conjugate momentum. A vertical stroke denotes covariant differentiation in the metric g_{ij}. The label x in $\mathcal{H}_x$ and $\mathcal{H}_x^i$ means that the right-hand side of Eqs. (3) and (4) are evaluated at the point x. For details see, for example, R. Arnowitt, S. Deser, and C. W. Misner in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

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Galaxy Formation in Anisotropic Cosmologies*

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We analyze the growth rates of perturbations of the generic dust-filled Bianchi type-I cosmology (which exhibits anisotropy but not rotation). Anisotropy induces coupling between gravitational wave and density modes and can enhance the power-law rate of growth of the density perturbations. A maximum growth rate for pregalaxy perturbations is $t^{8/3}$ (where t is cosmic time), so that no conclusive solution to the galaxy formation problem is found. The detailed structure of the flow of energy between gravitational wave and density modes and of the growth of rotational modes (decoupled from other perturbations) is presented. We also give a detailed discussion of the gauge-invariance properties of these perturbations.

I. INTRODUCTION

Following the pioneering work of Lifshitz[1] and Lifshitz and Khalatnikov,[2] several authors have considered the problem of perturbations of isotropic, homogeneous cosmological models.[3-7] The hope has been to find an instability leading to the gravitational growth of perturbations analogous to the Jeans instability[8] for stationary Newtonian systems. It is now well recognized that no exponential (in time) growth of the Jeans type can occur in isotropic models, but instead the growth is limited to a power of the cosmic time t. For instance, in the "flat" ($k=0$) dust (pressureless perfect fluid) Friedmann[9,10]-Robertson[11,12]-Walker[13] (FRW) model, the rate of growth of the quantity $\delta w/w$ is proportional to $t^{2/3}$ (where $\delta w/w$ is the relative perturbation in the energy density, measured by a comoving observer).

With the recent renewed interest[14-18] in anisotropic homogeneous cosmologies, it seems a natural step to consider perturbations in such models. The generic strongly anisotropic model has the property that in at least one direction there is no expansion near the initial singularity. This is true even though the over-all volume of the universe is increasing because of expansion in other directions. This lack of expansion in one direction can be expected to strongly affect the growth of perturbations. We have found such effects, and find they result in density growth which can be faster than in the isotropic case. In some dust models of the type considered here, $\delta w/w$ is proportional to t^{ν} with ν arbitrarily close to $\frac{8}{3}$.

A power-law density growth as found here is not spectacularly different from the isotropic case.

Hence the problem of galaxy formation in isotropic (FRW) models is not solved in any of our models, strictly speaking. However, the more vigorous growth rate exhibited here (which can yield rather remarkably large net growths over a finite period) may signal the existence of even stronger growth of perturbations in more realistic models not yet discussed in the literature. In this sense, our results yield hope and encouragement that the galaxy formation problem will be solved within the classical context, in that exponentially growing small perturbations might be found in such "more realistic" models.

The alternate viewpoint, in our opinion the viewpoint much more likely to be true, accepts rather modest t^{ν} growth rates as inevitable in any general-relativistic cosmological model. The problem at hand is then to reconcile the experimental fact of the existence of galaxies with the slow power-law growth rate. This problem is essentially unchanged between the isotropic and anisotropic models.

Doroshkevich, Zel'dovich, and Novikov[19] have given a discussion of part of the problem we consider here. Their treatment concentrated on the density perturbation and was based on a Newtonian formulation of the expected behavior. Their treatment gives very clear insight into the physics, so long as the velocities associated with the perturbation are small. This paper will concentrate on a full relativistic treatment of the problem, showing the effects of the density–gravitational-radiation interactions, and discussing the relativistic gauge-invariance problem (which does not arise in a nonrelativistic treatment).

II. GALAXY FORMATION IN ISOTROPIC MODELS

In an isotropic (FRW) model, an acceptable solution to the galaxy problem is a perturbation $\delta w/w \sim 10^{-2}$ at the time hydrogen recombines (at a temperature of $\sim 2\times10^{3}$ °K). At this time the age of the model is $\sim 3\times10^{5}$ years, and the horizon size holds about 10^{4} galactic mass in baryons. After recombination, there is essentially no pressure to oppose condensation growth and condensations the size of clusters of galaxies (and presumably galaxies also by fragmentation) form in $\sim 10^{8}$ years. The $t^{2/3}$ growth rate for condensations in an FRW model requires a time ratio of $\sim 10^{3}$ for a 10^{2} factor growth in $\delta w/w$, and a density contrast of unity is usually taken as the onset of rapid collapse due to nonlinear effects. The source of a perturbation of magnitude $\delta w/w \sim 10^{-2}$, or of the growth rate which produces this perturbation, is the galaxy formation problem we discuss here. We note that a realistic model, to take account of the observed isotropy of the 3 °K microwave radiation, must be very isotropic at and after the time of the last scattering of the microwave radiation.[14] If, as usual, we take this instant to be the recombination time, then anisotropy must have been small when the hydrogen recombines: The isotropic $t^{2/3}$ law therefore must hold after that time. (We will always quote the parameters appropriate for dust models, for simplicity.)

A 1% fluctuation on the order of the size enclosing one galaxy ($\sim 10^{70}$ particles) clearly cannot be a random fluctuation. It is of course possible that the random fluctuations occurred much earlier, when there was time enough for a $\delta w/w$ perturbation to grow from one of order 10^{-34} to one of order unity. This viewpoint accepts that the formation of galaxies is closely tied up with the beginnings of the universe itself, and that special initial conditions are necessary for the creation of galaxies of the size now in existence.

However, the initial singularity may be a chaotic state, as the idea of Misner[20] suggests. The galaxy formation problem then becomes one of finding mechanisms which damp out all but the modes of perturbation that lead to the observed scale of condensations. It is known, for instance, that a perturbation which has wavelength less than a horizon size oscillates if the pressure does not vanish.[21] In a realistic model, the damping processes of shock waves, etc. are expected to reduce the amplitude of this oscillating wave. A perturbation with wavelength longer than the horizon size, regardless of pressure terms, grows until the horizon has grown to include it, since no restoring forces have time to act until the horizon allows communication. This perturbed region acts like a dust FRW model, evolving independently of its surroundings. Thus, if horizons exist, the galaxy formation problem can be approached by a search for the dissipative and damping mechanisms which would select out particular condensation sizes. Recent discussions of this type have been given by Rees[22] and Peebles and Yu[23] and in references they give.

A comment should be made concerning initial conditions (that is, the spectrum of perturbations which exist near the initial singularity) with regard to the existence of horizons. Appeal is usually made to the statistical fluctuation spectrum as a function of the number, N, of particles: $\delta N/N \sim N^{-1/2}$. When the region under consideration is in good thermal equilibrium (in which case the region must be smaller than the horizon size), we can reasonably expect fluctuations of such a spectrum. But, if the region is of a scale much larger than a horizon size, we can take one of two viewpoints. If we assume the universe is in fact uniform even on a very large scale, then probabilistic arguments still lead to the estimate $\delta N/N \sim N^{-1/2}$. However, on such a scale, where there is no causal contact between parts of the region, there is also no justification for such a uniformity assumption. The perturbations on a scale larger than a horizon scale can be very large, and it is possible that a new range of variation is continually brought into play as the horizon expands (the chaotic cosmology).

Even with regard to the question of fluctuations within the horizon size, there is no justification for an assumption of thermal equilibrium. Thermal instabilities can play a very important role, resulting in perturbations much larger than the statistical perturbations $\delta N/N \sim N^{-1/2}$. These instabilities may dominate gravitational effects at later stages of the universe. We will not consider such perturbations, but only adiabatic ones, in which gravitational processes are dominant. As mentioned above, a reasonable model can have been strongly anisotropic only very early in its history, in accord with the isotropy of the black-body radiation observed now. Hence, the thermal instability effects which come into play during the large anisotropy phase of the universe can typically occur only on a very small length scale, since they can operate only within the horizon size. The one exception is the special model which has no horizon in some direction. In this case, thermal instabilities could be very important at all epochs in its history.

III. THE ANISOTROPIC MODEL

The anisotropic background model we use is of the Euclidean-homogeneous (Bianchi type-I) form[24,25]

$$ds^2 = -\frac{1}{\gamma}d\tau^2 + e^{2\alpha}e^{2\beta}{}_{ij}dx^i dx^j \tag{3.1}$$

and we work in the coordinate basis $\{d\tau, dx^i\}$. Here $\alpha = \alpha(\tau)$, and $\beta_{ij} = \beta_{ij}(\tau)$ is a 3×3 diagonal traceless matrix which describes the anisotropy. The function $\gamma = \gamma(\tau)$ is redundant and can be eliminated by changing the time coordinate to cosmic proper time t, defined by

$$dt = \gamma^{-1/2} d\tau .$$

The function γ is useful in understanding how equations behave under time-variable transformations, however, and we will keep γ in the formulation.

