbert space constructed there.

We now use the irreducibility assumption about the currents which says that anything which commutes with them is a function of the charges only. Thus the Poincaré-dilatation generators satisfying (2.19) can differ from expressions (2.22) only by an additive Hermitian function of the charges. However, H and P appear on the right-hand side of (2.18) and hence they are uniquely given by (2.22). In addition, the indeterminacy in M and D merely corresponds to the possibility of adding the generator of a gauge transformation to M and D. Thus we may take Eqs. (2.22) to be true without any loss of generality.

The equations of motion and transformation laws of the charged field are now found to be determined by the commutators of ψ with j, the only assumptions needed for this result being the existence of the generators and the irreducibility of the j's. Let us calculate the commutator of $\Theta_{\mu\nu}$ with ψ. According to Eq. (2.21) we need only

$$\left[\frac{1}{2c}:j_+{}^2(u):,\psi(u',v')\right]=-\frac{a+\bar{a}\gamma^5}{c}:j_+(u)\psi(u',v'):\delta(u-u')+\frac{i(a+\bar{a}\gamma^5)^2}{4\pi c}\psi(u',v')\delta'(u-u'), \tag{2.23a}$$

$$\left[\frac{1}{2c}:j_-{}^2(v):,\psi(u',v')\right]=-\frac{a-\bar{a}\gamma^5}{c}:j_-(v)\psi(u',v):\delta(v-v')+\frac{i(a-\bar{a}\gamma^5)^2}{4\pi c}\psi(u',v')\delta'(v-v'). \tag{2.23b}$$

This result follows from a straightforward use of our normal ordering, whereas elsewhere[13] limiting procedures are employed. In deriving (2.23) we have used the fact that Eqs. (2.10) are valid for positive and negative frequencies in u and v separately and that

$$-[\delta^{(+)}(x)]^2+[\delta^{(-)}(x)]^2=\frac{i}{2\pi}\delta'(x). \tag{2.24}$$

The positive and negative frequencies are defined by

$$A^{(\pm)}(\lambda)=\frac{1}{2\pi}\int_0^\infty dp\int_{-\infty}^\infty d\bar{\lambda}\,e^{\pm ip(\lambda-\bar{\lambda})}A(\bar{\lambda}),$$

where A can be either $j_+(u)$ or $j_-(v)$. Thus

$$\delta^{(\pm)}(x)=\frac{\pm i}{2\pi}\frac{1}{x\pm i\epsilon},\qquad \frac{d}{dx}\delta^{(\pm)}(x)=\frac{\mp i}{2\pi}\frac{1}{(x\pm i\epsilon)^2}, \tag{2.25}$$

$$[\delta^{(\pm)}(x)]^2=\frac{\mp i}{2\pi}\frac{d}{dx}\delta^{(\pm)}(x). \tag{2.26}$$

The space-time translations for the charged field ψ now follow from

$$\partial_u\psi(u,v)=\tfrac{1}{2}i[H+P,\psi(u,v)], \tag{2.27a}$$

$$\partial_v \psi(u,v) = \tfrac{1}{2} i[H-P, \psi(u,v)], \qquad (2.27b)$$

which yields

$$\partial_u \psi(u,v) = \frac{-i(a+\bar{a}\gamma^5)}{2c} :j_+(u)\psi(u,v):, \qquad (2.28a)$$

$$\partial_v \psi(u,v) = \frac{-i(a-\bar{a}\gamma^5)}{2c} :j_-(v)\psi(u,v):. \qquad (2.28b)$$

These equations are consistent with the original Eqs. (2.14) for

$$a - \bar{a} = gc. \qquad (2.29)$$

However, the equations for $\partial_u \psi_1$ and $\partial_v \psi_2$ are extra.[14] They provide additional information which we put in instead of employing the method of point separation in the equations of motion.

This can be nicely demonstrated for the case of free fields, where $g=0$, $a=\bar{a}=1$. In this case the Dirac equation takes then the form

$$\partial_u \psi_2 = 0,$$

$$\partial_v \psi_1 = 0,$$

and thus $\psi_1 \equiv \psi_1(u)$, $\psi_2 \equiv \psi_2(v)$. The extra equations are

$$\partial_u \psi_1(u) = -\frac{i}{c} :j_+(u)\psi_1(u):,$$

$$\partial_v \psi_2(v) = -\frac{i}{c} :j_-(v)\psi_2(v):,$$

with[15] $j_+(u) = 2 :\psi_1^\dagger(u)\psi_1(u):$, $j_-(v) = 2 :\psi_2^\dagger(v)\psi_2(v):$. Let us take the equation for $\partial_u \psi_1(u)$. By the method of separation of points, we have for the right-hand side,[15]

$$-\frac{i}{c} :j_+(u+\epsilon)\psi_1(u):$$

$$= -\frac{2i}{c} :\psi_1^\dagger(u+\epsilon)\psi_1(u+\epsilon)\psi_1(u):$$

$$= -\frac{2i}{c} :\psi_1(u)\psi_1^\dagger(u+\epsilon)\psi_1(u+\epsilon):$$

$$\approx -\frac{2i}{c} :\psi_1(u)\psi_1^\dagger(u+\epsilon)[\psi_1(u) + \epsilon\partial_u \psi_1(u)]:$$

$$= -\frac{2i}{c} :\psi_1(u)\psi_1^\dagger(u+\epsilon)\epsilon\partial_u \psi_1(u):$$

$$= -\frac{2i}{c}\left[\lim_{\epsilon\to 0} \epsilon\langle 0|\psi_1(u)\psi_1^\dagger(u+\epsilon)|0\rangle\right]\partial_u \psi_1(u)$$

$$= \frac{1}{\pi c}\partial_u \psi_1(u).$$

Thus in the free case the apparently new equations become an identity, provided $\pi c = 1$.

Let us now look at the Lorentz transformation and dilatation properties of ψ. By integrating Eqs. (2.23) with u and v and using Eq. (2.28), we get

$$\tfrac{1}{2} i[D+M, \psi(u,v)] = u\partial_u \psi(u,v) + \frac{(a+\bar{a}\gamma^5)^2}{8\pi c}\psi(u,v), \qquad (2.30a)$$

$$\tfrac{1}{2} i[D-M, \psi(u,v)] = v\partial_v \psi(u,v) + \frac{(a-\bar{a}\gamma^5)^2}{8\pi c}\psi(u,v). \qquad (2.30b)$$

So we obtain the important Lorentz transformation law for ψ,

$$i[M, \psi(u,v)] = \left(u\partial_u - v\partial_v + \frac{a\bar{a}}{2\pi c}\gamma^5\right)\psi(u,v), \qquad (2.31)$$

and we see that ψ will transform like a spinor only if

$$\frac{a\bar{a}}{\pi c} = \frac{(a+\bar{a})^2 - (a-\bar{a})^2}{4\pi c} = 1. \qquad (2.32)$$

This relation and Eq. (2.29) fix a and $\bar{a}$ in terms of the coupling constant g and the normalization constant c. Note that the Lorentz transformation properties of ψ are not determined by the equations of motion (2.28) and have to be introduced as an outside requirement. In fact if the right-hand side of the last equation were 2, ψ would be a vector, and so on.

Finally we observe that

$$i[D, \psi(u,v)] = \left(u\partial_u + v\partial_v + \frac{a^2+\bar{a}^2}{4\pi c}\right)\psi(u,v), \qquad (2.33)$$

so ψ has the anomalous dimension

$$d = \frac{a^2+\bar{a}^2}{4\pi c}$$

$$= \frac{(a+\bar{a})^2 + (a-\bar{a})^2}{8\pi c}, \qquad (2.34)$$

or by Eqs. (2.29) and (2.32)

$$d = \frac{1}{2} + \frac{g^2 c}{4\pi}. \qquad (2.35)$$

We see that the dimension is changed from the canonical dimension $\frac{1}{2}$ by the coupling strength $g^2c/4\pi$. In Sec. III the same anomalous dimension will appear in the operator product of $\psi(x)$ and $\psi^\dagger(x')$. We shall obtain operator products in two ways. In one of them we shall not use any equation of motion, but only a consistency condition on the product of four fields.

III. PRODUCTS OF SPINOR FIELDS IN TERMS OF CURRENTS

Let us first start with the Eqs. (2.28a) and (2.28b). They can be integrated to read

$$\psi(u,v) = :\exp\left(-\frac{i}{2c}(a+\bar{a}\gamma_5)\int_{u_0}^{u} j_+(u')du'\right)\exp\left(-\frac{i}{2c}(a-\bar{a}\gamma_5)\int_{v_0}^{v} j_-(v')dv'\right)\psi(u_0v_0):. \tag{3.1}$$

We are thus motivated to try, for the product $\psi_1(u,v)\psi_1^\dagger(u',v')$, an expression of the form

$$\psi_1(u,v)\psi_1^\dagger(u',v') = \tilde{F}(u,u',v,v'):\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u'}^{u} j_+(\xi)d\xi\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v'}^{v} j_-(\eta)d\eta\right):,$$

where hopefully $\tilde{F}$ is a c-number function dependent on the differences $u-u'$, $v-v'$. Let us therefore consider the expression

$$F = \exp\left(\frac{i}{2c}(a-\bar{a})\int_{v'}^{v} j_-^{(+)}(\eta)d\eta\right)\exp\left(\frac{i}{2c}(a+\bar{a})\int_{u'}^{u} j_+^{(+)}(\xi)d\xi\right)$$
$$\times\psi_1(u,v)\psi_1^\dagger(u',v')\exp\left(\frac{i}{2c}(a-\bar{a})\int_{v'}^{v} j_-^{(-)}(\eta)d\eta\right)\exp\left(\frac{i}{2c}(a+\bar{a})\int_{u'}^{u} j_+^{(-)}(\xi)d\xi\right). \tag{3.2}$$

It can be shown that F commutes with $j_+(u)$ and $j_-(v)$. One uses

$$[j_+^{(-)}(\xi), j_+^{(+)}(\xi')] = [j_-^{(-)}(\xi), j_-^{(+)}(\xi')] = -\frac{c}{\pi}\frac{1}{(\xi-\xi'-i\epsilon)^2}, \tag{3.3}$$

other commutators of the j's with themselves being zero. We conclude that F is a function of the charges only. Observe that by using the commutation relation (3.14) given below and their analogs for the annihilation parts of the currents given after Eq. (4.4), the quantity F, defined by Eq. (3.2) commutes with any ψ spacelike to both (u,v) and $(u'v')$. Therefore F must in fact be independent of the charges, so it is a c-number function of the differences $u-u'$, $v-v'$, $F=F(u-u',v-v')$. From Eq. (3.2), we thus obtain

$$\psi_1(u,v)\psi_1^\dagger(u',v') = F(u-u',v-v'):\exp\left[\frac{i}{2c}\left((a+\bar{a})\int_u^{u'} j_+(\xi)d\xi + (a-\bar{a})\int_v^{v'} j_-(\eta)d\eta\right)\right]:. \tag{3.4}$$

Our task is now to determine the function $F(u-u',v-v')$.

