along $\vec{\Omega}_{DS\,av}$ initially, so that $\vec{\Omega}_{DS\,av}\times\vec{n}_{LT}^{(1)}=0$, but it is not as convenient to do this.

[14]The equatorial orbit for which case $\sin(2\alpha)=0$ is of no interest because $\vec{\Omega}_{LT\,av}\times\vec{n}_{LT}^{(1)}=0$ as well as $\vec{\Omega}_{DS\,av}\times\vec{n}_{LT}^{(1)}=0$.

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Best-Fit Estimate of Relativistic Effects in Time-Delay Experiments*

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Time-delay experiments are analyzed within the frame of a curved space-time. Residuals from Newtonian best fits of relativistic data are used as a measure of the "relativistic effects." Radial transponder trajectories are considered. If the motion is towards the sun, the relativistic residuals are of the order of 100 m. If the motion is away from the sun, they are at the 10-km level and the fraction due to the second-order curvature of the metric is at the 1-km level. Those effects are significantly smaller than those calculated from the divergence of the Newtonian and relativistic predictions after exact fit of the initial measurements.

I. INTRODUCTION

In preceding papers,[1-4] time-delay experiments performed from the earth to a natural or to an artificial planet have been analyzed within the frame of a curved space-time. In Refs. 2, 3, and 4, the method used is essentially the following: If the motion has n constants of integration, it is assumed that an equal number of time-delay measurements are used to determine them, thus providing a perfect fit of hypothetical data at these points. Then one calculates the divergence of Newtonian and relativistic predictions for the other measurements. These divergences are interpreted as "relativistic effects." In some cases, this method gives results which are highly sensitive to the particular and arbitrary choice of the measurements used to determine the constants of the motion. The results can in fact vary by orders of magnitude.

In the present paper, a best-fit-analysis approach is followed in an attempt to simulate more closely the actual experiment. Essentially, one examines to which extent relativistic effects in the data could equally be explained within Newtonian theory by appropriate increments to the constants of the motion. More precisely a best fit of the relativistic terms in the relativistic expression for the time delay is performed with increments to the classical parameters of the expression. Residuals from this best fit are "relativistic effects" which cannot be explained in classical theory within the limitation of the problem. Such residuals can be compared with the expected accuracy of the measurements for an estimate of the possibility of determining the components of the curvature of the metric and thus test general relativity and other theories of gravitation.

This approach is applied here to time-delay experiments carried from the earth to artificial planets moving on radial trajectories towards the sun or away from it. The results for this simple model should suggest an upper limit to the order of magnitude of the relativistic effects to be seen on quasiradial sections of a "grand tour" trajectory or of a very high eccentricity orbit of comparable energy.

For simplicity, it is assumed that the earth is on a circular orbit and that the artificial planet trajectory is contained in the ecliptic plane. Classical perturbations such as the oblateness of the sun and the gravitational field of other planets are neglected. Relativistic effects due to the rotation of the sun (Lense-Thirring effects) are also neglected (they are at the cm level or lower and undetectable at the present time). The field of the sun is assumed to have spherical symmetry and is described by a generalized metric. Thus relativity corrections for theories which absorb the gravitational field in the curvature of space-time and where test particles travel along geodesics can be evaluated and compared.

In Sec. II, a solution to first order in GM/rc^2 is given for the motion of a test mass along a radial orbit. Previous results on the propagation of photons and on circular orbits are also included. In Sec. III, the Newtonian best fit of relativistic data and the calculation of relativistic residuals are discussed. In Sec. IV, time-delay experiments to transponders moving towards the sun and away from it are considered.