The unperturbed field equations are given in the Appendix. Here we merely present the solution. The dust background model (cosmological term $\Lambda = 0$) obeys

$$\beta_{ij} = b_{ij}u + c_{ij}, \tag{3.2}$$

with b_{ij} and c_{ij} traceless, diagonal arrays of constants. Here u is related to the proper time by

$$du = e^{-3\alpha}dt, \tag{3.3}$$

and the function α obeys

$$e^{-3\alpha/2} = -\left(\frac{w_0}{3b^2}\right)^{1/2} \sinh \tfrac{3}{2} bu \tag{3.4a}$$

$$= \left(\frac{3w_0}{4}\right)^{-1/2}\left[t\left(t + 4\frac{b}{w_0}\right)\right]^{-1/2}. \tag{3.4b}$$

Here

$$b^2 = \tfrac{1}{6} b_{ij} b_{ij}, \tag{3.5a}$$

and w_0 is defined by

$$T_{00} \equiv w = w_0 e^{-3\alpha}, \quad w_0 \text{ constant} \tag{3.5b}$$

giving the $R^{-3} = e^{-3\alpha}$ behavior of the energy density for dust. The integration constants are chosen in (3.3) and (3.4) so that $u = -\infty$ when $t = 0$, while $u = 0$ when $t = \infty$. The $T_{00} = G_{00}$ Einstein equation gives the behavior of the density w:

$$w = 3e^{-6\alpha}\left[\left(\frac{d\alpha}{du}\right)^2 - b^2\right] \tag{3.6}$$

$$= \tfrac{4}{3} t^{-1}\left(t + \frac{4b}{w_0}\right)^{-1}. \tag{3.7}$$

Equations (3.4b), (3.7), and (3.8) below show that except for an over-all scale factor in e^α which can be adjusted by a spatial coordinate (scale) change, the only constant entering the average expansion of the background model is the combination

$$t_b = 4b/w_0 .$$

This constant determines the epoch at which matter becomes dominant over the anisotropy in determining the expansion and is invariant under scale changes.

For each of the three eigenvalues β_i of the anisotropy matrix, we have

$$e^{\beta_i} = \left(\frac{t}{t + t_b}\right)^{s_i/3}, \tag{3.8a}$$

where

$$s_i = b_{ii}/b, \tag{3.8b}$$

and the constants c_{ij} have been set equal to zero by a spatial coordinate transformation. The parameters s_i determine the shape anisotropy of the model. They obey

$$s_1{}^2+s_2{}^2+s_3{}^2=6\,,\quad s_1+s_2+s_3=0\,. \tag{3.9}$$

Near the initial $t=0$ singularity, the background model is very anisotropic. At these times, $t \ll t_b$, the over-all expansion in the different directions is governed by

$$R_i \equiv e^{\alpha} e^{\beta}{}_i$$

$$\propto t^{(1+s_i)/3} \quad \text{(asymptotically as } t\to 0). \tag{3.10}$$

Equations (3.9) require that the s_i, if labeled in order of increasing size, obey

$$-2 \leqslant s_1 \leqslant -1 \leqslant s_2 \leqslant 1 \leqslant s_3 \leqslant 2\,. \tag{3.11}$$

(We will use other, more convenient, orderings of the s_i in the remaining sections of this paper.) We see from (3.10) that, except for the case $s_i=(-1,-1,2)$, there is always a direction in which there is actually *contraction* in the early anisotropic model $(t \ll t_b)$. In the case $s_i=(-1,-1,2)$ and in the case $s_i=(-2,1,1)$, the models are axisymmetric; and the perturbation scheme is simple to deal with, as we shall see below. The case $s_i=(-2,1,1)$ has an extreme blue shift (rapid contraction) in one direction. This case and the case $s_i=(-1,-1,2)$ limit the behavior of computer-generated results for general s_i.

IV. HORIZONS IN ANISOTROPIC MODELS

In the discussion of isotropic models, it was pointed out how crucial was the role of the horizon in determining when the growing phase of the perturbation stops and the oscillatory phase begins. Growth continues for all perturbations of size larger than the horizon size, but can be slowed or reversed by pressure or other effects when the horizon size becomes larger than the perturbation. The formula for the *coordinate* size of the horizon[26] in the anisotropic model with metric (3.1), in the ith coordinate direction, is

$$H_i=\int_0^t \frac{dt}{R_i} \quad (t=\text{proper time})$$

$$=\int_0^t e^{-\alpha} e^{-\beta}{}_i dt\,. \tag{4.1}$$

The coordinate size is relevant because the dust in our background model moves along lines $x^i=$ constant. Hence, the coordinate size directly measures the amount of mass contained within a horizon. The proper horizon size is related to H_i by

$$H_i^p=R_i H_i \quad \text{(no sum)}. \tag{4.2}$$

The integration defining H_i cannot in general be performed in closed form. However, it is clear that horizon sizes can be arbitrarily large, for certain directions at an early epoch.[27] For a direction with $s=2$, in the particular axisymmetric model which admits this value, the integral defining the horizon diverges at its lower limit. Following the discussion in Sec. II, we then expect there will be no growing modes (since every wavelength lies within the horizon); and we shall see in our discussion below that this is in fact the case. Waves with the same coordinate size (hence spanning approximately the same amount of matter) will become oscillatory at vastly different times, if their direction of propagation is different, in any one particular model, because of the wide range of horizon sizes possible. We will verify this directional effect in the next section and will see that we are correct in our implicit assumption in the statement above; that it is the horizon size in the direction of propagation of the perturbation which determines whether the perturbation is oscillatory.

V. THE PERTURBATION EQUATIONS

Our perturbation scheme uses a frame with $\delta g_{0\alpha}=0$, so that perturbed metric can be written

$$\begin{aligned} ds^2 &= -\frac{1}{\gamma}d\tau^2+e^{2\alpha}e^{2\beta}{}_{ij}(\delta^j{}_l+h^j{}_l)dx^i dx^l\,,\\ h^i{}_l &= h^i{}_l(\tau,x^k)\,. \end{aligned} \tag{5.1}$$

It is assumed that

$$e^{2\beta}{}_{ij}h^j{}_l=e^{2\beta}{}_{lj}h^j{}_i\,. \tag{5.2}$$

The gauge freedom still allowed will be discussed below.

The detailed calculation of the perturbed field equations is found in the Appendix. In this paper we will deal exclusively with dust perturbations of dust models, although formulas are given in the Appendix for general fluids. In isotropic models, the absence of pressure means that perturbations do not oscillate (no restoring forces) even when the scale is smaller than a horizon size. In our anisotropic models, there is a coupling between gravitational waves and density perturbations. This has the effect of giving an oscillatory character to the density perturbations once they are within the horizon size, with the curvature providing the springiness even if the pressure vanishes. Hence our consideration of only zero-pressure models has most of the qualitative features of a model with pressure, but the mathematics is simpler because the sound-wave modes do not enter.

The perturbed Einstein equations, Eqs. (A11)–(A13) in the Appendix, are Fourier analyzed to turn them into ordinary coupled differential equations. When a particular wave vector k_a is singled out, the perturbation $h^i{}_j$ is

$$h^i{}_j(x,\tau)=\mu^i{}_j(\tau)e^{ik_a x^a} \quad (k_a=\text{constant}). \tag{5.3}$$

The sum in the exponent is over three spatial dimensions, and k_1, k_2, k_3, are considered three time-independent constants (no metric enters the sum $k_a x^a$). All quantities appearing in the equations are Fourier analyzed in this way.

An analysis of the equations of motion shows that

$$(V_\perp)_i \propto e^{3\sigma\alpha} \propto R^{3\sigma},$$

where $(V_\perp)_i$ are the transverse Fourier-transformed components of the perturbed velocity:

$$(V_\perp)_i k_i = 0,$$

and σ is defined by the equation of state

$$p=\sigma w \quad (\sigma=0 \text{ in the case of dust}).$$

These transverse velocity components determine the rotation uniquely, and these components are decoupled from all other perturbed quantities and from each other. The longitudinal component of velocity, in dust models, is constant:

$$V_i k_i = \text{constant},$$

and can be set equal to zero by an infinitesimal coordinate transformation. This transformation amounts to a gauge transformation. All the gauge freedom in the "physical modes" of the problem, i.e., the density perturbation, the wave perturbations, and the rotation, are fixed by the choice $V_i k_i = 0$ which we make. (See Appendix. In particular, the rotation is gauge-invariant.) There is some gauge freedom remaining to be specified, as we shall se in Sec. X when the complete perturbed metric is reconstructed.