One way to do that is to use the equations of motion (2.28a) and (2.28b) directly on Eq. (3.2). We get

$$\partial_u F(u-u',v-v') = -\frac{(a+\bar{a})^2}{4\pi c(u-u'-i\epsilon)}F(u-u',v-v'), \tag{3.5a}$$

$$\partial_v F(u-u',v-v') = -\frac{(a-\bar{a})^2}{4\pi c(v-v'-i\epsilon)}F(u-u',v-v'), \tag{3.5b}$$

where

$$[\psi_1^\dagger(u',v'), j_+^{(-)}(u)] = \frac{i}{2\pi}\frac{a+\bar{a}}{u-u'-i\epsilon}\psi_1^\dagger(u'v'), \tag{3.6a}$$

$$[\psi_1^\dagger(u',v'), j_-^{(-)}(v)] = \frac{i}{2\pi}\frac{a-\bar{a}}{v-v'-i\epsilon}\psi_1^\dagger(u'v') \tag{3.6b}$$

have been used. Equations (3.5a) and (3.5b) can be integrated to give

$$F(u,v) = F_0(u-i\epsilon)^{-(a+\bar{a})^2/4\pi c}(v-i\epsilon)^{-(a-\bar{a})^2/4\pi c}. \tag{3.7}$$

The phase of $F(u,v)$ is determined by $F^*(u,v)=F(-u,-v)$ which is immediate from Eq. (3.4). Using also the positivity of $\psi(g)\psi^\dagger(g)$, with g a test function we get

$$F(u,v) = f_0(iu+\epsilon)^{-(a+\bar{a})^2/4\pi c}(iv+\epsilon)^{-(a-\bar{a})^2/4\pi c} \tag{3.8}$$

with f_0 positive. By definition the factors $(iu+\epsilon)$ and $(iv+\epsilon)$ have phases between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Another way of finding $F(u,v)$ is to look into the product

$$\psi_1(u_1,v_1)\psi_1^\dagger(u_2,v_2)\psi_1(u_3,v_3)\psi_1^\dagger(u_4,v_4) \tag{3.9}$$

and to use Eq. (3.4) once for the products $\psi_1(u_1,v_1)\psi_1^\dagger(u_2,v_2)$ and $\psi_1(u_3,v_3)\psi_1^\dagger(u_4,v_4)$ and once for $\psi_1^\dagger(u_2,v_2)\psi_1(u_3,v_3)$ [see (3.10) below] and then for $\psi_1(u_1,v_1)\psi_1^\dagger(u_4,v_4)$. A comparison between the two expressions yields the result (3.7) for F. We also get, with this method, a determination of $G(u,v)$ which appears in

$$\psi_1^\dagger(u',v')\psi_1(u,v)=G(u'-u,v'-v):\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u'}^{u} j_+(\xi)d\xi\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v'}^{v} j_-(\eta)d\eta\right):, \tag{3.10}$$

an equation which is derived in the same way as Eq. (3.4). Before we can compare the two expressions we have to commute several terms. Let us outline the calculation. We have, on the one hand,

$$\begin{aligned}
&[\psi_1(u_1,v_1)\psi_1^\dagger(u_2,v_2)][\psi_1(u_3,v_3)\psi_1^\dagger(u_4,v_4)]\\
&\quad=F(u_1-u_2,v_1-v_2):\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_1} j_+(\xi)d\xi\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_1} j_-(\eta)d\eta\right):\\
&\qquad\times F(u_3-u_4,v_3-v_4):\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_4}^{u_3} j_+(\xi')d\xi'\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_4}^{v_3} j_-(\eta')d\eta'\right):\\
&\quad=F(u_1-u_2,v_1-v_2)F(u_3-u_4,v_3-v_4)\left(\frac{(u_1-u_3-i\epsilon)(u_2-u_4-i\epsilon)}{(u_1-u_4-i\epsilon)(u_2-u_3-i\epsilon)}\right)^{(a+\bar{a})^2/4\pi c}\left(\frac{(v_1-v_3-i\epsilon)(v_2-v_4-i\epsilon)}{(v_1-v_4-i\epsilon)(v_2-v_3-i\epsilon)}\right)^{(a-\bar{a})^2/4\pi c}\\
&\qquad\times:\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_1} j_+(\xi)d\xi\right)\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_4}^{u_3} j_+(\xi')d\xi'\right)\\
&\qquad\times\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_1} j_-(\eta)d\eta\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_4}^{v_3} j_-(\eta')d\eta'\right):,
\end{aligned}\tag{3.11}$$

where we have used the following identity:

$$\begin{aligned}
&\exp\left(-i\mu\int_{\bar{u}}^{\bar{u}'} j^{(-)}(\xi)d\xi\right)\exp\left(-i\lambda\int_{u}^{u'} j^{(+)}(\xi')d\xi'\right)\\
&\qquad=\exp\left(-i\lambda\int_{u}^{u'} j^{(+)}(\xi')d\xi'\right)\exp\left(-i\mu\int_{\bar{u}}^{\bar{u}'} j^{(-)}(\xi)d\xi\right)\left(\frac{(\bar{u}'-u'-i\epsilon)(\bar{u}-u-i\epsilon)}{(\bar{u}'-u-i\epsilon)(\bar{u}-u'-i\epsilon)}\right)^{\lambda\mu c/\pi}
\end{aligned}\tag{3.12}$$

which holds for either j_+ or j_-. On the other hand,

$$\begin{aligned}
\psi_1(u_1,v_1)[\psi_1^\dagger(u_2,v_2)\psi_1(u_3,v_3)]\psi_1^\dagger(u_4,v_4)&=G(u_2-u_3,v_2-v_3)F(u_1-u_4,v_1-v_4)\\
&\quad\times\left(\frac{(u_1-u_3-i\epsilon)(u_2-u_4-i\epsilon)}{(u_1-u_2-i\epsilon)(u_3-u_4-i\epsilon)}\right)^{(a+\bar{a})^2/4\pi c}\left(\frac{(v_1-v_3-i\epsilon)(v_2-v_4-i\epsilon)}{(v_1-v_2-i\epsilon)(v_3-v_4-i\epsilon)}\right)^{(a-\bar{a})^2/4\pi c}\\
&\quad\times:\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_3} j_+(\xi)d\xi\right)\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_4}^{u_1} j_+(\xi')d\xi'\right)\\
&\quad\times\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_3} j_-(\eta)d\eta\right)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_4}^{v_1} j_-(\eta')d\eta'\right):,
\end{aligned}\tag{3.13}$$

where we have used

$$\begin{aligned}
&\psi_1(u_1,v_1)\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_3} j_+^{(+)}(\xi)d\xi\right)=\left[\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_3} j_+^{(+)}(\xi)d\xi\right)\psi_1(u_1,v_1)\right]\left(\frac{u_1-u_3-i\epsilon}{u_1-u_2-i\epsilon}\right)^{(a+\bar{a})^2/4\pi c},\\
&\psi_1(u,v)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_3} j_-^{(+)}(\eta)d\eta\right)=\left[\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_3} j_-^{(+)}(\eta)d\eta\right)\psi_1(u,v)\right]\left(\frac{v_1-v_3-i\epsilon}{v_1-v_2-i\epsilon}\right)^{(a-\bar{a})^2/4\pi c}
\end{aligned}\tag{3.14}$$

and the equations for $\psi_1^\dagger$ obtained by Hermitian conjugation of these. Comparing Eqs. (3.11) and (3.13) we see that the operator parts are identical, since

$$\int_{u_2}^{u_1}+\int_{u_4}^{u_3}=\int_{u_2}^{u_3}+\int_{u_4}^{u_1}.$$

Comparing the c-number parts we get, for

$$f(u,v)\equiv F(u,v)(u-i\epsilon)^{(a+\bar{a})^2/4\pi c}(v-i\epsilon)^{(a-\bar{a})^2/4\pi c},$$
$$g(u,v)\equiv G(u,v)(u-i\epsilon)^{(a+\bar{a})^2/4\pi c}(v-i\epsilon)^{(a-\bar{a})^2/4\pi c}, \tag{3.15}$$

the following equation:

$$f(u,v)f(u',v')=g(u'',v'')f(u+u'+u'',v+v'+v''). \tag{3.16}$$

The solution of this equation is

$$f(u,v)=g(-u,-v)$$
$$=c_1e^{c_2u+c_3v},$$

as one can easily convince oneself by taking a logarithm of both sides of Eq. (3.16) and then differentiating with respect to u and v. c_1, c_2, and c_3 are constants. From Lorentz covariance of the expression in Eq. (3.4) one deduces that $c_2=c_3=0$, and thus the solution (3.7) is obtained. We also get, as a side result, the solution for G,

$$G(u,v)=F(u,v). \tag{3.17}$$

IV. WIGHTMAN FUNCTIONS, COMMUTATORS, AND BILOCAL OPERATORS

In Sec. III we obtained our basic result, namely, the formulas that express the product of two spinor fields at arbitrary points in terms of currents,

$$\psi_1(u,v)\psi_1^\dagger(u',v')=f_0[i(u-u')+\epsilon]^{-(a+\bar{a})^2/4\pi c}[i(v-v')+\epsilon]^{-(a-\bar{a})^2/4\pi c}$$
$$\times:\exp\left[-\frac{i}{2c}\left((a+\bar{a})\int_{u'}^{u}j_+(\xi)d\xi+(a-\bar{a})\int_{v'}^{v}j_-(\eta)d\eta\right)\right]:, \tag{4.1a}$$

$$\psi_1^\dagger(u',v')\psi_1(u,v)=f_0[i(u'-u)+\epsilon]^{-(a+\bar{a})^2/4\pi c}[i(v-v')+\epsilon]^{-(a-\bar{a})^2/4\pi c}$$
$$\times:\exp\left[-\frac{i}{2c}\left((a+\bar{a})\int_{u'}^{u}j_+(\xi)d\xi+(a-\bar{a})\int_{v'}^{v}j_-(\eta)d\eta\right)\right]:, \tag{4.1b}$$

and the corresponding relation for ψ_2 obtained by $\bar{a}\to-\bar{a}$. Here f_0 is a positive real number whose value determines the normalization of ψ_1. We saw that these formulas follow algebraically from the commutation relations of the j's with the ψ's, and that the equations of motion of the ψ's are not needed to derive them. On the other hand, by differentiating them with respect to u and v, the equations of motion (2.28) and their integral form (3.1) result. They also provide a complete solution of the Thirring model. For once a representation of the j's acting in the vacuum sector is chosen, Eqs. (4.1) determine all the Wightman functions, as shown below.

We choose the Fock space representation of the commutation relations (2.8), with vacuum $|0\rangle$,

$$j_\pm^{(-)}(\eta)|0\rangle=0. \tag{4.2}$$

In terms of Fourier components, $\tilde{j}_\pm(p)$, with $\tilde{j}_\pm(-p)=\tilde{j}_\pm^\dagger(p)$, $p>0$, we have[16]

$$j_\pm(\eta)=\int_0^\infty\frac{dp}{(2\pi)^{1/2}}[\tilde{j}_\pm(p)e^{-ip\eta}+\tilde{j}_\pm^\dagger(p)e^{ip\eta}],$$
$$[\tilde{j}_\pm(p),\tilde{j}_\pm^\dagger(p')]=2cp\,\delta(p-p'),\quad p,p'>0 \tag{4.3}$$

other commutators being zero. Because of the charge selection rules, all nonvanishing Wightman functions contain equal numbers of ψ_1's and $\psi_1^\dagger$'s (and ψ_2's and $\psi_2^\dagger$'s), so all that remains to be done is to find out how to commute the spinor fields to group them in pairs like (4.1).

From Eqs. (4.1) we have directly

$$\psi_1(u_1,v_1)\psi_1^\dagger(u_2,v_2)=\left(\frac{i(u_2-u_1)+\epsilon}{i(u_1-u_2)+\epsilon}\right)^{(a+\bar{a})^2/4\pi c}\left(\frac{i(v_2-v_1)+\epsilon}{i(v_1-v_2)+\epsilon}\right)^{(a-\bar{a})^2/4\pi c}\psi_1^\dagger(u_2,v_2)\psi_1(u_1,v_1). \tag{4.4}$$

Using Eqs. (4.1a) and (3.17) and equations analogous to (3.17) with the annihilation part of the currents instead of creation

$$\psi_1(u_1,v_1)\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_3} j_+^{(-)}(\xi)d\xi\right)=\left[\exp\left(-\frac{i}{2c}(a+\bar{a})\int_{u_2}^{u_3} j_+^{(-)}(\xi)d\xi\right)\psi_1(u_1,v_1)\right]\left(\frac{u_2-u_1-i\epsilon}{u_3-u_1-i\epsilon}\right)^{(a+\bar{a})^2/4\pi c},$$

$$\psi_1(u_1,v_1)\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_3} j_-^{(-)}(\eta)d\eta\right)=\left[\exp\left(-\frac{i}{2c}(a-\bar{a})\int_{v_2}^{v_3} j_-^{(-)}(\eta)d\eta\right)\psi_1(u_1,v_1)\right]\left(\frac{v_2-v_1-i\epsilon}{v_3-v_1-i\epsilon}\right)^{(a-\bar{a})^2/4\pi c},$$

we get

$$\psi_1(u_1,v_1)\psi_1(u_2,v_2)\psi_1^\dagger(u_3,v_3)=\left(\frac{i(u_3-u_1)+\epsilon}{i(u_1-u_3)+\epsilon}\,\frac{i(u_1-u_2)+\epsilon}{i(u_2-u_1)+\epsilon}\right)^{(a+\bar{a})^2/4\pi c}\left(\frac{i(v_3-v_1)+\epsilon}{i(v_1-v_3)+\epsilon}\,\frac{i(v_1-v_2)+\epsilon}{i(v_2-v_1)+\epsilon}\right)^{(a-\bar{a})^2/4\pi c}$$
$$\times\psi_1(u_2,v_2)\psi_1^\dagger(u_3,v_3)\psi_1(u_1,v_1),$$

which combined with Eq. (4.4) yields

$$\psi_1(u_1,v_1)\psi_1(u_2,v_2)=\left(\frac{i(u_1-u_2)+\epsilon}{i(u_2-u_1)+\epsilon}\right)^{(a+\bar{a})^2/4\pi c}\left(\frac{i(v_1-v_2)+\epsilon}{i(v_2-v_1)+\epsilon}\right)^{(a-\bar{a})^2/4\pi c}\psi_1(u_2,v_2)\psi_1(u_1,v_1). \tag{4.5}$$

Because of the condition (2.32), $(a+\bar{a})^2(4\pi c)^{-1}=(a-\bar{a})^2(4\pi c)^{-1}+1$, required for a spinor Lorentz transformation law, one sees that (4.4) and (4.5) correspond to anticommutation laws for spacelike separation, $(u_1-u_2)(v_1-v_2)<0$.