II. MOTION ALONG A RADIAL ORBIT

In the present case where spherical symmetry is assumed, it is convenient to describe the gravitational field of the sun by the generalized metric as introduced by Eddington:[5]

$$c^2d\tau^2 = \left[1-2\alpha\left(\frac{r_0}{r}\right)+2\beta\left(\frac{r_0}{r}\right)^2\cdots\right]c^2dt^2 - \left[1+2\gamma\left(\frac{r_0}{r}\right)\cdots\right]d\sigma^2\,, \tag{1}$$

with

$$d\sigma^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 = dx^2+dy^2+dz^2\,. \tag{2}$$

$r_0 = GM/c^2$; α, β, and γ are dimensionless numbers. G is the gravitational constant, M is the mass of the sun, and c is the speed of light. r, θ, ϕ and x, y, z are the polar and rectangular coordinates. t is the time and τ the proper time. The origin is at the center of gravity of the sun and the orientation of the axis is fixed with respect to the stars.

The quantities α, β, and γ measure the curvature of the space-time. α is set to unity so that Newtonian theory is obtained in the zeroth-order approximation in r_0/r. β and γ are regarded as quantities to be determined by experiments. According to general relativity, their value is unity. Their value can be different in other relativistic theories of gravitation.[6]

As usual, the relativistic predictions will be evaluated to first order in (r_0/r) only. Thus the α, β, and γ terms of the metric need to be retained only.

The motion of photons and transponders is along geodesics of the metric given in Eq. (1). If the motion of the transponders is restricted to be in the ecliptic ($\theta=\pi/2$) plane and along a radial line (ϕ_r = const), the geodesic law of motion takes the following form[4]:

$$t = \pm\frac{T_E A\sqrt{-A}}{\pi\sqrt{8}\,(1+V)}(W^- - W_0^-) + t_0 \tag{3a}$$

for a bounded orbit ($A<0$) and

$$t = \pm\frac{T_E\sqrt{A^3}}{\pi\sqrt{8}\,(1+V)}(W^+ - W_0^+) + t_0 \tag{3b}$$

for an unbounded orbit ($A>0$), where $V=\frac{3}{2}-2/A-\gamma/A$. T_E is the period of the earth. The radius of the (circular) orbit of the earth is chosen as the unit of length. W_0 is the value of W for $\rho=1$. To $O(r_0/r)$, $-A$ is the classical ratio of the potential energy to the total energy at unit distance from the origin. The + and − signs refer to a motion away from and toward the sun. t_0 is the extrapolated time of passage of the transponder at 1 A.U. The quantities W^- and W^+ are

$$W^- = -\frac{(-1-A\rho-U\rho^2)^{1/2}}{A\rho} + \tfrac{1}{2}\sin^{-1}\left[\frac{A\rho+2}{A\rho(1-4U/A^2)^{1/2}}\right], \tag{4a}$$

$$W^+ = \frac{(1+A\rho+U\rho^2)^{1/2}}{A\rho} - \tfrac{1}{2}\ln\left[\frac{(1+A\rho+U\rho^2)^{1/2}+1)}{A\rho}+\frac{1}{2}\right], \tag{4b}$$

where $\rho=1/r$ and $U=-Ar_0(2+\beta+2\gamma)$. From now on, r and ρ will refer to the transponder radial coordinate. The Eqs. (3) and (4) differ from the Newtonian ones through the U and V terms. In addition to the position of the transponder, the position of the earth is needed and is written as

$$r_E = 1\,,\qquad \phi = 2\pi t/T_E + \phi_0\,, \tag{5}$$

where ϕ is reckoned from the radial trajectory of the transponder. The time of travel of the electromagnetic signals from the earth to the transponder is[7]

$$T = (R^0 + R')/c\,, \tag{6a}$$

where R^0 is the classical distance

$$R^0 = (r^2+1-2r\cos\phi)^{1/2}\,, \tag{6b}$$

and R' is the relativistic term

$$R' = (1+\gamma)r_0\ln\left(\frac{R^0+1-r\cos\phi}{r(R^0+\cos\phi-r)}\right). \tag{6c}$$

III. RESIDUALS IN NEWTONIAN BEST FIT OF RELATIVISTIC DATA

The experiments considered are illustrated in Figs. 1(a) and 1(b). Measurements of photon travel time from the earth to a transponder are made over a portion AB of the transponder orbit. The travel time and the timing of the measurements are made with an atomic time standard or equivalent.[8] Also, the measurements are expressed in units of the first one so that the unknown speed of light is eliminated.