In order to simplify the analysis at this point, we shall assume that k_i is an eigenvector of b_{ij}. Since the matrix b_{ij} is diagonal, this states that k_i has only a single component, which we conventionally call k_3 in the remainder of the discussion. We denote by s_3 the value of s in this direction, and we no longer constrain s_3 to be the largest of the s_i. Thus, s_3 has the range $-2 \le s_3 \le 2$.

We sketch the derivation presented in the Appendix. With the gauge fixed, and the choice of wave vector $k=k_3$ only, we define gravitational waves in the usual way as the transverse-traceless part of the metric perturbation.[28] The result is a 3×3 traceless symmetric matrix with vanishing $(i,3)$ components. Of the two linearly independent components of this matrix, one polarization mode of the radiation [that described by the (1, 2) perturbed metric components of the metric] is completely decoupled from other perturbations and evolves freely in the background geometry. This decoupling is a result of our assumption that only k_3 is nonvanishing. For general k_i both radiation polarization modes will be coupled to the density perturbations. However, it is useful, in fact, to have the free wave available as a standard solution to which the coupled waves can be compared. (The waves of both polarizations are decoupled in isotropic models.[1])

The remaining perturbation equations, then, reduce to two coupled second-order equations, for the density contrast $\delta=\delta w/w$, and for η, the variable describing the coupled gravitational waves. These equations are

$$\begin{aligned}\{(FK^2t^2)(t+t_b)^2+\tfrac{1}{4}s_3{}^2t_b{}^2\}\frac{d^2\delta}{dt^2}+\left\{\frac{(t+t_b)}{t}(FK^2t^2)\left[\frac{2s_3}{3}t_b+\tfrac{2}{3}(2t+t_b)\right]+t_b{}^2s_3{}^2\frac{(2t+t_b)}{2t(t+t_b)}\right\}\frac{d\delta}{dt}\\ -\frac{2}{3t}\frac{1}{(t+t_b)}\{(t+t_b)^2(FK^2t^2)-\tfrac{3}{4}t_b{}^2s_3{}^2\}\delta=\tfrac{1}{3}(s_1-s_2)(FK^2t^2)\left\{\tfrac{1}{2}s_3\left(\frac{t_b}{t}\right)^2\eta+\left(\frac{t_b}{t}\right)(t+t_b)\frac{d\eta}{dt}\right\}\end{aligned} \tag{5.4}$$

and

$$\frac{d^2\eta}{dt^2}+\frac{2t+t_b}{t(t+t_b)}\frac{d\eta}{dt}+\eta K^2F=\tfrac{1}{3}(s_1-s_2)\frac{t_b}{t(t+t_b)}\frac{d\delta}{dt}, \tag{5.5}$$

where

$$K\equiv k_3(\tfrac{3}{4}w_0)^{-1/3} \tag{5.6a}$$

is the "scale-invariant wave number." (That is,

K is the wave number invariant under changes of scale $\alpha \to \alpha + \text{constant}$.) Also

$$F \equiv [t(t+t_b)]^{-2/3}\left(\frac{t}{t+t_b}\right)^{-2s_3/3} \tag{5.6b}$$

(notice that $FK^2 = k^a k_a$). These equations cannot be solved analytically, in general.

The two equations (5.4) and (5.5) can always be combined into a single fourth-order equation except in the axially symmetric cases when the coupling constant $s_1 - s_2$ vanishes, and which we consider separately below. This is a profitable procedure only in the limit $t \ll t_b$, in which case the coefficients simplify considerably. In particular, as $t \to 0$, we find $FK^2t^2 \to 0$. From (3.10), (4.1), and (4.2), we see that $H_i^{\,b} \propto t$ if $s_i \neq 2$. Hence, the limit $FK^2t^2 \to 0$ implies $k^3 k_3 (H_3^{\,b})^2 \to 0$, i.e., $k_3 H_3 \to 0$ (since k_3 and H_3 have opposite scale properties). In the following, we will write the product k_3H_3 as simply kH. (The limit $kH \to 0$ means that the wavelength exceeds the horizon size for small t. This is always true sufficiently near the singularity, for nonaxisymmetric backgrounds, with $s_1 \neq s_2$.) The general solution to to the fourth-order equation can be found explicitly in this case. In particular, for the dust density contrast, we find

$$\delta = \frac{A}{t} + B + Ct^{2(2-s_3)/3}\left(1 + D\ln\frac{t}{t_b}\right), \quad A, B, C, D \text{ constants}. \tag{5.7}$$

Although this result was derived in a scheme with only $k_3 \neq 0$, it is valid for general k_i, since the limit $kH \ll 1$ removes all k dependence from the solution. For general k_i, s_3 in (5.7) must be replaced by the constant

$$s = \frac{b_{ij}k_ik_j}{bk_sk_s}.$$

The "wave", described by η, is not oscillatory in this limit, but is a sum of simple power laws obtained by inserting the roots given here for δ into (5.5).

By comparison, the solution for the dust perturbation in an isotropic background is

$$\delta = \frac{A}{t} + Bt^{2/3} = A'v^{-1/2} + B'v^{1/3}, \tag{5.8}$$

where v is the volume element.

VI. THE DIRECTIONAL DEPENDENCE OF DENSITY PERTURBATIONS

The solution given by (5.7) depends on the direction of the wave propagation vector, since s_3 enters and the 3-direction is defined to be the propagation direction. This directional dependence of the most rapidly growing modes is induced by the anisotropy, so that the background geometry dictates the growth rates in an anisotropy-dominated approximation. The directional dependence appears in the term $t^{2(2-s_3)/3}$ in (5.7).

The growth is most rapid when the 3-direction is such that $s_3 \leq -1$. As can be seen from (3.10), this inequality implies contraction in the 3-axis direction (i.e., the propagation direction). On the other hand, the universe is expanding in those directions for which $s > -1$, and the growth rate is lower in those cases, being lowest for that direction with the largest s. If the value of s is close to 2, the growth rate approaches its minimum value, i.e., no growth at all.

When considering the combined fourth-order equation, the values $s_3 = \pm 2$ are excluded. In these cases the coupling constant between waves and density perturbations is zero as (5.4) and (5.5) show. One would guess that the most rapidly growing density contrast should be found in the direction for which $s_3 \approx -2$ (but s_3 not precisely equal to -2). However, it turns out in computer calculation[27] that the coupling is then so small that in many cases (depending on the time interval under consideration) the larger coupling constant associated with $s_3 \neq -2$ gives a larger final density contrast despite the smaller exponent in the power rate of the density contrast δ. The role of the coupling constant is discussed further below.

The dust-filled model with $s = (-\sqrt{3}, 0, \sqrt{3})$ illuminates the importance of the directional effect. According to (5.7), the maximum growth rates of the density contrast corresponding to the axis directions given above are $t^{2.49}$, $t^{1.33}$, and $t^{0.18}$, respectively. We choose the anisotropy parameter t_b equal to 300 yr and the scale-invariant wave number $K = 10\ \text{yr}^{-1/3}$, which corresponds to a wavelength $\lambda \sim 10^5$ ly (ly = light-years) at the present time, $t = 10^{10}$ yr. The quantity kH reaches the value 1, and the oscillatory behavior begins in the three directions, when respectively $t = 18$ yr, $t = 0.3$ yr, and $t = 10^{-17}$ yr. At the time $t = 10^{-17}$ yr, the average growth rate in the third direction, $t^{0.18}$, drops further from this already low value. This oscillatory transition occurs 18 orders of magnitude in time earlier than the transition for the mode which starts out at the highest rate $t^{2.49}$.

VII. THE AXIALLY SYMMETRIC CASES

The axially symmetric cases with $s_1 = s_2$ and $s_3 = \pm 2$ are especially simple. The waves and the density perturbations are uncoupled because the coupling constant $s_1 - s_2$ is zero for waves traveling in the symmetry axis direction.

Equation (5.4) has the following (exact) solution for the density contrast in the model universe with $s_3 = 2$, for momentum vector k_i in the 3-direction:

$$\delta = \frac{A}{t} + B\left[\frac{5t_b}{t+t_b} + 3K^2 \frac{(t+t_b)^{5/3}}{t}\right]. \tag{7.1}$$

For large values of t ($t \gg t_b$), this result is identical with the isotropic solution (5.7). However, for $t \ll t_b$, the differences are pronounced and there are no growing terms. For a long wavelength, $k_3 \approx 0$, we find an approximately constant mode, but if $k_3 \neq 0$, all modes are dying in the limit $t \ll t_b$. The value $s_3 = 2$ defines the direction of infinite horizon size, and the density perturbations with wave vectors oriented in this direction do not grow at all when $t \ll t_b$. Statistical fluctuations associated with this infinitely long communication channel are possible, but these fluctuations would not grow. A small change in the wave direction would not change the situation considerably, since the coupling constant $s_1 - s_2$ would be small.

