in Eq. (9) and the residuals are not significantly different within 3° of superior conjunction, but then the corona effect can increase the errors in the measurements to a significant level.

V. CONCLUSION

Relativistic effects in time-delay experiments to transponders on radial trajectories in the field of the sun have been estimated through a Newtonian best fit of the relativistic terms in the relativistic expression of the time delay. For transponders moving toward the sun, the relativistic residuals are of the order of 100 m and so, rather difficult to detect. For transponders moving away from the sun, they are of the order of 5 km and larger than the errors in the measurements. In addition, the residuals in this case, due to the second-order curvature of the metric (β), are of the order of 1 km and also larger than the errors in measurements. The residuals due to the increase of the optical path for photons grazing the surface of the sun are at a 2-km level except very near superior conjunction ($<\pm 3°$). Except for the last one, these estimates of general relativistic effects as best-fit residuals are smaller than those obtained by simple evaluation of relativistic terms or from the divergence of relativistic and Newtonian predictions by orders of magnitude.

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[9]The yearly modulations of the measurements are not retained in the discussion of this experiment.

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Momentum Constraints as Integrability Conditions for the Hamiltonian Constraint in General Relativity*

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It is shown that if the Hamiltonian constraint of general relativity is imposed as a restriction on the Hamilton principal functional in the classical theory, or on the state functional in the quantum theory, then the momentum constraints are automatically satisfied. This result holds both for closed and open spaces and it means that the full content of the theory is summarized by a single functional equation of the Tomonaga-Schwinger type.

It is well known[1-3] that the whole message of general relativity is conveyed by the initial-value equations,[4]

$$\mathcal{H}_x[g_{ij};\pi^{kl}]=0\,, \tag{1}$$

$$\mathcal{H}_x^i[g_{ij};\pi^{kl}]=0\,, \tag{2}$$

where

$$\mathcal{H}_x = g^{-1/2}(g_{ik}g_{jl}-\tfrac{1}{2}g_{ij}g_{kl})\pi^{ij}\pi^{kl}-g^{1/2}R\,, \tag{3}$$

$$\mathcal{H}_x^i=-2\pi^{ij}{}_{|j}\,. \tag{4}$$

If Eqs. (1) and (2) hold on every three-dimensional spacelike cut through a space-time, then such a space-time is Ricci-flat, i.e., it satisfies all ten vacuum Einstein's equations. If one works in the language of the Hamilton-Jacobi equation,[5] then Eqs. (1) and (2) translate – via the prescription $\pi^{kl} \to \delta S/\delta g_{kl}$ – into four functional differential equations for Hamilton's principal functional S, namely,

$$g^{-1/2}(g_{ik}g_{jl}-\tfrac{1}{2}g_{ij}g_{kl})\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}}-g^{1/2}R=0\,, \tag{5}$$

$$\left(\frac{\delta S}{\delta g_{ij}}\right)_{|j}=0\,. \tag{6}$$

When gravity is quantized by the Dirac method,[6] Eqs. (1) and (2) become constraints imposed on the state functional ψ,[7]

$$g^{-1/2}(g_{ik}g_{jl}-\tfrac{1}{2}g_{ij}g_{kl})\frac{\delta^2\psi}{\delta g_{ij}\delta g_{kl}}+g^{1/2}R\psi=0\,, \tag{7}$$

$$\left(\frac{\delta\psi}{\delta g_{ij}}\right)_{|j}=0\,. \tag{8}$$

In a closed space, Eqs. (6) and (8) are equivalent to the statement that S and ψ are functionals of the three-geometry only, i.e., they can be regarded, for example, as functionals of the three invariants of the characteristic polynomial of $R^i{}_k$.[8] This conclusion is not correct in open spaces, when one must also prescribe the "location" of the three-dimensional slice at infinity.[9]

We shall prove now that, irrespective of whether the space is closed or open, *the three functional equations (6) need not be imposed separately but are a consequence of the single equation (5).* The quantum counterpart of this statement is also true: Eq. (7) implies Eq. (8). The significance of this result will be briefly discussed after the proof.

The argument runs as follows (we treat the classical case first): The fact that the composite functional $\mathcal{H}_x[g_{ij};\,\delta S/\delta g_{kl}]$ vanishes for every g_{ij} [Eq. (5)] implies that its functional derivatives also vanish. We have then

$$\frac{\delta\mathcal{H}_x}{\delta g_{ij}(y)}+\int d^3z\,\frac{\delta\mathcal{H}_x}{\delta\pi^{kl}(z)}\frac{\delta\pi^{kl}(z)}{\delta g_{ij}(y)}=0\,, \tag{9}$$

where $\pi^{kl}(z)$ is an abbreviation for $\delta S/\delta g_{kl}(z)$. Multiplying Eq. (9) by $\delta\mathcal{H}_{x'}/\delta\pi^{ij}(y)$ and integrating over y, we get

$$\int d^3y\,\frac{\delta\mathcal{H}_x}{\delta g_{ij}(y)}\frac{\delta\mathcal{H}_{x'}}{\delta\pi^{ij}(y)}=-\int d^3y\int d^3z\,\frac{\delta\mathcal{H}_x}{\delta\pi^{kl}(z)}\frac{\delta\mathcal{H}_{x'}}{\delta\pi^{ij}(y)}\frac{\delta^2 S}{\delta g_{ij}(y)\delta g_{kl}(z)}\,. \tag{10}$$