The three relations involved are thus

$$t_i = t_i(T_E, A, \rho_i, t_0, \beta, \gamma)\,, \tag{7a}$$

$$\phi_i = \phi_i(T_E, \phi_0, t_i)\,, \tag{7b}$$

$$T_{i1} = T_{i1}(\phi_i, \rho_i, \phi_1, \rho_1, \gamma)\,, \tag{7c}$$

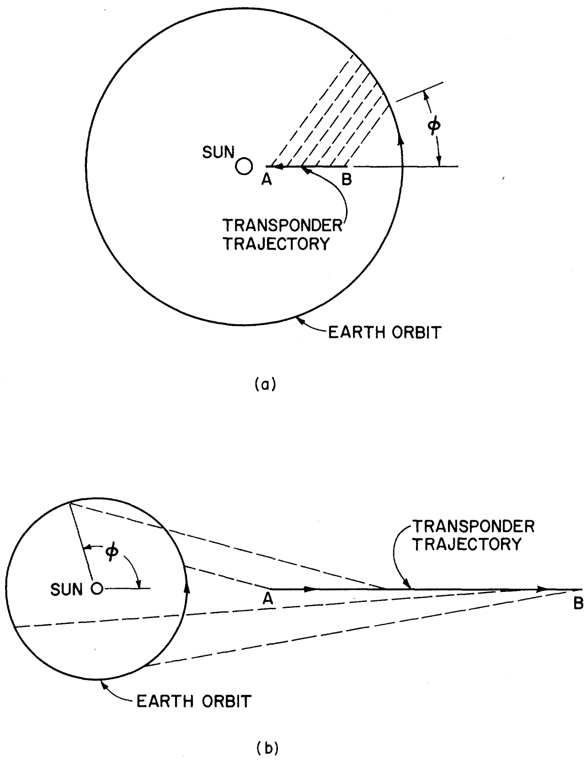

FIG. 1. Time-delay experiments to a transponder moving toward the sun (a) and away from it (b).

where the two first ones are Eqs. (3) and (5). T_{i1} is the ratio of the ith to the first measurement and is readily calculated from Eq. (6). If Eqs. (7b) and (7c) are combined, the system reduces to

$$t_i = t_i(T_E, A, \rho_i, t_0, \beta, \gamma), \tag{8a}$$

$$T_{i1} = T_{i1}(T_E, \phi_0, \rho_i, \rho_1, \gamma, t_1, t_i). \tag{8b}$$

As shown by Eqs. (8a) and (8b), the ratios T_{i1} are functions of the four constants of integration A, T_E, ϕ_0, t_0, the relativistic parameters β and γ, and the observables t_1 and t_i:

$$T_{i1} = T_{i1}(T_E, A, t_0, \phi_0, \beta, \gamma, t_i, t_1). \tag{9}$$

If the data T_{i1} are relativistic, an exact fit with Newtonian theory would require that increments T_E', A', t_0', and ϕ_0' to the corresponding first integrals compensate the absence of the terms in β and γ in Eq. (9). This will be written as

$$\sum_k A_{ik} x_k + B_i = 0, \tag{10a}$$

where the unknown $x_1, \ldots, x_4$ are the increments T_E', A', t_0', and ϕ_0'. The A's are the partials

$$A_{ik} = \partial T_{i1}/\partial x_k, \tag{10b}$$

where the index i refers to a particular measurement. B_i is the relativistic component of T_{i1} resulting from the presence of the quantity V in Eq. (3), U in Eq. (4), and R' in Eq. (6a). The A_{ik}'s and B_i's are long expressions and are omitted here for this reason. Equation (10) has four unknowns. If more than four observations are assumed to be available, values $\bar{x}_i$ of the parameters which best fit these observations are solved for by least-squares methods. The deviations d_i of the Newtonian best fit from the assumed relativistic measurements are then

$$d_i = A_{ik}\bar{x}_k + B_i. \tag{11}$$

The d_i's are dimensionless quantities. They are readily converted to meters:

$$d_i(\text{meters}) = 3\times 10^{11} R_1^0 d_i. \tag{12}$$

It is also possible to consider separately the residuals in the best fit of the effects resulting from the presence in the metric of the second-order curvature (β) or the space curvature (γ) by including in the quantity B_i appearing in Eq. (10a) the relativistic term in T_{i1} due to β or γ only.