The other axisymmetric universe ($s_3 = -2$) offers an opposite example. If the wave vector is along the symmetry 3-axis, we find

$$\delta = \frac{A}{t+t_b} + B\left[-\frac{5t_b}{t} + 3K^2 \frac{t^{5/3}}{t+t_b}\right]. \tag{7.2}$$

The roles of t and $t+t_b$ are interchanged (as is the sign of t_b/t) from (7.1). Here we have in the approximation $t \ll t_b$ a growth rate of $k_3^2 t^{5/3}$. Consequently, a short-wavelength mode grows faster than longer ones. The constant A can be so chosen that the term $A/(t+t_b)$ cancels the negative contribution of $-Bt_b/t$ at some early time to result in a positive, growing density contrast δ.

In the coupled case ($s_3 \neq \pm 2$), the behavior of the density contrast δ is more complicated than in the axially symmetric solutions. The two modes of the density contrast have to behave (in some limit, at least) like solutions (7.1) and (7.2) do, and an estimate can therefore be made of the form of the most rapidly growing term. This estimate yields the function $t^{(1-2s)/3}$ + constant in an anisotropy-dominated limit (small t), which agrees with (7.1) and (7.2). In the next section, we will find under what conditions this estimate is close to the actual solution.

VIII. BEHAVIOR OF SHORT-WAVELENGTH PERTURBATIONS IN ANISOTROPY-DOMINATED MODELS

The solution (5.7) for the behavior of δ near the singularity was obtained assuming both anisotropy dominance, $t \ll t_b$, and that the wavelength is larger than the horizon size, $kH \ll 1$. It is possible to consider an intermediate regime in which the background model is still anisotropy-dominated but the wavelength is short enough to fall well within the horizon size: $kH \gg 1$.

As η and δ are coupled, it is clear that their behavior is neither purely oscillatory nor smooth and slowly varying. We expect that in the very-short-wavelength limit, both η and δ will be oscillatory. At longer wavelengths, δ loses the superimposed oscillatory character. In general, we find the "wave" quantity η represents the more oscillatory modes, whereas δ represents smoother modes.

We proceed by making use of an averaging process, similar to those proposed by Brill and Hartle[29] and Isaacson.[30] We average (5.4) over a short interval of time, but over many wavelengths with a smoothly varying weighting function. We assume that the average value of the wave is approximately zero:

$$\langle \eta \rangle \approx 0 .$$

Computer calculation[27] shows this is a fairly good approximation for short wavelength. The assumption $kH \gg 1$ implies $FK^2t^2 \gg 1$. Because of this, in this limit (5.4) becomes independent of wavelength and is simply an equation describing δ. This equation shows that δ has an over-all period of order t, the high-frequency parts being averaged out by the averaging procedure.

The averaging procedure brings the right-hand side of (5.4) approximately to zero. On the left-hand side, the slowly varying coefficients are approximated according to the assumptions $t \ll t_b$, $FK^2t^2 \gg s_3^2$. The averaging yields a simple equation

$$\left\langle \frac{d^2\delta}{dt^2} \right\rangle + \frac{2}{3t}(1+s_3)\left\langle \frac{d\delta}{dt} \right\rangle = 0 . \tag{8.1}$$

Generally, $(d/dt)\langle\delta\rangle$ is equal to $\langle d\delta/dt\rangle$ (because the weighting function is slowly varying), and we can solve (8.1) for $\langle\delta\rangle$. The solution is

$$\langle\delta\rangle \propto t^{(1-2s_3)/3} + \text{constant} \quad (kH \gg 1,\ t \ll t_b). \tag{8.2}$$

The procedure which averages the right-hand side of (5.4) to zero yields a decoupled equation with solutions similar to (7.1) and (7.2). Although we can see that constant terms such as in (8.2) may eventually become negligible [as will the decaying terms, similar to the A term in (7.1)], we keep them to accord with possible numerical results.

It must be emphasized that (8.2) is a short-wavelength approximation. For longer wavelengths, but still in the large-anisotropy approximation, we expect the solution to go over to the solution given in (5.7) for waves with $kH \ll 1$. The averaging conditions assumed to obtain (8.2) are less accurate for longer wavelengths. In fact, (8.2) fails completely near $kH \approx 1$, since the oscillatory behavior does not occur for perturbations larger than the horizon size.

Eventually, the model will evolve to the isotropic regime ($t \gg t_b$) and the power-law growth of (5.8) will hold, since in this limit the coupling between waves and density, which contains $(s_1 - s_2)$ as a factor, vanishes.

Doroshkevich, Zel'dovich, and Novikov[19] derive the density contrast in the anisotropic approximation using the time development of the volume element and a Newtonian analysis. They give as their main result the two power laws $t^{2(2-s)/3}$ and $t^{(1-2s)/3}$, as found here, in the large-anisotropy regimes with $kH \ll 1$ and $kH \gg 1$, respectively.

IX. RECONSTRUCTION OF COMPLETE PERTURBED METRIC

The analysis so far has concentrated on the "physical" quantities associated with the perturbation equations. We have discussed the density contrast δ, the rotation $V_{[i}k_{j]}$, the coupled gravitational waves [$\eta = \frac{1}{2}(\mu^1{}_1 - \mu^2{}_2)$ in our case when k_i has only a 3-component], and the decoupled wave $\eta_f = \mu^1{}_2$. These quantities clearly are not sufficient to specify the metric completely. However, a complete specification can be obtained in the form of time integrations involving these dynamical quantities. This is done by means of the perturbed $G_{0i} = T_{0i}$ equations (A36) and (A37) in the Appendix. Initially, at least, in this section we will work in a general gauge (which, however, has $g_{0\alpha} = 0$). As is usual in these problems, we expand $\mu^a{}_b$ [see Eqs. (A33)]:

$$\mu^a{}_b = p^a{}_b + \tfrac{1}{2}(q^a k_b + q_b k^a) + s k^a k_b + r\delta^a{}_b\,, \tag{9.1}$$

where $p^a{}_a = q^a k_a = p^a{}_b k_a = 0$.

A straightforward calculation shows that the solution of (A37) is

$$q_i = -4w_0 (V_\perp)_i (k_3)^{-2} e^{2\alpha} e^{2\beta_i} \times \int_{t_{0i}}^{t} e^{-3\alpha} e^{2\beta}{}_{33} e^{2\beta_i} dt \quad \text{(no sum)}. \tag{9.2}$$

(In this section we work in terms of proper time t; the dot indicates d/dt.) The gauge transformation properties of the metric, (A49), give

$$\bar{q}_i = q_i - 2g_{il}F^l \tag{9.3}$$

as the gauge behavior of q_i. (Here F^l is a constant number, the Fourier amplitude of the spatial gauge function; see Appendix). Comparing (9.3) with (9.2), we see that a change in gauge merely amounts, as far as q_i is concerned, to a shift in the lower limit of the integral (9.2). Recalling that q_i is orthogonal to k_i, we see that, in fact, two of the three F^l may be fixed (i.e., F^1 and F^2) by a choice of the integration constant t_{0i} appearing in (9.2). We shall, however, temporarily assume a general gauge, and not make such a choice.

From (9.1) we see that $\mu = sk^3k_3 + 3r$, while $\mu^3{}_3 = sk^3k_3 + r$. Hence, $\mu^1{}_1 + \mu^2{}_2 = 2r$. We use this fact and insert (9.1) for $\mu^a{}_b$, so that the $G_{03} = T_{03}$ perturbed equation, (A36), becomes

$$(\mu^1{}_1 + \mu^2{}_2)^{\cdot} - \tfrac{3}{2}(\mu^1{}_1 + \mu^2{}_2)\dot{\beta}_{33} = -(\dot{\beta}_{11} - \dot{\beta}_{22})\eta + 2w V_3/k_3\,. \tag{9.4}$$

(Here, as before, $2\eta = \mu^1{}_1 - \mu^2{}_2$.) At this point, it is well to investigate the gauge properties of this equation. Using (A48), we find [L and R refer to the left- and right-hand sides of (9.4), $\bar{L}$ and $\bar{R}$ the gauge-transformed sides]

$$\bar{L} = L + 2F_0(2\ddot{\alpha} - \ddot{\beta}_{33}) - 3F_0(2\dot{\alpha} - \dot{\beta}_{33})\dot{\beta}_{33}\,, \tag{9.5}$$

while

$$\bar{R} = R - 2wF_0k_3 - F_0(\dot{\beta}_{11} - \dot{\beta}_{22})^2\,. \tag{9.6}$$

The gauge terms are written in (9.5) and (9.6) in the order they arise in (9.4). The quantity F_0 is a constant, the Fourier amplitude of one of the gauge functions. For more detail, see (A47). The combination $2F_0(-\ddot{\beta}_{33} - 3\dot{\alpha}\dot{\beta}_{33})$ vanishes in view of (3.2). The remaining gauge terms then are equal on the two sides of (9.4), in view of the zero-order field equation (A2) + (A3).