Similarly by commuting ψ_2 with expression (4.1a), we find

$$\psi_2(u_1,v_1)\psi_1(u_2,v_2)\psi_1^\dagger(u_3,v_3)=\left[\left(\frac{i(u_1-u_2)+\epsilon}{i(u_2-u_1)+\epsilon}\,\frac{i(u_3-u_1)+\epsilon}{i(u_1-u_3)+\epsilon}\right)\left(\frac{i(v_1-v_2)+\epsilon}{i(v_2-v_1)+\epsilon}\,\frac{i(v_3-v_1)+\epsilon}{i(v_1-v_3)+\epsilon}\right)\right]^{(a^2-\bar{a}^2)/4\pi c}$$
$$\times\psi_1(u_2,v_2)\psi_1^\dagger(u_3,v_3)\psi_2(u_1,v_1).$$

Because of the Klein transformation relating commutation of different fields, we may assume that ψ_1 and ψ_2 anticommute at (some) equal time. It follows that

$$\psi_2(u_2,v_2)\psi_1^\dagger(u_1,v_1)=-\left(\frac{i(u_1-u_2)+\epsilon}{i(u_2-u_1)+\epsilon}\,\frac{i(v_1-v_2)+\epsilon}{i(v_2-v_1)+\epsilon}\right)^{(a^2-\bar{a}^2)/4\pi c}\psi_1^\dagger(u_1,v_1)\,\psi_2(u_2,v_2), \tag{4.6}$$

$$\psi_1(u_1,v_1)\psi_2(u_2,v_2)=-\left(\frac{i(u_1-u_2)+\epsilon}{i(u_2-u_1)+\epsilon}\,\frac{i(v_1-v_2)+\epsilon}{i(v_2-v_1)+\epsilon}\right)^{(a^2-\bar{a}^2)/4\pi c}\psi_2(u_2,v_2)\psi_1(u_1,v_1). \tag{4.7}$$

Using formulas (4.1)–(4.7), all Wightman functions involving arbitrary numbers of ψ's and j's are easily calculated (and hence all matrix elements as well). Thus, the commutation relations of the j's and the ψ's determine a unique solution of the Thirring model for which the j's are irreducible.

Our basic formula (4.1) expresses the products of two spinor fields $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_1^\dagger(x')\psi_1(x)$ in terms of currents. Conversely it may be used to express the current as the product of two spinor fields as the space-time points approach each other. From Eq. (4.1) we see that the bilocal operator defined by

$$R[\psi_1^\dagger(x')\psi_1(x)]\equiv[i(u'-u)+\epsilon]^{(a+\bar{a})^2/4\pi c}[i(v'-v)+\epsilon]^{(a-\bar{a})^2/4\pi c}\psi_1^\dagger(x')\psi_1(x) \tag{4.8}$$

is regular as the points approach each other and in fact has the limiting value $R[\psi_1^\dagger(x)\psi_1(x)]=f_0$. Its derivatives are also seen to be regular, and we have

$$\lim_{u',v'\to u,v} i\frac{\partial}{\partial u}R[\psi_1^\dagger(u',v')\psi_1(u,v)]=\frac{a+\bar{a}}{2c}f_0\,j_+(u), \tag{4.9a}$$

$$\lim_{u',v'\to u,v} i\frac{\partial}{\partial v}R[\psi_1^\dagger(u',v')\psi_1(u,v)]=\frac{a-\bar{a}}{2c}f_0\,j_-(v). \tag{4.9b}$$

The differentiation may be made part of the limiting process and we obtain alternative expressions for the current in terms of spinor fields

$$j_+(u)=\frac{2c}{(a+\bar{a})f_0}\,i\lim_{u',v'\to u,v}\frac{1}{u-u'}\{R[\psi_1^\dagger(u',v')\psi_1(u,v)]-f_0\}, \tag{4.10a}$$

$$j_-(v)=\frac{2c}{(a+\bar{a})f_0}\, i \lim_{u',v'\to u,v} \frac{1}{v-v'}\left\{R[\psi_1^\dagger(u',v')\psi_1(u,v)]-f_0\right\}. \tag{4.10b}$$

The corresponding expressions with $\bar{a}\to-\bar{a}$ and $\psi_1\to\psi_2$ are also available [and are necessary for $j_-(v)$ when $a=\bar{a}$]. An equal-time limit with $u'+v'=u+v$ may be used to give a meaning to the equation of motion for ψ, Eq. (2.14) as the limit of a nonlocal equation involving ψ only.[17] This brings us back to the original formulation of the Thirring model conceived as a theory of a self-interacting spinor field.

The singularity structure of the products $\psi_1\psi_1$ and $\psi_1^\dagger\psi_1^\dagger$ are determined by Eq. (3.11) for the fourfold product $\psi_1(x_1)\psi_1^\dagger(x_2)\psi_1(x_3)\psi_1^\dagger(x_4)$. We see that

$$R[\psi_1(u,v)\psi_1(u',v')]\equiv[i(u-u')+\epsilon]^{-(a+\bar{a})^2/4\pi c}[i(v-v')+\epsilon]^{-(a-\bar{a})^2/4\pi c}\psi_1(u,v)\psi_1(u',v'), \tag{4.11a}$$

$$R[\psi_1^\dagger(u,v)\psi_1^\dagger(u',v')]\equiv[i(u-u')+\epsilon]^{-(a+\bar{a})^2/2\pi c}[i(v-v')+\epsilon]^{-(a-\bar{a})^2/4\pi c}\psi_1^\dagger(u,v)\psi_1^\dagger(u',v') \tag{4.11b}$$

are smooth as the points coincide, and similarly for ψ_2 with $\bar{a}\to-\bar{a}$. To obtain the remaining regular bilocal products it is sufficient to consider the fourfold product $\psi_1(x_1)\psi_1^\dagger(x_2)\psi_2(x_3)\psi_2^\dagger(x_4)$ which is easily expressed in terms of currents and normal ordered. One finds that

$$R[\psi_1(x)\psi_2^\dagger(x')]\equiv\{[i(u-u')+\epsilon][i(v-v')+\epsilon]\}^{(a^2-\bar{a}^2)/4\pi c}\psi_1(x)\psi_2^\dagger(x'), \tag{4.12a}$$

$$R[\psi_2(x)\psi_1^\dagger(x')]\equiv\{[i(u-u')+\epsilon][i(v-v')+\epsilon]\}^{(a^2-\bar{a}^2)/4\pi c}\psi_2(x)\psi_1^\dagger(x'), \tag{4.12b}$$

$$R[\psi_1(x)\psi_2(x')]\equiv\{[i(u-u')+\epsilon][i(v-v')+\epsilon]\}^{(\bar{a}^2-a^2)/4\pi c}\psi_1(x)\psi_2(x'), \tag{4.13a}$$

$$R[\psi_1^\dagger(x)\psi_2^\dagger(x')]\equiv\{[i(u-u')+\epsilon][i(v-v')+\epsilon]\}^{(\bar{a}^2-a^2)/4\pi c}\psi_1^\dagger(x)\psi_2^\dagger(x') \tag{4.13b}$$

are also smooth, and similarly for $\psi_1\longleftrightarrow\psi_2$ and $\bar{a}\longleftrightarrow-\bar{a}$. In Sec. V a Hilbert space will be constructed on which $\psi(x)$ acts as a local field. One may also show that the regular bilocal operators defined here are local fields for $x'=x$. Their dimension may be found by commuting with the dilatation operator, which agrees with simply counting linear dimensions in the defining equations. The resulting dimensions (in units of mass) are[18]

$$d[\psi(x)]=[(a+\bar{a})^2+(a-\bar{a})^2]/8\pi c=\tfrac{1}{2}+g^2c/4\pi, \tag{4.14a}$$

$$d\{R[\psi_i^\dagger(x)\psi_i(x')]\}=0,\quad i=1,2 \tag{4.14b}$$

$$d\{R[\psi_1(x)\psi_1(x)]\}=[(a+\bar{a})^2+(a-\bar{a})^2]/2\pi c=2+g^2c/\pi, \tag{4.14c}$$

$$d\{R[\psi_1^\dagger(x)\psi_2(x)]\}=d\{R[\psi_2^\dagger(x)\psi_1(x)]\}=\bar{a}^2/\pi c=\bar{a}/a, \tag{4.14d}$$

$$d\{R[\psi_1(x)\psi_2(x)]\}=a^2/\pi c=a/\bar{a}. \tag{4.14e}$$

Note that all bilocal operators are scalars, except for $R[\psi_1\psi_1]$, $R[\psi_1^\dagger\psi_1^\dagger]$ and $(1\to2)$, which are second-rank tensors.

V. CONSTRUCTION OF THE FIELD

We shall now construct explicitly the field $\psi(x)$, solution of (2.2).[19] Let $\tilde{f}(p)$ be a real function on the positive real axis, finitely differentiable and with support contained in $a\lesssim p\lesssim b$, for some $a,b>0$. Define

$$\phi_\pm(f)=\frac{2\pi}{\sqrt{c}}\int_{p>0}\frac{1}{\sqrt{p}}\tilde{f}(p)[\tilde{j}_\pm(p)+\tilde{j}_\pm(-p)]\,dp,$$

$$\pi_\pm(f)=(-)\frac{2\pi i}{\sqrt{c}}\int_{p>0}\frac{1}{\sqrt{p}}\tilde{f}(p)[\tilde{j}_\pm(p)-\tilde{j}_\pm(-p)]\,dp,$$

$$\tilde{j}(p)=\frac{1}{\sqrt{2\pi}}\int e^{ipx}j(x)\,dx,$$

$$\alpha_\pm(f)=\phi_\pm(f)+i\pi_\pm(f).$$

It is immediately verified that $\phi_\pm$, $\pi_\pm$ are Hermitian operators which satisfy canonical commutation relations $[16\pi^2\int_0^\infty dp\,\tilde{f}^2(p)=1]$ and that they commute with the charge operators $Q_\pm$. We write the Hilbert space as a direct sum,

$$\mathcal{H}=\bigoplus_{\underline{q}}\mathcal{H}_{\underline{q}},\quad \underline{q}=(q_+,q_-)$$

in such a way that $Q_\pm$ acts on $\mathcal{H}_{\underline{q}}$ as multiplication by $q_\pm$.[20] Denote by $\alpha_\pm^{\underline{q}}$ the restriction of $\alpha_\pm$ to the subspace $\mathcal{H}_{\underline{q}}$. The assumption of irreducibility of the $j_\pm(f)$ in each charge sector is now made precise by the requirement that the representation $R_{\underline{q}}$ generated by the $\alpha_\pm^{\underline{q}}$ (Ref. 21) be irreducible for each $\underline{q}$. The commutation properties of the field with $Q_\pm$ show that $\psi_{1,2}(f)$ maps (a subset of) $\mathcal{H}_{\underline{q}}$ into $\mathcal{H}_{\underline{q}-\underline{e}^{1,2}}$, where $e_\pm^1=a\pm\bar{a}$, $e_\pm^2=a\mp\bar{a}$. An irreducible representation of the fields can therefore be found in a space which is no larger than

$$\bigoplus_{\underline{q} \in C_{\alpha\beta} \subset R\times R} \mathcal{H}_{\underline{q}},$$

where $\underline{q} \in C_{\alpha\beta}$ if $q_+ = \alpha + n_1(a+\bar{a}) + n_2(a-\bar{a})$, $q_- = \beta + n_1(a-\bar{a}) + n_2(a+\bar{a})$ for some $\underline{n} \equiv (n_1, n_2) \in Z\times Z$.