IV. RESULTS

Three specific experiments have been considered. In all cases results are also briefly compared with results obtained from simple estimates of the relativistic terms in Eq. (9) and with results obtained from the divergence of relativistic and Newtonian predictions after exact fit of the first five measurements. This last number corresponds to the number of parameters (T_E, A, t_0, ϕ_0) of the present simplified problem plus 1 for the speed of light.

In the first experiment, the transponder is travelling toward the sun. Time-delay measurements are performed at equally spaced points of its trajectory from 0.51 to 0.01 A.U. (1 solar radius from the surface of the sun). It is assumed that $t_0 \simeq \phi_0 \simeq 0$. Thus the position of the earth is $\simeq 0°$ at the extrapolated time of passage of the transponder at 1 A.U. (close to actual conditions of launch and measurements sequence). The parameter A of the transponder orbit is given the value -100, which closely corresponds to "dropping" the transponder on a radial trajectory with no initial velocity. Consequently, ϕ, the position of the earth, varies from 18° to 28° during the interval of time covered by the measurements. In this case, the relativistic term in Eq. (9) increases as the transponder gets closer to the sun. The effect is similar to the one shown in Fig. 2 but is ~25 times larger in the region of interest (0.15 – 0.1 A.U.). The divergence of the Newtonian and general-relativity predictions following a perfect fit of the observations made at 0.51, 0.46, 0.41, 0.36, and 0.31 A.U. is shown on Fig. 2. The effect is at the

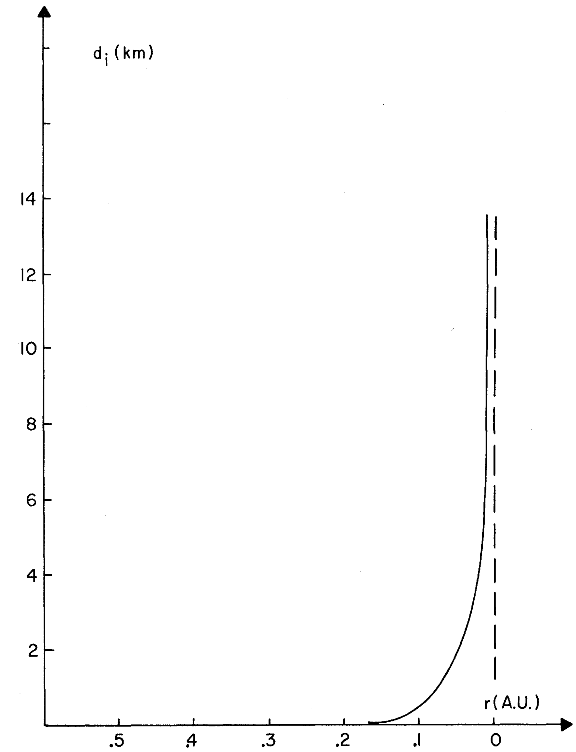

FIG. 2. Divergence of Newtonian and relativistic predictions in time-delay experiments to a transponder moving toward the sun.