Equation (9.4) can be integrated directly:

$$e^{-(3/2)\beta_{33}}(\mu^1{}_1 + \mu^2{}_2) = \int_{\infty}^{t} e^{-(3/2)\beta_{33}} \times [2wV_3k_3^{-1} - \tfrac{1}{2}(\dot{\beta}_{11} - \dot{\beta}_{22})(\mu^1{}_1 - \mu^2{}_2)]dt\,. \tag{9.7}$$

The lower limit in (9.7) is determined by requiring the solution have the gauge properties appropriate to the quantity $\mu^1{}_1+\mu^2{}_2$. If a gauge transformation is performed, it is found that the term arising from the left-hand side is, in fact, the antiderivative of terms which arise under the integral, using the known transformation properties of V_3 and η. [This is true only in virtue of the zero-order field equation, (A2)+(A3).] But on the right-hand side, the gauge terms give rise to the antiderivative evaluated at the limits of integration. To get equality of the gauge terms, we thus must choose the lower limit so that the antiderivative vanishes there. A glance at (A2)–(A4) shows that ∞ is the proper (and only) choice. We have a curiously distinct situation from that found above for the q_i. The general solution of a gauge-invariant equation does not have the proper gauge properties unless a particular value for the integration constant is chosen. [Recall that in the solution (9.2), picking a particular gauge picked out the particular integration constant.]

We have now essentially completed the construction of the metric, once the gauge is known. From the equations of motion, we obtain, for dust (see Appendix),

$$\delta=-\tfrac{1}{2}\mu+\int V_i k_j g^{ij}dt+\text{constant}. \tag{9.8}$$

Since $V_i=$ constant for dust, (9.8) gives $\mu=\mu^1{}_1+\mu^2{}_2+\mu^3{}_3$ in terms of δ. A particular choice of V_3 [the only component to enter (9.8)] fixes the gauge for the quantities δ, η, η_f, Ω_{ij}, and $(\mu^1{}_1+\mu^2{}_2)$ and can be thought of as fixing F_0 [see (A51)]. The general gauge transformation law for μ,

$$\overline{\mu}=\mu-2F_0\int k_3k^3dt+6F_0\dot{\alpha}-2F^3k_3, \tag{9.9}$$

shows that even with F_0 fixed, it is possible to change the constant in (9.8); a choice of the value of the constant amounts to a choice for F^3. Finally, we have seen that the gauge amplitudes F^1 and F^2 have been fixed by choosing the constant t_{0i} in the two equations (9.2).

To reconstruct the metric, our computation of δ, with (9.8), gives μ. Also, $\eta=\frac{1}{2}(\mu^1{}_1-\mu^1{}_2)$ and $\eta_f=\mu^1{}_2$ are known. From (9.7), we find $\mu^1{}_1+\mu^2{}_2$, and, finally, (9.2) gives $\mu^1{}_3$ and $\mu^2{}_3$. The entire construction of $\mu^b{}_a$ is then complete, and the metric perturbation $h^b{}_a$ is found by use of (5.3).

X. ROTATION TO SECOND ORDER

We have already mentioned that the rotation ω_{ij} is decoupled from other perturbations and is gauge-invariant. The gauge invariance is straightforward to explain. The usual expression for the gauge properties of any tensor T,

$$T\to\overline{T}=T-\mathcal{L}_\xi T$$

(where $\mathcal{L}_\xi$ indicates the Lie derivative with respect to the gauge field ξ), shows, when expanded out order by order, that the nth-order perturbation to T depends only on the gauge to the $n-m-1$ order, if the first m orders of T vanish. Hence, the first-order rotation is gauge-invariant, because the unperturbed rotation is zero, while the second-order rotation discussed in this section depends only on the first-order gauge.

That rotation is decoupled from other perturbations to first order can be understood by considering the kinematical equation giving the evolution of the rotation (for dust):

$$\begin{aligned}\omega_{\alpha\beta,\sigma}u^\sigma&=-(\omega_{\beta\sigma,\alpha}+\omega_{\sigma\alpha,\beta})u^\sigma\\&=2\omega_{\sigma[\alpha}u^\sigma{}_{,\beta]}\,.\end{aligned} \tag{10.1}$$

[This is simply the equation curl(curl u) = 0.[21]] The rotation tensor is a skew-symmetric tensor satisfying

$$\omega_{\alpha\beta}u^\beta=0\,. \tag{10.2}$$

Because (10.1) is homogeneous in $\omega_{\alpha\beta}$, the rotation to any order solves an equation involving only quantities of lower order. This circumstance allows us to obtain the rotation to second order. In this section, we assume a general gauge (with $\delta g_{0\alpha}=0$) to first order. The $(0i)$ second-order components are obtained directly by writing (10.2) out in terms of components:

$$\omega_{j0}u^0=-\omega_{ji}u^i\,. \tag{10.3}$$

Because in zero order both the rotation and the spatial components of the velocity vanish, (10.3) gives

$$\omega^{(1)}_{j0}=0\,. \tag{10.4}$$

Since $u^0=1$ (through first order) if proper time t is used, so that $\gamma=1$, then

$$\omega^{(2)}_{j0}=-\omega^{(1)}_{ji}e^{-2\beta}{}_{il}e^{-2\alpha}\delta u_l\,. \tag{10.5}$$

We have written out the right-hand side of (10.5) to show the explicit time dependence. (Both the first-order rotation $\omega^{(1)}_{ij}$ and the first-order spatial velocities δu_i are constant in time if the background model contains dust, although they may be position dependent.)

To obtain the components $\omega_{ij}^{(2)}$, it is necessary to expand (10.1). We find

$$\omega_{ab,0}^{(2)} = \tfrac{1}{2} e^{-2\alpha} e^{-2\beta}{}_{cd}[\delta u_{d,a}\delta u_{b,c} - \delta u_{d,b}\delta u_{a,c}] - \omega_{ab,c}^{(1)}\delta u_d e^{-2\alpha} e^{-2\beta}{}_{cd}. \tag{10.6}$$

Again, the only time dependence is via the zero-order metric, so this equation can be straightforwardly integrated to give

$$\omega_{ab}^{(2)} = \Lambda_{ab} + \left[\int e^{-2\alpha} e^{-2\beta}{}_{cd} dt\right] \times [\tfrac{1}{2}\delta u_{d,a}\delta u_{b,c} - \tfrac{1}{2}\delta u_{d,b}\delta u_{a,c} - \omega_{ab,c}^{(1)}\delta u_d], \tag{10.7}$$

where Λ_{ab} are antisymmetric functions of x^i only. It is straightforward to check that (10.6) is gauge-invariant. The gauge properties of the solution (10.7) show that the arbitrary functions Λ_{ab} may be chosen as the integration constants in the solution in one particular gauge as indicated. These functions then change, however, under a first-order gauge transformation.

It should be emphasized that no use was made in this section of the Fourier-transform technique. Therefore the specialization of the wave vector to $k_i = (0, 0, k_3)$, which we set previously, was not done here. The results here are completely general.

XI. CONCLUSIONS

The calculation of perturbation growth rates in an anisotropic cosmology presents enormous technical difficulties. The major physical reason behind these difficulties is the coupling of density perturbations to gravitational waves. Further complication may be expected if a background model with rotation were chosen, and the use of such a rotating background (for example, a Bianchi type-IX model[17,18]) should be expected to yield important new clues as to the evolution of protogalaxies.

The growth of density perturbations in the nonrotating, anisotropic, type-I models discussed here demonstrates the flow of energy between gravitational waves and protogalactic conglomerations. Detailed results from numerical calculations will be presented by one of us separately,[27] but they bear out the analytic approximation results presented in the present paper. The net effect on the growth rates of a density perturbation is to modify the power law exponent. The growth rate in an isotropic (FRW) dust model is $t^{2/3}$, and in the anisotropic models can be from t^0 to $t^{8/3}$, depending on direction and on the model, t being cosmic time.

Although no spectacular change in the power-law exponent is seen, the total effect on a density perturbation, integrated over a suitable cosmic epoch, defined by the growth of the horizon, can be enormous.