We consider in detail only the case $a \neq \pm\bar{a}$ (the other cases correspond to the "decoupling" of one of the currents $j_\pm$, and are to be treated separately, along the same lines but with obvious simplifications. For the choice $a\bar{a} = \pi c$ [Eq. (2.32)] we cannot have $a = -\bar{a}$). When $a \neq \pm\bar{a}$, one can classify the representations giving α, β, and $\underline{n}$. We assume that there exists a representation which has a vector Ω invariant under space-time translations. By convention, we associate $\alpha = 0$, $\beta = 0$, and $\underline{n} = \underline{0}$ to this representation and denote by $R_{\underline{n}}$ the representations characterized by $(0, 0, \underline{n})$, and by $\mathcal{H}_{\underline{n}}$ the corresponding subspaces. The Hilbert space we consider is then $\mathcal{H} = \oplus_{\underline{n}} \mathcal{H}_{\underline{n}}$.

The generators of space-time translations are represented formally by (2.22a) and (2.22b), which we rewrite in the form

$$H+P = \int_{p>0} p\, \alpha_+^\dagger(p)\alpha_+(p)dp\,,$$
$$H-P = \int_{p>0} p\, \alpha_-^\dagger(p)\alpha_-(p)dp\,, \qquad (5.1)$$

with

$$\alpha_\pm(p) = \frac{\tilde{j}(p)}{(cp)^{1/2}}.$$

Let $\alpha_\pm^{\underline{0}}$ be the restriction of $\alpha_\pm$ to $\mathcal{H}_{\underline{0}}$. From $H\Omega = P\Omega = 0$ and (5.1), one derives $\alpha_\pm^{\underline{0}}(f)\Omega = 0$, for all functions f of the class considered. $R_{\underline{0}}$ is therefore a Fock representation. We study next the structure of the representations $R_{\underline{n}}$. They are completely described by the following.

Lemma 1. Let $\tilde{J}_+(p)$ be functions on $[0,\infty)$, square-integrable, Hölder-continuous (from the right) of order $\epsilon > 0$ at the origin, and such that $\tilde{J}_\pm(0) = 1$.

Let

$$A^{\underline{n}}_{\underline{J},\pm}(f) \equiv \alpha_\pm^{\underline{n}}(f)\left(\frac{c}{8\pi}\right)^{1/2} - \rho_\pm^{\underline{n}}\int \frac{\tilde{f}(p)}{\sqrt{p}}\tilde{J}_\pm(p)dp\,,$$
$$\rho_\pm^{\underline{n}} = n_1 e_\pm^1 + n_2 e_\pm^2\,. \qquad (5.2)$$

The representation of the canonical commutation relations generated on $\mathcal{H}_{\underline{n}}$ by the $A^{\underline{n}}_{\underline{J},\pm}$ is then of Fock type.

A proof of this Lemma is to be found in Appendix A. Here we will only add the following remarks, which are easy consequences of Lemma 1.

Remark 1. The statement in the Lemma holds true for each $\underline{\tilde{J}}$ in the class considered $[\underline{\tilde{J}} \equiv (\tilde{J}_+, \tilde{J}_-)]$. This is a consequence of the fact that

$$\int |\underline{\tilde{J}}(p) - \underline{\tilde{J}}'(p)|^2 \frac{dp}{p} < \infty \quad \text{for any pair } \underline{\tilde{J}}, \underline{\tilde{J}}'.$$

Remark 2. Using Eq. (5.2) it is possible to extend the definition of $\alpha_\pm^{\underline{n}}(f)$ to all f's for which

$$\int |\tilde{f}(p)|^2 dp < \infty\,, \quad \int_0^\epsilon \frac{\tilde{f}(p)}{\sqrt{p}}dp < \infty,$$

for some $\epsilon > 0$.

Remark 3. Since $(1/\sqrt{p})\tilde{J}(p)$ are not in $\mathfrak{L}_2(0,\infty)$, the representations $R_{\underline{n}}$ and $R_{\underline{n}'}$ are not equivalent for $\underline{n} \neq \underline{n}'$. In particular, $R_{\underline{n}}$ is not of Fock type if $\underline{n} \neq \underline{0}$.

Remark 4. The functions $\tilde{J}_\pm(p, x_0)$, $\tilde{J}_\pm(p\,|\,\lambda)$ defined by

$$\tilde{J}(p; x_0) \equiv \frac{1}{\sqrt{p}}(1 - e^{ipx_0})\tilde{J}_\pm(p)\,, \quad -\infty < x_0 < +\infty$$

$$\tilde{J}(p\,|\,\lambda) \equiv \frac{1}{\sqrt{p}}[\tilde{J}_\pm(p) - \tilde{J}_\pm(\lambda p)]\,, \quad 0 < \lambda < \infty$$

are square-integrable in p and continuous in x_0, λ at $x_0 = 0$, $\lambda = 1$ uniformly in p over the compacts. Therefore translations in space-time, Lorentz transformations, and dilatations are implemented by strongly continuous groups of unitary operators, in each $\mathcal{H}_{\underline{n}}$. The generators of space-time translations are the ones given in (5.1) where the normal-ordering prescription is given in terms of the $\alpha_\pm^{\underline{n}}$, and coincides with the prescription in terms of $A^{\underline{n}}_{\pm,\underline{J}}$, as seen from (5.2). Also, the energy-momentum density $\Theta_{\mu\nu}(x,t)$ as given in (2.20) exists on each $\mathcal{H}_{\underline{n}}$ as a bilinear form, and also as an unbounded operator when smeared, at fixed t, with a twice-differentiable function of x.

Remark 5. Also the generators M, D of the Lorentz boosts and of dilatations exist; indeed, using the fact that $i\sqrt{p}(\partial/\partial p)\sqrt{p}$ defines in $L^2(R_1^+)$ a self-adjoint operator, one can easily check that (2.22) defines on each $\mathcal{H}_{\underline{n}}$ a self-adjoint operator, provided $\int u J_\pm^2(u) < \infty$.

Before proceeding further, we want to show that $Q_\pm$ is indeed, in a well-defined sense,[22] the integral of $j_\pm(x,t)$ at fixed t. Recall that, so far in this section, $Q_\pm$ are operators the eigenvalues of which are used to label the representations $R_{\underline{n}}$ which occur in $\mathcal{H}$.

Lemma 2. In each sector $\mathcal{H}_{\underline{n}}$ there is a dense set D of vectors such that, for $\phi \in D$

$$\lim_{n\to\infty}\left(\phi, \int f_n(x) j_\pm(x,t)dx\,\phi\right) = (\phi, Q_\pm\phi)$$

for all t, for every sequence f_n such that

$$\int p|\tilde f_n(p)|^2 dp < \infty, \quad \tilde f_n(0)=1$$

and which converges to L $[L(x)=1, \forall x]$ in the sense that

$$\lim_{n\to\infty}\int \tilde f_n(p)\tilde g(p)dp = \tilde g(0)$$

for all functions $\tilde g(p)$ continuous at $p=0$.

Proof. Let D be the set of finite linear combinations of those vectors in $\mathcal{H}_{\underline{n}}$ which are obtained by applying polynomials in $A^{\underline{n}\dagger}_{\underline{J},\pm}(f)$ to coherent states $\|g\rangle\!\rangle_n$ where $g, f\in \mathcal{L}_2$ and continuous at $p=0$.[23] Every $\xi\in D$ is in the domain $\int f_m(x) j_\pm(x,t)dx$; one computes

$$\left(\xi, \int f_m(x)j_\pm(x,t)dx\,\xi\right) = (\xi, Q_\pm \xi)\int_{p>0}\tilde f_m(p)e^{ipt}\tilde J_\pm(p)dp + \left(\xi, \int_{p>0}\sqrt{p}\,\tilde f_m(p)A^{\underline{n}}_{\underline{J},\pm}(p)e^{-ipt}dp\,\xi\right)$$
$$+\left(\xi, \int_{p>0}\sqrt{p}\,\tilde f_m(-p)(A^{\underline{n}}_{\underline{J},\pm})^\dagger(p)e^{ipt}dp\,\xi\right).$$

The first integral on the right-hand side converges to 1, while the two other terms converge to zero, since, if $\tilde J(p)$ is a function continuous at $p=0$, such is also $\sqrt{p}\,\tilde J(p)$ and moreover $\lim_{p\to 0}\sqrt{p}\,\tilde J(p)=0$. This concludes the proof of Lemma 2.

It is now convenient to define two families $U_{\underline{J}}, V_{\underline{J}}$ of unitary operators on $\mathcal{H}$. U_J maps $\mathcal{H}_{\underline{n}}$ onto $\mathcal{H}_{\underline{n}-\underline{\delta}^1}$ (δ^i_j is the Kronecker δ) and is such that[24]

$$U_{\underline{J}}A^{\underline{n}}_{\underline{J},\pm}U^{-1}_{\underline{J}} = A^{\underline{n}-\underline{\delta}^1}_{\underline{J},\pm}, \quad U_{\underline{J}}\Omega^{\underline{n}}_{\underline{J}} = \Omega^{\underline{n}-\underline{\delta}^1}_{\underline{J}}.$$

The operator $V_{\underline{J}}$ is similarly defined, with δ^2 instead of $\underline{\delta}^1$. $U_{\underline{J}}$ and $V_{\underline{J}}$ are uniquely defined by this equation, since the representations $R^{(\underline{J})}_{\underline{n}}$, $R^{(\underline{J})}_{\underline{n}-\underline{\delta}^1}$, $R^{(\underline{J})}_{\underline{n}-\underline{\delta}^2}$ are equivalent and irreducible; here $R^{(\underline{J})}_{\underline{n}}$ is the representation of the canonical commutation relations generated by $A^{\underline{n}}_{\underline{J},\pm}(f)$ (Ref. 21). From (5.2) it is evident that, as a set of operators on $\mathcal{H}_{\underline{n}}$, $R^{(\underline{J})}_{\underline{n}}$, and $R_{\underline{n}}$ coincide. The operators $U_{\underline{J}}, V_{\underline{J}}$ all commute.

The free group generated by $U_{\underline{J}}$ and $V_{\underline{J}}$ (fixed $\underline{J}$) acts transitively on the sectors $\mathcal{H}_{\underline{n}}$; from Lemma 1, Remark 3, it follows then[25] that the algebra of operators generated by $U_{\underline{J}}, V_{\underline{J}}$, $\exp[i\phi_\pm(f)]$, $\exp[i\pi_\pm(f)]$ is irreducible.

Since $R^{(\underline{J})}_{\underline{n}}$ and $R^{(\underline{J}')}_{\underline{n}}$ are equivalent, the operator $U_{\underline{J}}{}^{-1}U_{\underline{J}'}$ leaves each $\mathcal{H}_{\underline{n}}$ invariant; in fact, its reduction to $\mathcal{H}_{\underline{n}}$ belongs to $R_{\underline{n}}$, and is given explicitly by

$$U_{\underline{J}}{}^{-1}U_{\underline{J}'}|_{\mathcal{H}_{\underline{n}}} = \exp\left[\frac{1}{4(2\pi c)^{1/2}}\left((a+\bar a)\int\frac{\alpha^{\underline{n}}_+(p)}{\sqrt{p}}[\tilde J_+(p)-\tilde J'_+(p)]dp + (a-\bar a)\int\frac{\alpha^{\underline{n}}_-(p)}{\sqrt{p}}[\tilde J_-(p)-\tilde J_-(p')]dp' - \text{H.c.}\right)\right]$$

(notice that $\int du[J_\pm(u)-J'_\pm(u)]=0$).