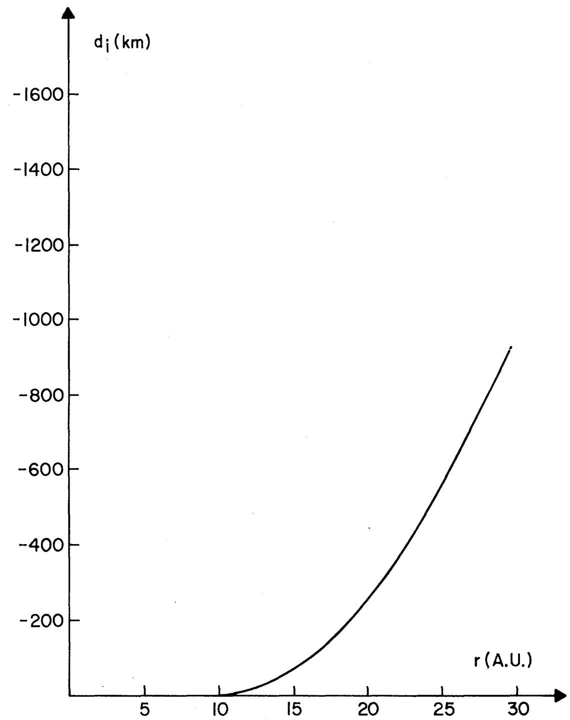

FIG. 4. Divergence of Newtonian and relativistic predictions in time-delay experiments to a transponder moving away from the sun.

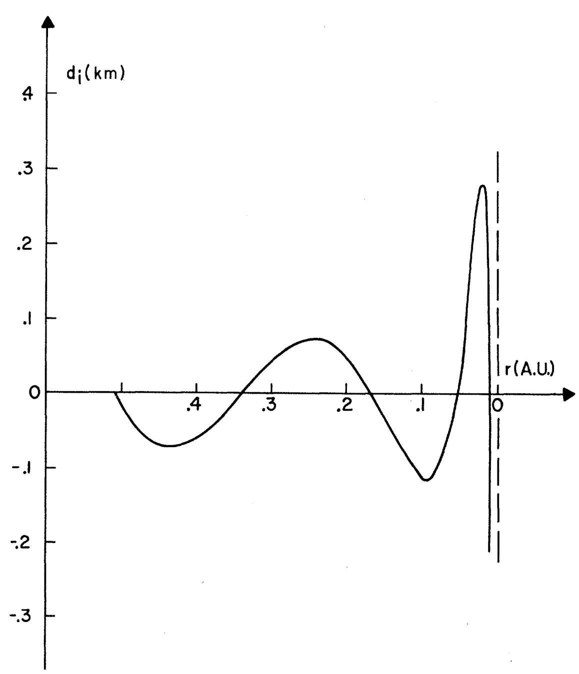

FIG. 3. Relativistic residuals in a Newtonian fit of relativistic time-delay measurements to a transponder moving toward the sun.

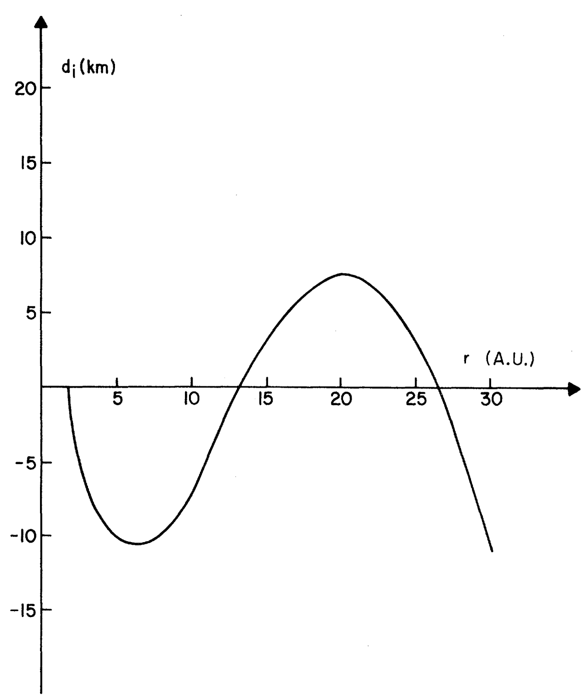

FIG. 5. Relativistic residuals in a Newtonian fit of relativistic time-delay measurements to a transponder moving away from the sun.