The calculations presented here are for dust, and the only stop to the growth of a perturbation is due to the stealing of energy by gravitational radiation. In a fluid model, pressure provides another source of springiness to stop perturbation growth (and make it oscillatory), but this mechanism comes in only when the horizon size is large enough to allow communication via sound waves among all parts of the perturbation. Here this growth phase is limited by the outflow of gravitational radiation (due to the anisotropy-induced coupling to density modes), and an additional limit can be imposed by stopping calculation when the horizon size becomes too large.

The calculations here and elsewhere[27] presented are expected to be valid up to the time the horizon size is large enough to include pressure effects. The integration time for the growth of perturbations is thus roughly proportional to the horizon size in a particular direction in a particular model. It is found that those directions with the largest horizon (longest growth period for density perturbations) have the lowest coupling between gravitational waves and density modes, and have the slowest power-law growth rates. Medium-length horizon distances, however, do show, in numerical results,[27] a large enhancement, due both to increased power-law rate of growth due to gravitational wave effects and to increased growth period, over the isotropic case.

No solution to the galaxy formation problem – that of providing a density contrast of $\delta \sim 10^{-2}$ by the hydrogen recombination time $t \sim 3\times10^5$ yr – is claimed. The growth rates calculated here are simply not high enough to support such a claim. However, the effects here discussed do suggest that further investigation – especially of perturbations of rotating cosmologies – is desirable.

APPENDIX

The unperturbed metric is

$$ds^2 = -\frac{1}{\gamma} d\tau^2 + e^{2\alpha} e^{2\beta}{}_{ij} dx^i dx^j, \tag{A1}$$

where

$$\gamma=\gamma(\tau), \quad \alpha=\alpha(\tau), \quad \beta_{ij}=\beta_{ij}(\tau),$$

and where β_{ij} is diagonal and traceless: $\beta_{tt}=0$.

The functional form of either α or γ can be specified by choice of time-coordinate τ. The zeroth-order field equations are $(,0=d/d\tau)$

$$\gamma R_{00}=-\tfrac{3}{2}\alpha_{,0}\gamma_{,0}-3\gamma\alpha_{,00}-3\gamma\alpha_{,0}{}^2-\gamma\beta_{ij,0}\beta_{ij,0}$$
$$=\tfrac{1}{2}(w+3p), \tag{A2}$$

$$\tfrac{1}{3}R^k{}_k=\tfrac{1}{2}\alpha_{,0}\gamma_{,0}+\gamma\alpha_{,00}+3\gamma\alpha_{,0}{}^2$$
$$=\tfrac{1}{2}(w-p), \tag{A3}$$

$$2e^{-2\alpha}e^{-\beta}{}_{jb}e^{-\beta}{}_{ia}(R_{ab}-\tfrac{1}{3}g_{ab}R^k{}_k)$$
$$=\gamma_{,0}\beta_{ij,0}+2\gamma\beta_{ij,00}+6\gamma\alpha_{,0}\beta_{ij,0}$$
$$=0. \tag{A4}$$

Here w is the energy density; p is the pressure of the fluid; and the unperturbed fluid velocity field is

$$u^\alpha=(\gamma^{1/2},0,0,0).$$

The solution of (A4) is (3.2), where u is defined by

$$du=\gamma^{-1/2}e^{-3\alpha}d\tau. \tag{A5}$$

From the field equations comes the relation

$$w_{,0}+3\alpha_{,0}(w+p)=0, \tag{A6}$$

an equation which is equivalent to the Bianchi identity. This equation results in

$$w=w_0e^{-3(1+\sigma)\alpha} \tag{A7}$$

if

$$p=\sigma w, \quad \sigma=\text{constant}. \tag{A8}$$

In this case an important, although simple, class of solutions results, with $\alpha(u)$ being given by

$$e^{3(\sigma-1)\alpha/2}=A\sinh[(1-\sigma)\tfrac{3}{2}bu],$$
$$A=\text{constant}, \quad b=\tfrac{1}{6}b_{ij}b_{ij}.$$

The perturbed metric is

$$ds^2=-\frac{1}{\gamma}d\tau^2+e^{2\alpha}e^{2\beta}{}_{ik}(\delta^k{}_j+h^k{}_j)dx^idx^j, \tag{A9}$$

where the small functions $h^k{}_j$ are functions of τ and of x^i:

$$h^k{}_j=h^k_j(\tau,x^i).$$

We assume that

$$e^{2\beta}{}_{is}h^s{}_j=e^{2\beta}{}_{js}h^s{}_i$$

as required by the symmetry of the metric. The (0μ) components of the metric are not perturbed:

$$\delta g_{0\mu}=0. \tag{A10}$$

Because of these conditions, the perturbation in the fluid velocity is

$$\delta u^0=\delta u_0=0, \quad \delta u_i \text{ is nonzero}.$$

The first-order field equations are (w and p are unperturbed quantities, δw and δp are the perturbations)

$$g^{00}\delta R_{00}:\ -\tfrac{1}{2}\gamma_{,0}h_{,0}-\gamma(h_{,00}+2\alpha_{,0}h_{,0}+2\beta_{ij,0}h^i{}_{j,0})=\delta w+3\delta p, \tag{A11}$$

$$\delta R_{0i}:\ h^j{}_{i,j0}-h_{,i0}-2\beta_{it,0}h^j{}_{t,j}+2\beta_{jt,0}h^t{}_{i,j}+\beta_{ij,0}h_{,j}-\beta_{ab,0}h^a{}_{b,i}=-2\gamma^{-1/2}(w+p)\delta u_i, \tag{A12}$$

$$g^{ik}\delta R_{kj}:\ \tfrac{1}{2}e^{-2\alpha}e^{-2\beta}{}_{is}[h^a{}_{s,ja}+h^a{}_{j,sa}-e^{2\beta}{}_{sk}e^{-2\beta}{}_{ab}h^k{}_{j,ab}-h_{,sj}]+\tfrac{1}{4}\gamma_{,0}h^i{}_{j,0}+\tfrac{1}{2}\gamma h^i{}_{j,00}+\tfrac{1}{2}\gamma h_{,0}(\alpha_{,0}\delta^i{}_j+\beta_{ij,0})$$
$$+\tfrac{3}{2}\gamma\alpha_{,0}h^i{}_{j,0}+\gamma\beta_{is,0}h^s{}_{j,0}-\gamma\beta_{kj,0}h^i{}_{k,0}=\tfrac{1}{2}(\delta w-\delta p)\delta^i{}_j, \tag{A13}$$

where we have used the notation

$$h\equiv h^i{}_i.$$

We make a Fourier analysis of these equations and pick out one particular wave vector k_i by writing

$$\begin{aligned}
&h^i{}_j(x,\tau)=\mu^i{}_j(\tau)e^{ik_nx^n}, \quad k_i=\text{constant},\\
&\delta w=W(\tau)e^{ik_nx^n},\\
&\delta p=P(\tau)e^{ik_nx^n},\\
&\delta u_i=iV_i(\tau)e^{ik_nx^n},\\
&\mu=\mu^s{}_s(\tau).
\end{aligned} \tag{A14}$$

The resulting field equations are ordinary differential equations, involving only functions of τ:

$$-\tfrac{1}{2}\gamma_{,0}\mu_{,0}-\gamma(\mu_{,00}+2\alpha_{,0}\mu_{,0}+2\gamma^{-1/2}e^{-3\alpha}b_{ij}\mu^{i}{}_{j,0})=W+3P, \tag{A15}$$

$$\mu^{s}{}_{i,0}k_s-\mu_{,0}k_i+\gamma^{-1/2}e^{-3\alpha}(-2b_{it}\mu^{s}{}_{t}k_s+2b_{st}\mu^{t}{}_{i}k_s+b_{is}\mu k_s-b_{ab}\mu^{a}{}_{b}k_i)=-2\gamma^{-1/2}(w+p)V_i, \tag{A16}$$

$$-e^{-2\alpha}e^{-2\beta}{}_{is}(\mu^{t}{}_{s}k_jk_t+\mu^{t}{}_{j}k_sk_t-e^{-2\beta}{}_{ab}e^{2\beta}{}_{su}k_ak_b\mu^{u}{}_{j}-\mu k_sk_j)+\tfrac{1}{2}\gamma_{,0}\mu^{i}{}_{j,0}+3\gamma\alpha_{,0}\mu^{i}{}_{j,0}+\gamma\mu^{i}{}_{j,00}+\gamma\alpha_{,0}\mu_{,0}\delta^{i}{}_{j}$$
$$+\gamma^{1/2}e^{-3\alpha}(\mu_{,0}b_{ij}-2b_{sj}\mu^{i}{}_{s,0}+2b_{is}\mu^{s}{}_{j,0})=(W-P)\delta^{i}{}_{j}. \tag{A17}$$

Here we made use of (3.2) to write

$$\beta_{ij,0}=b_{ij}\gamma^{-1/2}e^{-3\alpha},\quad b_{ij}=\text{constant}. \tag{A18}$$

To further analyze these equations, we use the projection operator

$$k_{ij}\equiv\delta_{ij}-k_ik_j(k_sk_s)^{-1}. \tag{A19}$$

This operator has the properties

$$k_{is}k_{sj}=k_{ij}=k_{ji},\quad k_{is}k_s=0,$$
$$k_{is}l_s=l_i\ \text{ for any } l_i \text{ such that } l_sk_s=0.$$

It is numerically equal to the operator

$$\delta^{i}{}_{j}-g^{is}k_sk_j(g^{st}k_sk_t)^{-1}=\delta^{i}{}_{j}-e^{-2\beta}{}_{is}k_sk_j(e^{-2\beta}{}_{tu}k_tk_u)^{-1}$$

if k_i is an eigenvector of β_{ij}. This last operator, although mathematically useful, is not in general constant, whereas k_{ij} is independent of time.