Next, we interchange the labels x and x' and subtract the resulting equation from Eq. (10) to obtain

$$[\mathcal{H}_x,\mathcal{H}_{x'}]\big|_{\pi^{kl}=\delta S/\delta g_{kl}}=\int d^3y\int d^3z\,\frac{\delta\mathcal{H}_x}{\delta\pi^{kl}(z)}\frac{\delta\mathcal{H}_{x'}}{\delta\pi^{ij}(y)}\left(\frac{\delta^2 S}{\delta g_{kl}(z)\delta g_{ij}(y)}-\frac{\delta^2 S}{\delta g_{ij}(y)\delta g_{kl}(z)}\right), \tag{11}$$

where [,] denotes the Poisson bracket. Using now the fact that the second variational derivatives of a functional commute, Eq. (11) reduces to

$$[\mathcal{H}_x,\mathcal{H}_{x'}]\big|_{\pi^{kl}=\delta S/\delta g_{kl}}=0\,. \tag{12}$$

On the other hand, the Poisson bracket $[\mathcal{H}_x,\mathcal{H}_{x'}]$ can also be calculated directly,[2] which gives

$$[\mathcal{H}_x,\mathcal{H}_{x'}]=(\mathcal{H}^i_x+\mathcal{H}^i_{x'})\delta_{,i}(x,x')\,. \tag{13}$$

Now, we substitute into Eq. (13) the variational derivatives $\delta S/\delta g_{ij}$ for the momenta π^{ij} and, taking into account Eq. (12), we get

$$2\left(\frac{\delta S}{\delta g_{ij}(x)}\right)_{|j}\delta_{,i}(x,x')+\left(\frac{\delta S}{\delta g_{ij}(x)}\right)_{|ji}\delta(x,x')=0\,. \tag{14}$$

Equation (14) means that

$$2\left(\frac{\delta S}{\delta g_{ij}(x)}\right)_{|j}\xi_{,i}(x)+\left(\frac{\delta S}{\delta g_{ij}(x)}\right)_{|ji}\xi(x)=0\,, \tag{15}$$

for arbitrary $\xi(x)$. Since ξ and $\xi_{,i}$ can be adjusted independently at any point, it follows that

$$\left(\frac{\delta S}{\delta g_{ij}(x)}\right)_{|j}=0\,. \tag{16}$$

The momentum constraints in the form (16) are thus integrability conditions for the Hamilton-Jacobi equation (5).

The proof of the analogous result in the quantum theory is even simpler.[10] One only needs to realize that the equation

$$\mathcal{H}_x\psi=0 \tag{17}$$

implies that

$$(\mathcal{H}_x\mathcal{H}_{x'}-\mathcal{H}_{x'}\mathcal{H}_x)\psi=0 \tag{18}$$

and, due to the fact that Eq. (13) holds for commutators as well as for Poisson brackets, it follows that

$$\mathcal{H}^i_x\psi=0 \tag{19}$$

by exactly the same reasoning that led from Eq. (12) to Eq. (16).

It should be pointed out here that, in the quantum case, different factor orderings for $\mathcal{H}$ amount to different factor orderings for $\mathcal{H}^i$ on the right-hand side of Eq. (13). It is natural to adopt an ordering for $\mathcal{H}$ which yields

$$\mathcal{H}^i = -2\pi^{ij}{}_{,j} - g^{il}(2g_{lj,k} - g_{jk,l})\pi^{jk},$$

because only then is Eq. (19) equivalent to the statement (8) that the state functional is invariant under localized coordinate transformations [a good ordering for $\mathcal{H}$ in this sense is the one given in Eq. (3)]. Nevertheless, for any consistent choice of factor ordering [i.e., such a choice that Eq. (18) does not give rise to a new constraint besides $\mathcal{H}^i = 0$] our conclusion remains true: $\mathcal{H}\psi = 0$ implies $\mathcal{H}^i\psi = 0$. Note also that in spite of the fact that we have been using the metric representation (i.e., treating S and ψ as functionals of g_{ij}), this is by no means necessary, and the implication $\mathcal{H}\psi = 0 \Rightarrow \mathcal{H}^i\psi = 0$, or its analog in terms of the Hamilton functional, holds irrespective of the representation in which the equations are written.