We turn now to construction of the field $\psi(x)$, given the Hilbert space $\mathcal{H}=\oplus_{\underline{n}}\mathcal{H}_{\underline{n}}$ and the representations $R_{\underline{n}}$ described above.[26] The crucial remark is the following: If $\psi(f)$ exists, then $U_{\underline{J}}{}^{-1}\psi_1(f)$ and $V_{\underline{J}}{}^{-1}\psi_2(f)$ must commute with the $Q_\pm$, and must therefore be "functions of the $j_\pm$'s and of the $Q_\pm$'s," due to the irreducibility of $R_{\underline{n}}$ for each $\underline{n}$. We write

$$U_{\underline{J}}{}^{-1}\psi_1(f) \equiv \mathfrak{F}_{\underline{J}}(f), \quad V_{\underline{J}}{}^{-1}\psi_2(f) \equiv \mathcal{G}_{\underline{J}}(f) \tag{5.3}$$

and obtain equations for $\mathfrak{F}_J$, $\mathcal{G}_J$ using (2.10a) and (2.10b). We prove that these equations admit a unique solution (unique apart from normalization); this solution is then used in (5.3) to define $\psi_{1,2}(f)$.

We shall give details only of the determination of $\mathfrak{F}_J(f)$, the results for $\mathcal{G}_J(f)$ will only be quoted.

It is expedient to begin by determining an "approximant" to ψ_1, denoted by $\psi_{1,\Lambda}(u,v)$ which satisfies

$$[j_+(u), \psi_{1,\underline{\Lambda}}(u',v')] = -(a+\bar a)\psi_{1,\underline{\Lambda}}(u',v')\delta_{\Lambda_+}(u-u'),$$
$$[j_-(v), \psi_{1,\underline{\Lambda}}(u',v')] = -(a-\bar a)\psi_{1,\underline{\Lambda}}(u',v')\delta_{\Lambda_-}(v-v'), \tag{5.4}$$

where

$$\delta_\Lambda(x) = \frac{1}{2\pi}\int_{|p|<\Lambda} e^{ipx}dp, \quad \underline{\Lambda}=(\Lambda_1,\Lambda_2).$$

For fixed (u,v), $\psi_{1,\underline{\Lambda}}(u,v)$ will turn out to be a bounded operator; the field $\psi_1(u,v)$ will then be obtained by taking the strong limit, when $\Lambda_+,\Lambda_-\to\infty$, on a suitable domain, of $\int\psi_{1,\Lambda}(u,v)f(u,v)du\,dv$, for suitable test functions $f(u,v)$. From (5.4) and the transformation of $A^{\underline{n}}_{\underline{J}\pm}$ under U_J it follows that $\mathfrak{F}_{\underline{\Lambda}}(u,v)$ must satisfy

$$[j_+(u), \mathfrak{F}_{\underline{\Lambda}}(u',v')] = (a+\bar{a})[J_+(u) - \delta_{\Lambda_+}(u-u')]\,\mathfrak{F}_{\underline{\Lambda}}(u',v'),$$
$$[j_-(v), \mathfrak{F}_{\underline{\Lambda}}(u',v')] = (a-\bar{a})[J_-(v) - \delta_{\Lambda_-}(v-v')]\,\mathfrak{F}_{\underline{\Lambda}}(u',v'). \tag{5.5}$$

The operator $\mathfrak{F}_\Lambda(u,v)$ leaves $\mathcal{H}_{\underline{n}}$ invariant for each $\underline{n}$; the solution of (5.5) is then known to be

$$\mathfrak{F}_\Lambda(u,v) = C_{\underline{\Lambda}}(\underline{Q};u,v) : \exp\left(\int h^{\underline{J}}_{\Lambda_+}(u,u')j_+(u')du' + \int h^{\underline{J}}_{\Lambda_-}(v,v')j_-(v')dv'\right):, \tag{5.6}$$

where

$$h^{\underline{J}}_{\Lambda_\pm}(u,u') = \frac{a\pm\bar{a}}{2ci}\left(\int_{-\infty}^{u'} J(u'')du'' - \theta_{\Lambda_\pm}(u'-u)\right).$$

It can be checked that $\int |\tilde{h}_{\Lambda_\pm}(u,p)|\,dp < \infty$, since $p\tilde{h}(u,p) = \tilde{J}_\pm(p) - \tilde{\delta}_{\Lambda_\pm}(p)\,e^{ipu} = O(p^\epsilon)$ at $p=0$, for some $\epsilon > 0$; the exponential in (5.6) defines therefore a unitary operator.

We now substitute (5.6) in (5.3), and recast the expression thereby obtained for $\psi_{1,\Lambda}(u,v)$, in the form

$$\psi_{1,\underline{\Lambda}}(u,v) = \nu_{\underline{\Lambda}}(\underline{Q};u,v) : \exp\left(\int h_{\Lambda_+}(u,u')j_+(u')du' + \int h_{\Lambda_-}(v,v')j_-(v)dv'\right)U_{\underline{J}}:, \tag{5.7}$$

where the expression between colons is defined to be

$$\exp\left(\int h_{\Lambda_+}(u,u')j_+^{(+)}(u')du' + \int h_{\Lambda_-}(v,v')j_-^{(+)}(v')dv'\right)U_{\underline{J}}\,\exp\left(\int h_{\Lambda_+}(u,u')j_+^{(-)}(u')du' + \int h_{\Lambda_-}(v,v')j_-^{(-)}(v')dv'\right) \tag{5.8}$$

and $j^{(+)}, j^{(-)}$ are the positive- and negative-frequency parts of j. As defined in (5.7), the normalization of field depends on $\underline{J}$, for fixed $\nu_{\Delta}(Q;u,v)$. One has, however, the following identity: If $\tilde{J}_\pm, \tilde{J}'_\pm$ have support in $\{p \mid p < \Lambda\}$, then

$$\exp\left(\frac{(a+\bar{a})^2}{32\pi c}\int [|\tilde{J}_+(p)|^2 - |\tilde{J}'_+(p)|^2]\frac{dp}{p} + \frac{(a-\bar{a})^2}{32\pi c}\int [|\tilde{J}_-(p)|^2 - |\tilde{J}'_-(p)|^2]\frac{dp}{p}\right)\psi_{1,\Lambda}(u,v;\underline{J})$$
$$= \nu_\Lambda(Q;u,v;J)\nu_\Lambda^{-1}(Q;u,v,J')\psi_{1,\Lambda}(u,v;J') \tag{5.9}$$

also, one can verify that

$$e^{iPa - iHa_0}U_{\underline{J}} = U_{\underline{J}(a,a_0)}\,e^{iPa - iHa_0}, \tag{5.10}$$

where

$$[\tilde{J}_\pm(a,a_0)](p) = e^{ip(a\pm a_0)}\tilde{J}_\pm(p).$$

Using (5.10), the requirement that $\psi_{1,\Lambda}(u,v)$ transform correctly under space-time translation leads to

$$\psi_{1,\Lambda}(u,v) = \nu_{\underline{\Lambda}}(\underline{Q};\underline{J}) : \exp\left(\int h_{\Lambda_+}(u,u')j_+(u')du' + \int h_{\Lambda_-}(v,v')j_-(v')dv'\right)U_{\underline{J}}:. \tag{5.11a}$$

In the same way, one obtains the following expression for $\psi_{2,\underline{\Lambda}}$ which is an approximant of ψ_2:

$$\psi_{2,\Lambda}(u,v) = \mu_{\underline{\Lambda}}(\underline{Q};\underline{J}) : \exp\left(\int h'_{\Lambda_+}(u,u')j_+(u')du' + \int h'_{\Lambda_-}(v,v')j_-(v')dv'\right)V_{\underline{J}}:, \tag{5.11b}$$

where h'_Λ is obtained from $h_{\Lambda_\pm}$ replacing $\bar{a}$ with $-\bar{a}$. Notice that the dependence of $\nu_{\underline{\Lambda}}(\underline{Q};\underline{J})$, $\mu_{\underline{\Lambda}}(\underline{Q};\underline{J})$ on $\underline{J}$ is only through the exponential factor in (5.9); in particular, $\nu_{\underline{\Lambda}}$, $\mu_{\underline{\Lambda}}$ do not change when $\underline{J}$ is transformed by translations, Lorentz transformations, and dilatations.

The dependence of $\nu_{\underline{\Lambda}}, \mu_{\underline{\Lambda}}$ on $\underline{Q}$ is not relevant to determine the transformation properties of $\psi_\Lambda \equiv (\psi_{1,\Lambda}, \psi_{2,\Lambda})$ under Lorentz transformations and dilatations, since $\underline{Q}$ is an invariant. Also, the commutation properties of $\psi_{1,\underline{\Lambda}}$ with itself (e.g., at spacelike distances) and of $\psi_{2,\underline{\Lambda}}$ with itself, do not depend on $\nu_{\underline{\Lambda}}, \mu_{\underline{\Lambda}}$. The commutation properties of $\psi_{1,\underline{\Lambda}}$ with $\psi^\dagger_{1,\underline{\Lambda}}$, or of $\psi_{2,\underline{\Lambda}}$ with $\psi^\dagger_{2,\underline{\Lambda}}$, do depend on $\nu_{\underline{\Lambda}}, \mu_{\underline{\Lambda}}$. It can be easily checked [compare, e.g., Eq. (4.4) and following lines] that, when $a\bar{a} = n\pi c$, the usual commutation properties at spacelike separations of $\psi_{\underline{\Lambda}}(u,v)$ with itself and with $\psi^\dagger_{\underline{\Lambda}}(u',v')$ are obtained if and only if, for fixed J,

(a) $\nu(m_1m_2)\nu^\dagger(m_1m_2)=\nu(m_1+1m_2)\nu^\dagger(m_1+1m_2)$,

(b) $\mu(m_1m_2)\mu^\dagger(m_1m_2)=\mu(m_1m_2+1)\mu^\dagger(m_1m_2+1)$,

(c) $\nu(m_1m_2)\mu(m_1+1\,m_2)=\mu(m_1m_2)\nu(m_1m_2+1)(-)^n$,

(d) $\nu(m_1m_2)\mu^\dagger(m_1+1m_2-1)=\mu^\dagger(m_1m_2-1)\nu(m_1m_2-1)(-)^n$,

where

$$m_1\equiv\frac{(a+\bar a)Q_+-(a-\bar a)Q_-}{4a\bar a},$$

$$m_2=\frac{(a+\bar a)Q_--(a-\bar a)Q_+}{4a\bar a},$$

and relations (a), (b), (c), and (d) follow from considering $(\psi_1,\psi_1^\dagger)$, $(\psi_2,\psi_2^\dagger)$, (ψ_1,ψ_2), and $(\psi_1,\psi_2^\dagger)$, respectively (we have suppressed the J dependence in ν and μ). Relations (a) and (b) imply

$$\nu(m_1m_2)=\rho(m_2)e^{i\phi(m_1m_2)},$$

$$\mu(m_1m_2)=\sigma(m_1)e^{i\psi(m_1m_2)},$$

where ρ,σ and ϕ,ψ are Hermitian. Combining with (c) we get that ρ and σ must be constants and that (modulo 2π)

(e) $\phi(m_1m_2)-\phi(m_1m_2+1)=\psi(m_1m_2)-\psi(m_1+1m_2)+n\pi$.

Relation (d) renders no new information. From (e) we get, that for a given ϕ,

(f) $\psi(m_1m_2)=-\sum_{m=0}^{m_1}[\phi(m-1\,m_2)-\phi(m-1\,m_2+1)]+\tilde\psi(m_2)+m_1n\pi$.