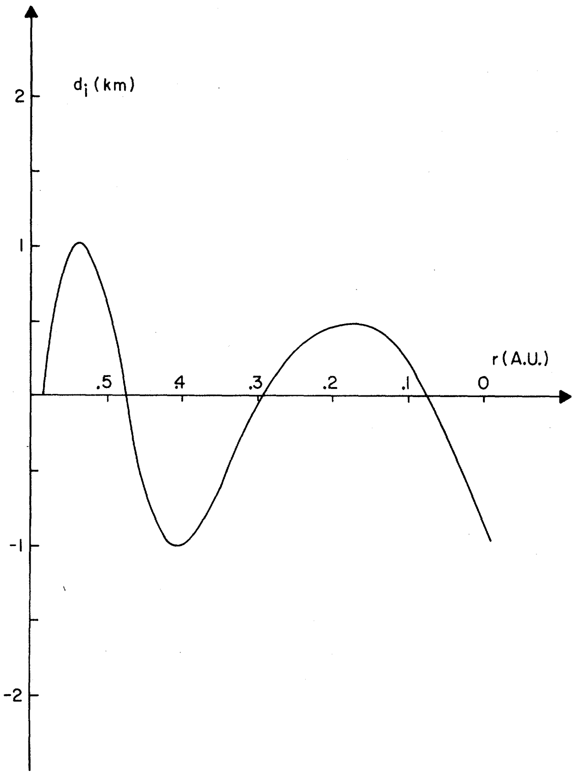

FIG. 6. Relativistic residuals due to the β term of the metric in a Newtonian fit of relativistic measurements to a transponder moving away from the sun.

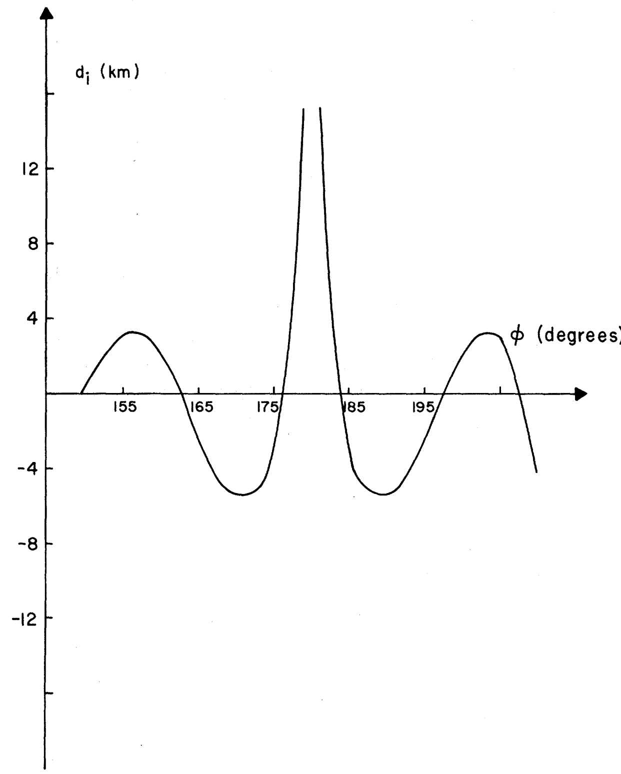

FIG. 7. Relativistic residuals in a Newtonian fit of relativistic time-delay measurements near superior conjunction of the transponder.

2-km level and reaches 14 km at 0.01 A.U. This is still a rather large effect. However, if a best fit of relativistic data is attempted with Newtonian theory, the residuals as shown on Fig. 3 are of the order of 100 m and the maximum residual is near 0.01 A.U. and is about 300 m, down by a factor ~25 from the divergence effect. In this experiment, very similar results are obtained for the γ component of the relativistic effect. The term in β in Eq. (9) increases in a fashion similar to the one shown on Fig. 2. It reaches 44 km at 0.01 A.U. The Newtonian fit of this effect is, however, good enough to keep the relativistic residuals below the 1-m level, an ~10^4 reduction factor.