We use the constant operator k_{ij} first to define the "perpendicular" components of V_i:

$$(V_\perp)_i\equiv V_sk_{is}. \tag{A20}$$

Equation (A16) (projected by use of k_{ij}) and (A17) (projected by use of k_ik_{jk}) together result in

$$p_{,0}(V_\perp)_i+(w+p)(V_\perp)_{i,0}=0. \tag{A21}$$

If (A7) holds, that is, if $p=\sigma w$, $\sigma=\text{const}$, then

$$(V_\perp)_i\propto w^{-\sigma/(1+\sigma)}\propto e^{3\sigma\alpha}\propto R^{3\sigma}. \tag{A22}$$

(Here $R=e^{\alpha}$ is the "radius of the universe.") Thus $(V_\perp)_i$ is constant if $\sigma=0$ (dust).

The interpretation of $(V_\perp)_i$ as determining rotation comes from the definition of the rotation tensor:

$$\omega_{\mu\nu}\equiv\tfrac{1}{2}(u_{\beta,\alpha}-u_{\alpha,\beta})(\delta^{\alpha}{}_{\mu}+u^{\alpha}u_{\mu})(\delta^{\beta}{}_{\nu}+u^{\beta}u_{\nu}).$$

The unperturbed rotation vanishes. The first-order rotation has only spatial components and is given by

$$(\delta\omega)_{ij}=\tfrac{1}{2}(u_{j,i}-u_{i,j}).$$

The Fourier component with wave number k_i is defined by

$$(\delta\omega)_{ij}=i\Omega_{ij}e^{ik_sx^s} \tag{A23}$$

and Ω_{ij} is given by

$$\Omega_{ij}=\tfrac{1}{2}(V_jk_i-V_ik_j)=\tfrac{1}{2}[(V_\perp)_jk_i-(V_\perp)_ik_j].$$

Thus $(V_\perp)_i$ determines Ω_{ij}, and it is in turn determined by Ω_{ij}:

$$(V_\perp)_i=-2\Omega_{ij}k_j(k_ak_a)^{-1}.$$

The projections of (A16) in the direction of k_i and of (A17) in the directions k_ik_j yield, after use of (A15) and (A6),

$$0=\gamma^{-1/2}Pk_sk_s+p_{,0}V_ik_i+(w+p)(V_ik_i)_{,0}. \tag{A24}$$

In dust models, both P and p vanish, so that

$$V_ik_i=\text{constant}. \tag{A25}$$

The constancy of V_ik_i in these models is indicative of the lack of sound waves when the pressure vanishes.

Both (A21) and (A24) may be derived directly from the first-order field equations, as we have indicated. They are equivalent to the perturbed Bianchi identities, however, and may also be derived from them.

From Eqs. (A15)–(A17) above, or directly by considering the perturbation of (A6), one obtains

$$\frac{4}{3(1+\sigma)}\delta\Theta+\gamma^{1/2}[e^{3\alpha(1+\tilde\sigma)}\delta w]_{,0}=0, \tag{A26}$$

where $P=\tilde\sigma W$, $p=\sigma w$, and the perturbed expansion $\delta\Theta$ is

$$\delta\Theta=\delta u^{i}{}_{,i}+\tfrac{1}{2}\gamma^{1/2}h_{,0}. \tag{A27}$$

In the case of dust, we have $V_i=\text{constant}$, and we further can choose a gauge so that $V_ik_i=0$. [See (A51) below.] The Fourier-transformed (A26) then becomes

$$\delta\equiv W/w=-\tfrac{1}{2}\mu+\text{constant},\quad \sigma=\tilde\sigma=0,\quad V_ik_i=0. \tag{A28}$$

Further analysis of the field equations yields coupled equations in several components of $\mu^{i}{}_{j}$. One of these components is decoupled if k_i is an eigenvector of b_{ij} and hence of β_{ij}. If that assumption is made, then

$$e^{-2\beta}{}_{ij}k_ik_{jk}=0=b_{it}k_ik_{tj},\quad\text{etc}. \tag{A29}$$

[The c_{ij} of (3.2) may be set equal to zero by a zeroth-order spatial linear coordinate transformation; and we assume this transformation has been carried out.]

We define the following parts of $\mu^{i}{}_{j}$:

$$\begin{aligned}
&\mu \equiv \mu^s{}_s\,,\\
&r \equiv \tfrac{1}{2}[\mu - \mu^s{}_t k_s k_t (k_a k_a)^{-1}]\,,\\
&q_i \equiv 2\mu^s{}_t k_s k_{it}(k_a k_b e^{-2\alpha} e^{-2\beta}{}_{ab})^{-1}\,,\\
&\eta^i{}_j \equiv \mu^s{}_t (k_{is} k_{tj} - \tfrac{1}{2} k_{ts} k_{ij})\,.
\end{aligned} \tag{A30}$$

As in the text, we assume b_{ij} to be diagonal and k_i to have only a 3-component:

$$(b_{ij}) = \mathrm{diag}(s_1 b,\ s_2 b,\ s_3 b)\,, \tag{A31}$$

with

$$\begin{aligned}
&s_1 + s_2 + s_3 = 0\,,\\
&s^2{}_1 + s^2{}_2 + s^2{}_3 = 6\,,\\
&b = \text{constant};\\
&(k_i) = (0,\ 0,\ k_3)\,.
\end{aligned} \tag{A32}$$

Definitions (A30) become

$$\begin{aligned}
&\mu = \mu^1{}_1 + \mu^2{}_2 + \mu^3{}_3\,,\\
&r = \tfrac{1}{2}(\mu^1{}_1 + \mu^2{}_2)\,,\\
&q_1 = 2e^{2\alpha} e^{2\beta}{}_{33} \mu^3{}_1 / k_3\,,\\
&q_2 = 2e^{2\alpha} e^{2\beta}{}_{33} \mu^3{}_2 / k_3\,,\\
&q_3 = 0\,,\\
&\eta^i{}_3 = \eta^3{}_i = 0\,,\\
&\eta^1{}_1 = -\eta^2{}_2 = \tfrac{1}{2}(\mu^1{}_1 - \mu^2{}_2)\,,\\
&\eta^1{}_2 = \mu^1{}_2\,.
\end{aligned} \tag{A33}$$

There are thus two algebraically independent components in q_i and two in $\eta^i{}_j$; these four plus μ and r make up the six components of the $\mu^i{}_j$ (which is symmetric when the upper index is lowered using the unperturbed g_{ij}). We define

$$\eta \equiv \eta^1{}_1\,, \quad \eta_f \equiv \eta^1{}_2\,. \tag{A34}$$

When Eqs. (A15)–(A17) are rewritten in terms of μ, r, q_i, and $\eta^i{}_j$, assuming (A31) and (A32), the result is as follows:

Equation (A15) becomes

$$-\tfrac{1}{2}\gamma_{,0}\mu_{,0} - \gamma(\mu_{,00} + 2\alpha_{,0}\mu_{,0}) + 2\gamma^{1/2} e^{-3\alpha} b[(s_1 - s_2)\eta_{,0} - (s_1 + s_2)\mu_{,0} + 3(s_1 + s_2) r_{,0}] = W + 3P\,. \tag{A35}$$

Equation (A16) breaks into two sets of equations:

$$(i=3)\quad r_{,0} - \tfrac{3}{2}\gamma^{-1/2} e^{-3\alpha} b s_3 r = \gamma^{-1/2}(k_3)^{-1}(w + p) V_3 - \tfrac{1}{2}\gamma^{-1/2} e^{-3\alpha} b\eta(s_1 - s_2) \tag{A36}$$

and

$$(i \neq 3)\quad \tfrac{1}{2} q_{i,0} - \alpha_{,0} q_i - b s_i \gamma^{-1/2} e^{-3\alpha} q_i = -2\gamma^{-1/2}(k_3)^{-2}(w+p)(V_\perp)_i e^{2\alpha} e^{2\beta}{}_{33}\ \ (\text{no sum})\,. \tag{A37}$$

Equation (A37) gives a first integral of (A41) below.