Let us now use the freedom of a unitary transformation

$$\psi_1\to e^{iF(m_1m_2)}\psi_1\,e^{-iF(m_1m_2)}=e^{i[F(m_1m_2)-F(m_1+1m_2)]}\psi_1,$$

$$\psi_2\to e^{iF(m_1m_2)}\psi_2\,e^{-iF(m_1m_2)}=e^{i[F(m_1m_2)-F(m_1\,m_2+1)]}\psi_2.$$

We choose F in such a way that

$$F(m_1m_2)-F(m_1+1m_2)+\phi(m_1m_2)=0,$$

which brings us to ν depending on J only. Solving for F,

$$F(m_1m_2)=\sum_{m=0}^{m_1}\phi(m-1\,m_2)+\tilde F(m_2).$$

As for ψ_2, we have in the exponential i times

$$F(m_1m_2)-F(m_1m_2+1)+\psi(m_1m_2)=\tilde F(m_2)-\tilde F(m_2+1)+\tilde\psi(m_2)+m_1n\pi.$$

Choosing $\tilde F$ such that the last combination is $m_1n\pi$ we finally get that without loss of generality

$$\nu(\underline{Q};\underline{J})=\nu_0(\underline{J}),\quad \mu(\underline{Q},\underline{J})=\mu_0(\underline{J})\,e^{im_1n\pi}=\mu_0(\underline{J})\exp\left(in\pi\frac{(a+\bar a)Q_+-(a-\bar a)Q_-}{4a\bar a}\right),\tag{5.12}$$

where [compare (5.9)]

$$\nu_0(J')\exp\left(\frac{(a+\bar a)^2}{32\pi c}\int[|\tilde J_+(p)|^2-|\tilde J'_+(p)|^2]\frac{dp}{p}+\frac{(a-\bar a)^2}{32\pi c}\int[|\tilde J_-(p)|^2-|\tilde J'_-(p)|^2]\frac{dp}{p}\right)=\nu_0(J)$$

and a similar formula, with $\bar a\to-\bar a$, for $\mu_0(\underline{J})$. From (5.11), (5.12), and the definition of $h_{\underline{\Lambda}}, h'_{\underline{\Lambda}}$ one can see that $\psi_{\underline{\Lambda}}(u,v)\equiv(\psi_{1,\underline{\Lambda}}(u,v),\psi_{2,\underline{\Lambda}}(u,v))$ satisfies the differential equation

$$\begin{aligned}\frac{\partial}{\partial u}\psi_{\underline{\Lambda}}(u,v)&=\frac{-i}{2c}(a+\bar a\gamma_5):j_+^{\Lambda}(u)\psi_{\underline{\Lambda}}(u,v):,\\ \frac{\partial}{\partial v}\psi_{\underline{\Lambda}}(u,v)&=\frac{-i}{2c}(a-\bar a\gamma_5):j_-^{\Lambda}(v)\psi_{\underline{\Lambda}}(u,v):,\end{aligned}\tag{5.13}$$

where

$$j_{\pm}^{\Lambda}(u)=\left(\frac{c}{2\pi}\right)^{1/2}\int_{0<p<\Lambda_{\pm}}\sqrt{p}\,e^{ipu}\alpha_{\pm}(p)dp+\left(\frac{c}{2\pi}\right)^{1/2}\int_{0<p<\Lambda_{\pm}}\sqrt{p}\,e^{-ipu}\alpha_{\pm}^{\dagger}(p)dp\,.$$

Under Lorentz transformations and dilatations, the field $\psi_{\underline{\Lambda}}(u,v)$ transforms according to [compare (2.15) and (2.17)]

$$e^{iM\theta}\psi_{\underline{\Lambda}}(u,v)e^{-iM\theta}=\exp\left(\frac{(a+\gamma_5\bar{a})^2-(a-\gamma_5\bar{a})^2}{8\pi c}\theta\right)\psi_{\underline{\Lambda}'}(e^{\theta}u,e^{-\theta}v)\,, \tag{5.14a}$$

$$e^{iD\lambda}\psi_{\underline{\Lambda}}(u,v)e^{-iD\lambda}=\exp\left(\frac{(a+\gamma_5\bar{a})^2+(a-\gamma_5\bar{a})^2}{8\pi c}\lambda\right)\psi_{e^{-\lambda}\underline{\Lambda}}(e^{\lambda}u,e^{\lambda}v)\,, \tag{5.14b}$$

where, in (5.14a), $\Lambda'=(e^{-\theta}\Lambda_+\,e^{\theta}\Lambda_-)$.

To prove (5.14a) one selects $J_{\pm}$ so that

$$\operatorname{supp}\tilde{J}_+\subset\{p\,|\,|p|<\Lambda_+\}\cap\{p\,|\,|p|<e^{-\theta}\Lambda_+\},\quad \operatorname{supp}\tilde{J}_-\subset\{p\,|\,|p|<\Lambda_-\}\cap\{p\,|\,|p|<e^{\theta}\Lambda_-\}.$$

One also uses

$$e^{iM\theta}U_{\underline{J}}=U_{\underline{J}'}e^{iM\theta},\quad J_{\pm}'(p)=J(e^{\pm\theta}p)\,. \tag{5.15}$$

Equation (5.14b) is proved similarly. We find again [compare (2.31)] that a "spinorlike" factor in (5.14) is only obtained if $a\bar{a}=(2n+1)2\pi c$.

One can now compute the product of any number of $\psi_{i,\underline{\Lambda}_i}$, $\psi^{\dagger}_{j,\underline{\Sigma}_j}$, $i,j=1,2$ possibly with different cutoffs. We may expect (the proof is outlined in Appendix B) that these (bounded) operators be approximants for the corresponding product of $\psi_i(u,v)$, $\psi_i^{\dagger}(u,v)$, $i=1,2$. One finds without difficulty that, e.g.,

$$\begin{aligned}\psi_{1,\underline{\Lambda}}(u,v)\psi^{\dagger}_{1,\underline{\Lambda}}(u',v')=F_{\Lambda}(u-u',v-v';J):\exp\Big(&\int[h_{\Lambda_+}(u,u'')+\bar{h}_{\Lambda_+}(u',u'')]j_+(u'')du''\\&+\int[h_{\Lambda_-}(v,v'')+\bar{h}_{\underline{\Lambda}}(v',v'')]j_-(v'')dv''\Big):,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\psi_{1,\underline{\Lambda}}(u,v)\psi_{2,\underline{\Lambda}}(u',v')=G_{\Lambda}(u-u',v-v';J):\exp\Big(&\int[h_{\Lambda_+}(u,u'')+h'_{\Lambda_+}(u',u'')]j_+(u'')du''\\&+\int[h_{\Lambda_-}(v,v'')+h'_{\Lambda_-}(v',v'')]j_-(v'')dv''\Big)V_{\underline{J}}U_{\underline{J}}:,\end{aligned} \tag{5.17}$$

where the : : symbol is defined as in (5.8). Here

$$\begin{aligned}&F_{\underline{\Lambda}}(u-u',v-v';J)=\exp\left(\frac{(a+\bar{a})^2}{4\pi c}\,\xi^+_{\Lambda_+}(u-u')+\frac{(a-\bar{a})^2}{4\pi c}\,\xi^-_{\Lambda_-}(v-v')\right)|\nu_0(J)|^2\,,\\&\xi^{\pm}_{\Lambda_{\pm}}(u)=\int_0^{\Lambda_{\pm}}[e^{i\kappa u}-|\tilde{J}_{\pm}(\kappa)|^2]\frac{d\kappa}{\kappa}\,,\\&G_{\underline{\Lambda}}(u-u',v-v';J)=\exp\left(\frac{\bar{a}^2-a^2}{4\pi c}[\,\xi^+_{\Lambda_+}(u-u')+\xi^-_{\Lambda_-}(v-v')]\right)\nu_0(J)\mu_0(J)\,,\\&h_{\Lambda_{\pm}}(u,u'')+\bar{h}_{\Lambda_{\pm}}(u',u'')=\frac{a+\bar{a}}{2ic}[\,\theta_{\Lambda_{\pm}}(u'-u'')-\theta_{\Lambda_{\pm}}(u-u'')]\,.\end{aligned} \tag{5.18}$$

In analogy with Eq. (4.13a) we define

$$R[\psi_{1,\underline{\Lambda}}(u,v)\psi_{2,\underline{\Lambda}}(u',v')]\equiv\psi_{1,\underline{\Lambda}}(u,v)\psi_{2,\underline{\Lambda}}(u',v')G_{\underline{\Lambda}}^{-1}(u-u',v-v';\underline{J})\,.$$

One can now see [compare (5.16) and (5.17) with (5.11)] that the proof of convergence, when $\Lambda_+,\Lambda_-\to\infty$ of $\psi_{1,\Lambda}(u,v)\psi_{2,\Lambda}(u,v')$ is no more difficult than the proof of convergence of $\psi_{1,\Lambda}$ itself; one must only notice that [compare (4.7) and (4.13a)]

$$\lim_{\Lambda_{\pm}\to\infty}G_{\underline{\Lambda}}(u-u',v-v')=\{[i(u-u')+\epsilon][i(v-v')+\epsilon]\}^{(a^2-\bar{a}^2)/4\pi c}C(\underline{J}).$$

Only slightly more complicated is the proof that, on a suitable domain

$$\psi_{1,\underline{\Lambda}}(u,v)\psi_{2,\underline{\Lambda}}(u',v') \xrightarrow{s} \psi_1(u,v)\psi_2(u',v') \qquad (5.19)$$

(in the sense of distributions), and that the (strong) limit of $R[\psi_{1,\underline{\Lambda}}(u,v)\psi_{2,\underline{\Lambda}}(u',v')]$ is "smooth" at $u=u'$, $v=v'$ in the sense that, taking expectation values between vectors from a suitable dense set in $\mathcal{H}$, one obtains many times differentiable functions of $u-u'$, $v-v'$ at $u=u'$, $v=v'$.

One recovers therefore the behavior of the product $\psi_1(u,v)\psi_2(u',v')$ at the light cone and at the origin [compare (4.7) and (4.13a)]; one can also prove that $R[\psi_1(u,v)\psi_2(u,v)]$ is a local (distribution-valued) operator. Similar remarks apply for $\psi_1(u,v)\psi_1^\dagger(u',v')$ (notice that here $R[\psi_1(u,v)\psi_1^\dagger(u,v)] = C$) and in general for any product of any number of fields ψ_1, $\psi_1^\dagger$, ψ_2, and $\psi_2^\dagger$ and for their behavior when some of the coordinates become lightlike or coincide. The results for the behavior at the light cone coincide of course with the ones given in Sec. IV. Convergence proofs are outlined in Appendix B; accepting them here, it follows from (5.14a) and (5.14b) that the field $\psi(u,v)$ has the right transformation properties under Lorentz transformations and dilatations. From (5.13) it also follows that $\psi(u,v)$ satisfies the correct Thirring equations, since, from (5.16), it follows that (4.10a) and (4.10b) hold as an operator equation [on a dense domain on which $\int j_+(u)f(u)du$, $\int j_-(v)g(v)dv$ as defined by (4.10a) and (4.10b), coincide with the operators which appear in (2.3), and determine them uniquely].

The product of any number of fields $\psi(f)$, $\psi^\dagger(g)$ can be defined, on a dense domain in $\mathcal{H}$ containing the vacuum. Therefore the Wightman functions exist, and coincide with the ones obtained in Sec. IV (and in Klaiber's paper[5]). Since we work consistently within the Hilbert space $\mathcal{H}$ which has positive metric, this result provides an independent proof that Klaiber's kernels are of positive type.