The implication we have just proved has some interesting consequences. First of all, it amounts to a reduction in the number of equations from $4\times\infty^3$ equations (5), (6) to $1\times\infty^3$ equations (5). Any solution of the Hamilton-Jacobi equation (5) solves automatically the momentum constraints (6). In a closed space, this means that the Hamilton-Jacobi equation implies by itself that the functional S which *a priori* depends on six metric coefficients g_{ij} is only a functional of three-geometry. A similar result holds in the quantum theory: One can base the whole of quantum geometrodynamics (modulo factor ordering) on the single Tomonaga-Schwinger-type equation (7), and one never needs to worry about the additional constraints (8), because they are satisfied automatically.

Secondly, the result presented in this note should be further compared with the recent work of Moncrief[11] who has shown that the validity of the Hamiltonian constraint at one point $\bar{x}$, together with the momentum constraints $\mathcal{H}^i_x$ at all points x, implies that the Hamiltonian constraint holds at every point. This amounts to a reduction from $4\times\infty^3$ equations $\mathcal{H}_x = 0$, $\mathcal{H}^i_x = 0$ to $3\times\infty^3$ equations $\mathcal{H}^i_x = 0$ plus a single equation $\mathcal{H}_{\bar{x}} = 0$.

Moncrief's results depend only on the fact that $\mathcal{H}$ is a scalar density. They continue to hold if, for example, the intrinsic curvature term in $\mathcal{H}$ given by Eq. (3) is omitted (in which case the theory is no longer general relativity). On the contrary, our result depends much more sensitively on the detailed structure of the theory. If the intrinsic curvature term is dropped, the Poisson bracket $[\mathcal{H}_x, \mathcal{H}_{x'}]$ vanishes identically, and our proof breaks down. One sees therefore that the condition $\mathcal{H} = 0 \Rightarrow \mathcal{H}^i = 0$ restricts[12] very severely the form of $\mathcal{H}$ and is precisely satisfied by the $\mathcal{H}$ of the general theory of relativity.

Note added in proof. It can be shown on rather general grounds that the structure of the commutators $[\mathcal{H}, \mathcal{H}']$, $[\mathcal{H}, \mathcal{H}_{i'}]$, $[\mathcal{H}_i, \mathcal{H}_{j'}]$ is exactly the same for every parametrized field theory in a Riemannian space-time, irrespective of whether such a Riemannian space-time is a prescribed background or is determined by the theory itself as in general relativity. (By "parametrized" we mean a theory in which the variables defining the 3-surface on which the state of the field is given are treated on the same footing as the field variables themselves.) This result, anticipated by Dirac,[13] will be discussed in detail by one of us (C.T.) in a forthcoming paper. It has the immediate consequence that the conclusions of the present paper (and also the related work of Moncrief[11]) are valid for the gravitational field in interaction with other fields as well as for field theories on a prescribed Riemannian background.

One of us (V.M.) established the classical (Hamilton-Jacobi) theorem after studying its quantum-mechanical counterpart given by Thomas.[10] The other coauthor derived both theorems independently, unaware also of the related work of Moncrief.[11]

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Galaxy Formation in Anisotropic Cosmologies*

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We analyze the growth rates of perturbations of the generic dust-filled Bianchi type-I cosmology (which exhibits anisotropy but not rotation). Anisotropy induces coupling between gravitational wave and density modes and can enhance the power-law rate of growth of the density perturbations. A maximum growth rate for pregalaxy perturbations is $t^{8/3}$ (where t is cosmic time), so that no conclusive solution to the galaxy formation problem is found. The detailed structure of the flow of energy between gravitational wave and density modes and of the growth of rotational modes (decoupled from other perturbations) is presented. We also give a detailed discussion of the gauge-invariance properties of these perturbations.

I. INTRODUCTION

Following the pioneering work of Lifshitz[1] and Lifshitz and Khalatnikov,[2] several authors have considered the problem of perturbations of isotropic, homogeneous cosmological models.[3-7] The hope has been to find an instability leading to the gravitational growth of perturbations analogous to the Jeans instability[8] for stationary Newtonian systems. It is now well recognized that no exponential (in time) growth of the Jeans type can occur in isotropic models, but instead the growth is limited to a power of the cosmic time t. For instance, in the "flat" ($k=0$) dust (pressureless perfect fluid) Friedmann[9,10]-Robertson[11,12]-Walker[13] (FRW) model, the rate of growth of the quantity $\delta w/w$ is proportional to $t^{2/3}$ (where $\delta w/w$ is the relative perturbation in the energy density, measured by a comoving observer).

With the recent renewed interest[14-18] in anisotropic homogeneous cosmologies, it seems a natural step to consider perturbations in such models. The generic strongly anisotropic model has the property that in at least one direction there is no expansion near the initial singularity. This is true even though the over-all volume of the universe is increasing because of expansion in other directions. This lack of expansion in one direction can be expected to strongly affect the growth of perturbations. We have found such effects, and find they result in density growth which can be faster than in the isotropic case. In some dust models of the type considered here, $\delta w/w$ is proportional to t^{ν} with ν arbitrarily close to $\frac{8}{3}$.

A power-law density growth as found here is not spectacularly different from the isotropic case.