Equation (A17) yields

$$\delta R^3{}_3:\ \ 2e^{-2\alpha} e^{-2\beta_{33}} r (k_3)^2 + \tfrac{1}{2}\gamma_{,0}\mu_{,0} - \gamma_{,0} r_{,0} + 4\gamma\alpha_{,0}\mu_{,0} - 6\gamma\alpha_{,0} r_{,0} + \gamma\mu_{,00} - 2\gamma r_{,00} + \gamma^{1/2} e^{-3\alpha} b s_3 \mu_{,0} = W - P\,, \tag{A38}$$

$$\delta R^1{}_1 + \delta R^2{}_2:\ \ e^{-2\alpha} e^{-2\beta_{33}}(k_3)^2 r + \tfrac{1}{2}\gamma_{,0} r_{,0} + 3\gamma\alpha_{,0} r_{,0} + \gamma r_{,00} + \gamma\alpha_{,0}\mu_{,0} - \tfrac{1}{2}\gamma^{1/2} e^{-3\alpha} b s_3 \mu_{,0} = W - P\,, \tag{A39}$$

$$\delta R^1{}_1 - \delta R^2{}_2:\ \ \gamma\eta_{,00} + (\tfrac{1}{2}\gamma_{,0} + 3\gamma\alpha_{,0})\eta_{,0} + (k_3)^2 e^{-\beta_{33}} e^{-2\alpha}\eta + \tfrac{1}{2}\gamma^{1/2} e^{-3\alpha}\mu_{,0} b(s_1 - s_2) = 0\,, \tag{A40}$$

$$\delta R^3{}_j\ (j \neq 3):\ \ \gamma(e^{-2\alpha} e^{-2\beta_{33}} q_j)_{,00} + [\tfrac{1}{2}\gamma_{,0} + 3\gamma\alpha_{,0} + 2\gamma^{1/2} e^{-3\alpha} b(s_3 - s_j)](e^{-2\alpha} e^{-2\beta_{33}} q_j)_{,0} = 0\,, \tag{A41}$$

$$\delta R^1{}_2:\ \ e^{-2\alpha} e^{-2\beta_{33}}(k_3)^2 \eta_f + \tfrac{1}{2}\gamma_{,0}\eta_{f,0} + 3\gamma\alpha_{,0}\eta_{f,0} + \gamma\eta_{f,00} + 2\gamma^{1/2} e^{-3\alpha} b(s_1 - s_2)\eta_{f,0} = 0\,. \tag{A42}$$

To obtain (5.4) in the text, set $P = p = 0$ and $\gamma = 1$. It is also necessary to use a particular (partially) fixed gauge. As shown in (A51), V_3 may be set to zero by a gauge transformation, and we do so in the derivation of (5.4). Equation (A36) and its derivative will be used to eliminate $r_{,00}$ and $r_{,0}$ in terms of lower derivatives and of η. We form the following equation:

$$(\mathrm{A38}) - (\mathrm{A39}) = 0\,, \tag{A43}$$

and eliminate $r_{,00}$ and $r_{,0}$. Then we form (A38) + 2(A39) (dot means derivative with respect to proper time):

$$\ddot{\mu} + 6\dot{\alpha}\dot{\mu} + 4e^{-2\alpha} e^{-2\beta_{33}}(k_3)^2 r = 3W. \tag{A44}$$

We multiply (A43) through by $e^{-2\alpha} e^{-2\beta_{33}}(k_3)^2$ and use (A44) to eliminate r. The resultant equation contains $\dot{\eta}$, η, W, $\dot{\mu}$, and $\ddot{\mu}$ but no undifferentiated factors μ. We can, therefore, use (A28) to eliminate μ, rewrite in terms of $\delta = W/w$, and insert the explicit time dependences from (3.4), (3.7), and (3.8). The result is (5.4). Equation (5.5) follows quite simply from (A40), by inserting $\gamma = 1$ and the explicit time dependences from (3.4), (3.7), and (3.8), and eliminating $\dot{\mu}$ in favor of $\dot{\delta}$.

In general, we see that q_i is not coupled to the other metric perturbations, μ, r, η, and η_f. Coupling does appear if k_i is not an eigenvector of b_{ij}, but as we have seen, the rotational perturbation described by $(V_\perp)_i$ is still independent of other perturbations. We also see that η_f is decoupled. Again, coupling appears if k_i does not have the eigenvector property assumed. From (A35), (A36), (A38), (A39), and (A40), we see that μ, r, η, W, and P are all dependent on one another, although η becomes decoupled from the rest when $s_1 = s_2$.

The metric perturbations described above are not gauge-invariant. We have partially chosen a gauge by the conditions

$$\delta g_{00} = \delta g_{0i} = 0 . \tag{A45}$$

Four more gauge transformations are consistent with these conditions, these being the infinitesimal coordinate transformations

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu ,$$

$$\delta g_{\mu\nu} \to \delta \bar{g}_{\mu\nu} = \delta g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu} .$$

(The semicolon denotes covariant differentiation in the unperturbed metric.) The ξ^μ consistent with (A45) are

$$\begin{aligned} \xi^0 &= -\gamma^{1/2} f_0(x^i) , \\ \xi^i &= -f_{0,p} \int \gamma^{-1/2} e^{-2\alpha} e^{-2\beta}{}_{ip}\, d\tau + f^i(x^i) . \end{aligned} \tag{A46}$$

The four functions f_0 and f^i are functions of the spatial variables x^i only. The f^i correspond to arbitrary infinitesimal coordinate transformations in a fixed $\tau = \text{const}$ hypersurface, and f_0 corresponds to an arbitrary reshaping of the $\tau = \text{const}$ hypersurface.

The Fourier-analyzed forms of f_0 and f^i are

$$f_0(x^i) = F_0 e^{ik_s x_s} , \quad f^i(x^i) = F^i e^{ik_s x_s} , \qquad F_0, F^i \ \text{const} . \tag{A47}$$

Their effect on $\mu^i{}_j$ is

$$\begin{aligned} \mu^i{}_j \to \bar{\mu}^i{}_j = \mu^i{}_j + F_0 \Big\{ & -k_j k_s \int \gamma^{-1/2} e^{-2\alpha} e^{-2\beta}{}_{is}\, d\tau - e^{-2\beta}{}_{is} k_s e^{2\beta}{}_{jt} k_u \int \gamma^{-1/2} e^{-2\alpha} e^{-2\beta}{}_{tu}\, d\tau + 2\gamma^{1/2} (\alpha_{,0} \delta^i{}_j + \beta_{ji,0}) \Big\} \\ & - F^i k_j - e^{2\beta}{}_{js} F^s e^{-2\beta}{}_{it} k_t . \end{aligned} \tag{A48}$$

The effects on μ, r, η, η_f, and q_i, when k_i is an eigenvector of β_{ij} with only $k_3 \neq 0$, are

$$\begin{aligned} \mu \to \bar{\mu} &= \mu + 2F_0 \Big\{ -(k_3)^2 \int \gamma^{-1/2} e^{-2\alpha} e^{-2\beta_{33}} d\tau + 3\gamma^{1/2} \alpha_{,0} \Big\} - 2F^3 k_3 , \\ r \to \bar{r} &= r - F_0 \{ -2\gamma^{1/2} \alpha_{,0} + b s_3 e^{-3\alpha} \} , \\ \eta \to \bar{\eta} &= \eta + b(s_1 - s_2) e^{-3\alpha} F_0 , \\ \eta_f \to \bar{\eta}_f &= \eta_f , \\ q_i \to \bar{q}_i &= q_i - 2 e^{2\alpha} e^{2\beta}{}_{il} F^l \quad (i = 1, 2) . \end{aligned} \tag{A49}$$

Thus F_0 and F^3 affect only the density and coupled gravitational wave perturbations. The latter, η, is gauge-independent when $s_1 = s_2$, as is η_f no matter what s_i is. The two variables q_i are affected by F^i, but the rotation is not:

$$(V_\perp)_i \to (\bar{V}_\perp)_i = (V_\perp)_i . \tag{A50}$$

The effect of a gauge transformation on V_3 in a dust model is

$$V_3 \to \bar{V}_3 = V_3 - F_0 k_3 . \tag{A51}$$

Since V_3 is constant according to (A25), V_3 may be removed by fixing the gauge.

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