Finally, since much interesting work has been done recently in the general area of "reconstructing the fields from the observables," which is in essence what has been done in this section for the Thirring model, a few comments are in order. If J is a bounded interval on the real line, we denote by $a(J)$ the smallest norm-closed-algebra (see, e.g., Ref. 25) of operators which contain all $e^{i\phi(\tilde f)}$, $e^{i\pi(\tilde f)}$ support $\tilde f \subset J$. We denote by $a(J^\perp)$ the smallest norm-closed-algebra of operators which contains all $a(J')$, $J'\cap J=\phi$. We denote by $a^{\underline{n}}(J)$ $[a^{\underline{n}}(J^\perp)]$ the restriction of $a(J)$ $[a(J^\perp)]$ to $\mathcal{H}^{\underline{n}}$. From (5.2) one sees that, if $\operatorname{supp}\tilde J_\pm \subset J$, $\operatorname{supp}\tilde f \subset J^\perp$

$$U_{\underline{J}}^{\kappa} V_{\underline{J}}^{h} \begin{Bmatrix} \phi^{\underline{n}}(\tilde f) \\ \pi^{\underline{n}}(\tilde f) \end{Bmatrix} = \begin{Bmatrix} \phi^{\underline{n}-\kappa\underline{\delta}^1-h\underline{\delta}^2}(\tilde f) \\ \pi^{\underline{n}-\kappa\underline{\delta}^1-h\underline{\delta}^2}(\tilde f) \end{Bmatrix} U_{\underline{J}}^{\kappa} V_{\underline{J}}^{h}, \quad \kappa, h \in Z$$

and therefore U_J and V_J provide a "canonical" unitary equivalence between the $a^{\underline{n}}(J)$ for different $\underline{n}$. However, since from (5.2) for any $\tilde f$,

$$U_{\underline{J}}^{\kappa} V_{\underline{J}}^{h} \begin{Bmatrix} \phi^{\underline{n}}(\tilde f) \\ \pi^{\underline{n}}(\tilde f) \end{Bmatrix} = \begin{Bmatrix} \phi^{\underline{n}-\kappa\underline{\delta}^1-h\underline{\delta}^2}(\tilde f)+c_1(\tilde f)I \\ \pi^{\underline{n}-\kappa\underline{\delta}^1-h\underline{\delta}^2}(\tilde f)+c_2(\tilde f)I \end{Bmatrix} U_{\underline{J}}^{\kappa} V_{\underline{J}}^{h}$$

and since $I\in a(J)$ for all J, it is seen that the $U_{\underline{J}}$ and $V_{\underline{J}}$ provide a unitary equivalence between $a^{\underline{n}}(J')$ and $a^{\underline{n}'}(J')$, for any pair $\underline{n}',\underline{n}$, even if $J\cap J'\neq\phi$. Therefore, also the commutants $[a^{\underline{n}}(J)]'$ and $[a^{\underline{n}'}(J)]'$ are unitarily equivalent, for each J, and one has what is called "strong unitary equivalence."[27] Still, as we have seen, one can construct a countable number of inequivalent Bose fields $\psi^{(2N)}(u,v)$ (corresponding to $a\bar a = 2\pi Nc$) and a countable number of inequivalent Fermi fields $\psi^{(2N+1)}(u,v)$, each of which (a) is local relative to the observables [this follows from (2.9a) and (2.9b)], (b) covariant, (c) is irreducible, and (d) admits precisely $[a(R_1)]''$ as the set of observables.[28]

VI. CONCLUSION AND COMMENTS

The Thirring model is conventionally defined in terms of the equations of motion, whereas we have expressed our solution in terms of c, a, and $\bar a$ which appear in the commutation relations. The relation between the two is

$$a-\bar a = gc\,, \qquad (6.1)$$

to agree with the equation of motion, and

$$\frac{(a+\bar a)^2}{4\pi c} - \frac{(a-\bar a)^2}{4\pi c} = 1\,, \qquad (6.2)$$

to ensure a spinor transformation law. The choice of c is a convention which normalizes the currents to have strength $c^{1/2}$, as seen by Eqs. (2.3) and (4.3). Since the combination gj is what appears in the equation of motion, it is the quantity $gc^{1/2}$ or g^2c that has physical meaning and which measures the strength of the coupling. The solution of Johnson[4] corresponds to the choice

$$c = \frac{1}{\pi[1-(g/2\pi)^2]}\,. \qquad (6.3)$$

We would also like to note that requiring the combination

$$\psi_1(u,v)\psi_1^\dagger(u',v') + \lambda\psi_1^\dagger(u',v')\psi_1(u,v)$$

to vanish for spacelike separations, $(u-u')(v-v') < 0$, we get that

$$[(a+\bar a)^2 - (a-\bar a)^2](4\pi c)^{-1} = n$$

has to be an integer with $\lambda = (-1)^{n+1}$. Thus odd values of n correspond to anticommutators and even values to commutators, which is an instance of the connection between spin and statistics. Our final results are

$$\psi_1(u,v)\psi_1^\dagger(u',v')=\frac{f_0}{[i(u-u')+\epsilon][i(u-u')+\epsilon]^{g^2c/4\pi}[i(v-v')+\epsilon]^{g^2c/4\pi}}$$
$$\times :\exp\left[-i\left(\frac{g^2}{4}+\frac{\pi}{c}\right)^{1/2}\int_{u'}^{u} j_+(\xi)d\xi - i\frac{g}{2}\int_{v'}^{v} j_-(\eta)d\eta\right]:,$$
$$\psi_1^\dagger(u',v')\psi_1(u,v)=\frac{f_0}{[i(u'-u)+\epsilon][i(u'-u+\epsilon]^{g^2c/4\pi}[i(v'-v)+\epsilon]^{g^2c/4\pi}}$$
$$\times :\exp\left[-i\left(\frac{g^2}{4}+\frac{\pi}{c}\right)^{1/2}\int_{u'}^{u} j_+(\xi)d\xi - i\frac{g}{2}\int_{v'}^{v} j_-(\eta)d\eta\right]:, \tag{6.4}$$

and similarly for ψ_2 with $u \leftrightarrow v$ and $j_+ \leftrightarrow j_-$. We note that the singularity may be reexpressed in terms of the invariant separation

$$[i(u-u')+\epsilon]^{-g^2c/4\pi}[i(v-v')+\epsilon]^{-g^2c/4\pi}=(-x^2+i\epsilon x^0)^{-g^2c/4\pi} \tag{6.5}$$

with $2x^0=(u-u')+(v-v')$, $2x^1=(u-u')-(v-v')$, and $x^2=(x^0)^2-(x^1)^2$.

We have here a very nice example of an exact short-distance expansion, as suggested by Wilson[2,3] and an exact light-cone expansion, as postulated recently by several authors.[6-8]

As for short distances, the most singular matrix element is the vacuum expectation value. The singularities of all other matrix elements are connected by one regular operator. The light-cone expansion here contains one singular function only, and one regular bilocal operator. Such a structure supports the postulates in Refs. 6 and 7. We note that the singularity on the light cone is a dynamical one, depending on the interaction.

The Thirring model as originally formulated was a theory of a spinor field, in which the current was defined as a product or, more precisely, as the limit of a product of spinor fields. Our method may be summarized as the reversal of this procedure. We began with the current and its properties, and expressed the bilocal operators $\psi_1(x)\psi_1^\dagger(x')$ and $\psi_2(x)\psi_2^\dagger(x')$ in terms of the current. Ultimately the charged spinor field itself was constructed as an operator which intertwines different inequivalent representations of the current commutation relations.

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APPENDIX A: PROOF OF LEMMA 1

To avoid a cumbersome notation, which would obscure the simplicity of the proof, we shall concentrate on the case $n=(1,0)\equiv\underline{\delta}^1$. The proof of the general case follows along the same lines. We also neglect all indices and write

$$A_\pm(f)=\alpha_\pm(f)\left(\frac{c}{8\pi}\right)^{1/2}-(a\pm\bar{a})\int\frac{\tilde{f}(p)}{\sqrt{p}}\tilde{J}_\pm(p)dp\,.$$

The irreducibility of the $A_\pm$'s is obvious, given that of the $\alpha_\pm$'s. We notice that $\mathcal{H}_{\delta^1}$ contains all vectors of the form

$$\int\psi_1^\dagger(u,v)f(u)g(v)dudv'\,\Omega\equiv|f,g\rangle,\quad f,g\in S\,. \tag{A1}$$

[Recall that we assume at this stage that $\psi(u,v)$ exists, and obtain necessary conditions on the representations $R_{\underline{n}}$. Later on, we shall prove that these conditions are also sufficient for the existence of the field $\psi(u,v)$.]

We now compute the expectation value of the number operators N_+,N_- (associated to $A_\pm$) in a vector of the form (A1) and show that it is finite. According to Ref. 29, this proves the assertion in the Lemma.

Using (5.2) one can write

$$\langle f,g|(N_+ + N_-)|f,g\rangle = \int_{\kappa>0} d\kappa \int dp\, dq\, dp'\, dq'\, \bar{\tilde{f}}(p)\tilde{f}(p')\bar{\tilde{g}}(q)\tilde{g}(q') \times \langle \Omega, \tilde{\psi}_1(p,q)[A_+^\dagger(\kappa)A_+(\kappa) + A_-^\dagger(\kappa)A_-(\kappa)]\psi_1^\dagger(p',q')\Omega\rangle . \tag{A2}$$

Using the commutation properties of $A_\pm$ and ψ_1, one rewrites (A2) as the sum of two terms (corresponding to $\pm$),

$$\frac{(a\pm\bar{a})^2}{4}\int |\tilde{g}(q)|^2 \frac{1}{\kappa}[\bar{\tilde{f}}(p+\kappa)\tilde{f}(p+\kappa) - \tilde{J}_\pm(\kappa)\bar{\tilde{f}}(p)\tilde{f}(p+\kappa) - \tilde{J}_\pm(\kappa)\bar{\tilde{f}}(p+\kappa)\tilde{f}(p) + |\tilde{f}(p)|^2|J_\pm(\kappa)|^2]\tilde{G}(p,q)dp\, d\kappa\, dq\,, \tag{A3}$$

where for the $(-)$ case one has also yet to interchange f and g.

If one takes $\tilde{g},\tilde{f}\in D$, the coefficient of $\tilde{G}(p,q)$ in the integral is in $D_{(p)}\times D_{(q)}\times L^1_{(\kappa)}$, since $\tilde{J}_\pm(\kappa)$ is Hölder-continuous at $\kappa=0$ of order $\epsilon>0$. The kernel $\tilde{G}(p,q)$ is given explicitly in (3.8) and is seen to be in $S'_{(p)}\times S'_{(q)}$; the integral in (A3) is therefore unconditionally and absolutely convergent. It follows that $\langle f,g|(N_+ + N_-)|f,g\rangle<\infty$. This concludes the proof of Lemma 1, for the case $\underline{n}=\underline{\delta}^1$.

The general case is treated along the same lines, using instead of (A1) vectors of the form

$$\psi_1^\dagger(f_1,g_1)\psi_2^\dagger(f'_m,g'_m)|\Omega\rangle \equiv |(f_1,g_1)\ldots,\ \ldots(f'_m,g'_m)\rangle . \tag{A4}$$

APPENDIX B

We plan to outline here the proof of the strong convergence (on a suitable domain, and in the sense of distributions) of the sequence of operators $\psi_{1,\underline{\Lambda}},\psi_{2,\underline{\Lambda}}$ (and of their products) when $\Lambda_+,\Lambda_-\to\infty$. To ensure that the essential steps in the proof are not obscured by an elaborated notation, we shall give details only for the convergence of $\{\psi_{1,\underline{\Lambda}}\}$ on the vacuum Ω, and take $\Lambda_+=\Lambda_-=\Lambda$. We consider therefore the sequence of vectors in $\mathcal{H}$,

$$\int \psi_{1,\Lambda}(u,v)f(u,v)du\,dv\,\Omega \equiv |f\rangle_\Lambda\,, \tag{B1}$$

and prove that it converges in the topology of $\mathcal{H}$ (and therefore to a vector in $\mathcal{H}$, since $\mathcal{H}$ is complete), for fixed $f\in S$. We shall use the decomposition, for $\Lambda<\Sigma$,

$$\mathcal{H}_\Sigma = \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Sigma|\Lambda}\,, \tag{B2a}$$

and correspondingly

$$\psi_{1,\Sigma}(u,v) = \psi_{1,\Lambda}(u,v)\otimes\psi_{1\ \Sigma|\Lambda}(u,v)\,. \tag{B2b}$$