In the second experiment, the transponder is travelling away from the sun. Time-delay measurements are made at equally spaced points of its trajectory from 2 A.U. to 30 A.U. (orbit of Neptune). Again, it is assumed that $t_0 \simeq \phi_0 \simeq 0$. It is also assumed that the transponder energy is such that it covers the distance to the orbit of Neptune in ~9 yr ($A = 25$). The general-relativistic term in Eq. (9) is a smoothly increasing quantity which reaches 900 km at 30 A.U.[9] The divergence of Newtonian theory and general relativity following a perfect fit at 2, 4, 6, 8, and 10 A.U. is a very similar quantity (Fig. 4). If a best fit of relativistic data is attempted with Newtonian theory, the residuals as shown on Fig. 5 are at the 5-km level, down by a factor of ~200 from the previous quantities. As was the case in the previous experiment, similar remarks apply to the relativistic effects resulting from the parameter γ of the metric. The term in β in Eq. (9) increases smoothly to reach 25 km at 30 A.U. The Newtonian fit is not as good as it was in the first experiment however, and the residuals shown in Fig. 6 are at the 1-km level.

In the third experiment considered, the transponder is again travelling away from the sun. In order to test for the "slowing down" of photons grazing the surface of the sun, time-delay measurements are made at equally spaced points of its trajectory between 3.6 and 4.4 A.U. With the assumption $t_0 \simeq \phi_0 \simeq 0$, this corresponds to superior conjunction of the transponder. Again, $A = 25$. If the relativistic term of Eq. (9) is used as an estimate of the relativistic effect 15° and 5° before or after superior conjunction, the results are ~14 km and 30 km. The residuals from a Newtonian best fit of the data shown in Fig. 7 are, however, down to 4 and 5 km at these points. The relativistic term in Eq. (9) and the residuals are not significantly different within 3° of superior conjunction, but then the corona effect can increase the errors in the measurements to a significant level.

V. CONCLUSION

Relativistic effects in time-delay experiments to transponders on radial trajectories in the field of the sun have been estimated through a Newtonian best fit of the relativistic terms in the relativistic expression of the time delay. For transponders moving toward the sun, the relativistic residuals are of the order of 100 m and so, rather difficult to detect. For transponders moving away from the sun, they are of the order of 5 km and larger than the errors in the measurements. In addition, the residuals in this case, due to the second-order curvature of the metric (β), are of the order of 1 km and also larger than the errors in measurements. The residuals due to the increase of the optical path for photons grazing the surface of the sun are at a 2-km level except very near superior conjunction ($<\pm 3°$). Except for the last one, these estimates of general relativistic effects as best-fit residuals are smaller than those obtained by simple evaluation of relativistic terms or from the divergence of relativistic and Newtonian predictions by orders of magnitude.

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[7]This formula is identical to Eq. (7) of Ref. 1.

[8]Since the time standard is on a circular orbit, atomic time and coordinate time differ by a constant factor and so the two times can, in the present calculation, be substituted for one another.

[9]The yearly modulations of the measurements are not retained in the discussion of this experiment.

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Momentum Constraints as Integrability Conditions for the Hamiltonian Constraint in General Relativity*

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It is shown that if the Hamiltonian constraint of general relativity is imposed as a restriction on the Hamilton principal functional in the classical theory, or on the state functional in the quantum theory, then the momentum constraints are automatically satisfied. This result holds both for closed and open spaces and it means that the full content of the theory is summarized by a single functional equation of the Tomonaga-Schwinger type.

It is well known[1-3] that the whole message of general relativity is conveyed by the initial-value equations,[4]

$$\mathcal{H}_x[g_{ij};\pi^{kl}]=0, \qquad (1)$$

$$\mathcal{H}_x^i[g_{ij};\pi^{kl}]=0, \qquad (2)$$

where

$$\mathcal{H}_x=g^{-1/2}(g_{ik}g_{jl}-\tfrac{1}{2}g_{ij}g_{kl})\pi^{ij}\pi^{kl}-g^{1/2}R, \qquad (3)$$

$$\mathcal{H}_x^i=-2\pi^{ij}{}_{|j}\,. \qquad (4)$$

If Eqs. (1) and (2) hold on every three-dimensional spacelike cut through a space-time, then such a