Equation (B2a) formalizes the fact that $\mathcal{H}_{\underline{n}}$ is, for each $\underline{n}$, a Fock space for $A^{\underline{n}}_{\underline{J},\pm}$ and has a corresponding tensor-product decomposition relative to subsets of independent "degrees of freedom." Equation (B2b) follows from (B2a) and from the fact that $U_{\underline{J}}$ maps $\mathcal{H}^{(\underline{\Lambda})}_{\underline{n}}$ onto $\mathcal{H}^{(\underline{\Lambda})}_{\underline{n}-\underline{\delta}^1}$. Because of (B2b), one has, for every $\Sigma>\Lambda$,

$$F_\Sigma(u,v) = F_\Lambda(u,v)F_{\Sigma|\Lambda}(u,v)\,,$$

where, setting,[30]

$$\rho = \frac{(a+\bar{a})^2}{4\pi c}\,,\quad \sigma = \frac{(a-\bar{a})^2}{4\pi c}\,,$$

$$F_{\Sigma|\Lambda}(u,v) = \exp\left(\rho\int_\Lambda^\Sigma \frac{dp}{p}e^{ipu} + \sigma\int_\Lambda^\Sigma \frac{dp}{p}e^{ipv}\right). \tag{B3}$$

Since $\mathfrak{F}_{\Lambda_0}(u,v)$ is easily seen to be bounded and infinitely differentiable for fixed Λ_0, $0<\Lambda_0<\infty$, we shall only have to prove that

$$\int [F_{\Sigma|\Lambda_0}(u,v) - F_{\Lambda|\Lambda_0}(u,v)]X(u,v)du\,dv \equiv M_{\Sigma,\Lambda}(X) \tag{B4}$$

converges to zero when $\Lambda\to\infty$, $\Sigma>\Lambda$ for every $X\in S$. One has

$$M_{\Sigma,\Lambda}(X) = \sum_{\substack{n,m>0;\\ n+m\geq 1}} \frac{\rho^m}{m!}\frac{\sigma^n}{n!}\int_{R^{\Sigma|\Lambda_0}_{m,n} - R^{\Lambda|\Lambda_0}_{m,n}} dp_1\cdots dq_n \frac{\tilde{X}(p_1+\cdots+p_m, q_1+\cdots+q_n)}{p_1\cdots q_n}\,, \tag{B5}$$

where $R^{\Sigma|\Lambda_0}_{m,n}$ is the subset of R^{m+n}_+ defined by

$$\Lambda_0\leq p_i<\Sigma,\quad \Lambda_0\leq q_j<\Sigma,\quad i=1,\ldots,m,\quad j=1,\ldots,n.$$

It is convenient to write $M_{\Sigma,\Lambda}(X)$ as the sum of three series, corresponding to $m=0$; $n=0$; $m,n\geq 1$, respectively, each of which is shown to converge to zero when $\Lambda\to\infty$. We shall outline the proof for the third series ($m,n\geq 1$), denoted by $M^{(3)}_{\Sigma,\Lambda}(X)$. Using the inequality

$$\left(\frac{p_1+\cdots+p_m}{m}\right)^N \geq (p_1{}^N\cdots p_m{}^N)^{1/m}$$

valid for all $p_i \in R_+$; $m, n \in Z_+$ one has, for every $N>0$,

$$|M^{(3)}_{\Sigma,\Lambda}(X)| \leq K_N \sum_{m,n\geq 1} \frac{\rho^m}{m!}\frac{\sigma^n}{n!}\int\cdots \times\int \frac{dp_1}{p_1{}^{1+N/m}}\cdots\frac{dq_n}{q_n{}^{1+N/n}}\frac{1}{m^N n^N}, \tag{B6}$$

where

$$K_N \equiv \sup_{\xi,\eta} |\xi^N \eta^N \tilde{X}(\xi,\eta)| < \infty .$$

On the right-hand side of (B6) the integrand is positive; the integral can then be majorized by the sum of the integrals over the regions $O_i, \underline{O}_j$ defined by

$$O_i: \{\Lambda \leq p_i < \Sigma; \Lambda_0 \leq p_k, K \neq i; \Lambda_0 < q_j \ \forall j\},$$

$$\underline{O}_j: \{\Lambda < q_j \leq \Sigma; \Lambda_0 \leq q_k, k\neq j; \Lambda_0 < p_i \ \forall i\}.$$

The right-hand side of (B6) is then majorized by

$$K_N \sum_{m,n\geq 1} \frac{\rho^m}{m!}\frac{\sigma^n}{n!}\frac{1}{m^N n^N}\left[m\int_\Lambda^\Sigma \frac{dp}{p^{1+N/m}}\left(\int_{\Lambda_0}^\infty \frac{dp}{p^{1+N/m}}\right)^{m-1} \times\left(\int_{\Lambda_0}^\infty \frac{dq}{q^{1+N/n}}\right)^n + n \longleftrightarrow m\right]$$

which, in turn, is majorized by

$$K_N \sum_{m,n\geq 1} \frac{\rho^m}{m!}\frac{\sigma^n}{n!}\frac{1}{m^N n^N}\left[\frac{m^{m+1}}{N^m}\frac{n^n}{N^n}\left(\frac{\Lambda_0}{\Lambda}\right)^{N/m}\frac{1}{\Lambda_0^N} + n \longleftrightarrow m\right]. \tag{B7}$$

The series in (B7) is, for N sufficiently large (depending on ρ, σ) absolutely convergent, uniformly in Λ for $\Lambda > \epsilon > 0$. Each term in the series converges to zero when $\Lambda \to \infty$; the series converges therefore to zero when $\Lambda \to \infty$. We have in this way established that, when $\Lambda \to \infty$, the vectors

$$\int \psi_{1,\Lambda}(u,v) f(u,v) du\, dv\, \Omega$$

form a Cauchy sequence. For all $f \in S$ this Cauchy sequence defines a vector which we shall denote by

$$\int \psi_1(u,v) f(u,v) du\, dv\, \Omega;$$

linearity and continuity in f are easily demonstrated using (B7) and the corresponding properties of

$$\int \psi_{1,\Lambda}(u,v) f(u,v) du\, dv\, \Omega$$

(a crucial remark is that N, K_N can be chosen to be independent of g for all g in some neighborhood of f, in the topology of S). In the same way, one proves the sequence

$$\int \psi_{1,\Lambda}(u,v) f(u,v) du\, dv\, |X\rangle$$

converges strongly, when $\Lambda \to \infty$, for all $X \in D \equiv U_{\underline{n},p,m} D^{\underline{n}}_{m,p}$.

Here $D^{\underline{n}}_{m,p}$ denotes the set of finite linear combinations of vectors obtained by applying a polynomial of order $\underline{p}$ in the $\alpha^\dagger_\pm(f)$, $f \in \mathfrak{L}_2$ to coherent vectors $\|g)\!)$ for which $\int |g(p)|^2 dp < \infty$.

Since convergence can be proved to be uniform on each $D^{\underline{n}}_{m,p}$, uniformly in a sufficiently small neighborhood of f, it follows that these Cauchy sequences define on D a closed operator, continuous in f, which we shall denote by

$$\int \psi_1(u,v) f(u,v) du\, dv .$$

In the same way, on the same domain, one can define the closed operator

$$\int \psi_2(u,v) f(u,v) du\, dv .$$

Finally, through majorizations similar to the ones given in (B5), one proves that the vectors

$$\psi^*_{i_1,\Lambda_{i1}}(f_1)\cdots\psi^*_{i_n,\Lambda_{in}}(f_n)\Omega, \quad \psi^* = \psi, \psi^\dagger, i = 1, 2 \tag{B8}$$

converge, when $\Lambda_{i_1}, \ldots, \Lambda_{i_k} \to \infty$ $(1 \leq k \leq n)$, uniformly in the remaining Λ_j's.

This extends the domain of definition of the operator $\int \psi(u,v) f(u,v) du\, dv$ to a domain $\mathfrak{M}$, dense in $\mathcal{H}$ and invariant under action of $\psi^*(f)$ and of $a^*_\pm(g)$, $g \in \mathfrak{L}_2$; also, the uniform convergence of the sequence in (B8) shows that its limit, on $\mathfrak{M}$, is precisely the product of the limits of the component factors.

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[15]This is the only place in the article where normal ordering refers to the positive- and negative-frequency components of the free-spinor field rather than to the frequency components of the currents j.

[16]Use of the vector field (4.3) avoids the technical infrared difficulties of the scalar field, introduced in Ref. 5, whose derivative would be (4.3).

[17]To be consistent with this viewpoint the normal ordering in Eq. (2.14) should be replaced by, e.g.,

$$:j_+(u)\psi_2(u,v):=\lim_{l\to 0}\tfrac{1}{2}[j_+(u+l)\psi_2(u,v)+\psi_2(u,v)j_+(u-l)]$$

as may be verified using the commutator of $j^{(\pm)}$ with ψ.

[18]In an interesting recent work, A. H. Mueller and T. L. Trueman, Phys. Rev. D 4, 1635 (1971), argue that anomalous dimensions in the massive Thirring model are the same as in the massless model, but that the anomalous dimension of ψ changes with the gauge transformation of the massive Thirring model $\psi \to \exp(-ig\chi)\psi$. They conclude that the anomalous dimension is without physical significance. However, the local fields $R[\bar\psi\psi]=R[\psi_2^\dagger\psi_1]+R[\psi_1^\dagger\psi_2]$ and $R[\bar\psi\gamma_5\psi]=R[\psi_2^\dagger\psi_1]-R[\psi_1^\dagger\psi_2]$ have anomalous dimension $\bar a/a$ and are invariant under the gauge transformation. So if the limit is smooth as the mass goes to zero, the massive Thirring model has an observable anomalous dimension because $\partial_\mu\epsilon^{\mu\nu}j_\nu$ is proportional to $R[\bar\psi\gamma_5\psi]$.

[19]This section and the related Appendixes (A and B) are primarily the work of one of us (G.F.D.A.).

[20]We shall not distinguish between $\mathcal{H}_{\underline{a}}$ and its canonical image as a subspace of $\mathcal{H}$.

[21]We shall always by this wording refer to the Weyl algebra.

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[23]A coherent $\|g))_{\underline{n},\pm}$ is a state which satisfies

$$A^{\underline{n}}_{\underline{J},\pm}(h)\|g))_{\underline{n},\pm}=(g,h)\|g))_{\underline{n},\pm}.$$

[24]Ω^n_J is the (unique) vector in $\mathcal{H}_n$ such that $A^{\underline{n}}_{\underline{J},\pm}(f)\Omega^{\underline{n}}_{\underline{J}}=0, \forall f$.

[25]Diximier, *C*-Algebras* (Gouthier-Villars, 1969), Proposition 5.1.2.

[26]It should be noted that, for $\mathcal{H}_{\underline{a}}=\mathcal{H}_{\alpha+n_1\underline{e}^1+n_2\underline{e}^2}$, $\alpha\in R\times R$, one can in the same way construct a field $\psi(x;\underline{\alpha})$ solution of the Thirring equations. The "solution" $\psi(x;\underline{\alpha})$ is not equivalent to $\psi(x)\equiv\psi(x;\underline{0})$ if $\underline{\alpha}+n_1\underline{e}^1+n_2\underline{e}^2\neq 0$ for any choice of $n_1, n_2\in Z$. In fact, both fields will be irreducible, but in the Hilbert space of $\psi(x;\underline{\alpha})$ there is no vector invariant under space-time translations.

[27]We do not know whether strong duality, i.e., $[a^{\underline{n}}(J)]'=[a^{\underline{n}}(J)]''$ holds.

[28]This last property follows from Lemma 1, Remark 3. See also the paragraphs immediatly preceding Eq. (5.3), and Ref. 25.

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[30]We take $\tilde J_\pm(p)$ to vanish for $p\geq\Lambda$.

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A Basic Discontinuity Equation*

Kevin E. Cahill†
Department of Physics, Wesleyan University, Middletown, Connecticut 06457

and

Henry P. Stapp
Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 31 March 1972)

We derive from field theory a formula that expresses the discontinuity of any multiparticle scattering function across any normal threshold cut in terms of specified limits of scattering functions for other processes. The special case that expresses any inclusive cross section as a discontinuity has been used extensively in recent work on high-energy processes. Other cases of the general formula also appear to have important implications, which are briefly discussed.

I. INTRODUCTION

Recent studies of high-energy processes based on the work of Mueller have exploited a formula that expresses any inclusive cross section as a discontinuity in an appropraite multiparticle scattering function.[1] This formula is a special case of the general discontinuity equation displayed in Fig. 1. We describe this equation in detail in the next section. It has been discussed earlier