Applications of the Dual-Parton Model to High-Energy Multiparticle Processes*

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The dual-parton model of Kraemmer, Nielsen, and Susskind is reviewed and applied to high-energy production processes. Most, but not all, of the predictions are not new and have been given by many practitioners of the multiperipheral and parton models. What is new is a unified presentation of these predictions in terms of the fragmentation of the hadronic string into segments, each retaining the properties it had before collision. The method is applied to transverse-momentum distributions and correlations and to charge distributions of secondary hadrons.

I. PARTONS AND FRAGMENTATION

Several years ago it was suggested that a particularly clear description of hadrons moving near the speed of light might be possible because of a Galilean subgroup of the Poincaré group.[1] This subgroup guarantees that the description of a system from the infinite-momentum frame is entirely nonrelativistic with respect to motions in the two-dimensional plane perpendicular to the direction of large momentum. In particular it was argued that any many-particle description of hadrons would only make sense in this frame. Subsequently, Feynman has shown that a great many features of strong interactions can be understood within the framework of a many "parton" picture at infinite momentum.[2]

In this paper I will explore the relationship between what goes on inside a free hadron on the one hand and what goes on during and after a high-energy collision on the other. We shall follow Ref. 1 and assume that a high-energy hadron is a collection of constituents, each with a $\mathfrak{z}$ component of momentum $k_{\mathfrak{z}}$ and a transverse momentum K. The total momentum of the hadron is $(p_{\mathfrak{z}}, P)$. Following Ref. 1 we describe the longitudinal motion by defining a longitudinal fraction η. We arbitrarily choose some very large momentum, either some momentum in the problem or else just some number, L, with which to compare $\mathfrak{z}$ momenta. The longitudinal fraction of any system with $\mathfrak{z}$ momentum $k_{\mathfrak{z}}$ is

$$\eta = k_{\mathfrak{z}}/L . \tag{1}$$

The longitudinal fraction of the hadron is

$$\eta_H = p_{\mathfrak{z}}/L ,$$

and it is composed of the longitudinal fractions of the constituent partons.

Each parton is thus described by a triplet consisting of its longitudinal fraction and its transverse momentum (η, K). In terms of these variables the relativistic phase space is

$$d\Omega = \frac{d\eta dK}{\eta} . \tag{2}$$

Now it is characteristic of high-energy collision products that while their transverse momentum is strongly bounded below a few hundred MeV the longitudinal distribution of secondaries roughly follows phase space, meaning that for $\eta \to 0$ the number of secondaries in the interval $d\eta$ goes as $(\eta)^{-1}$. In view of this, Feynman has suggested that the partons themselves are similarly distributed. We shall follow Feynman's suggestion and assume the parton density in η space behaves like

$$\frac{dN}{d\eta} \sim \frac{1}{\eta} \tag{3}$$

at or near $\eta = 0$.

If we consider the action of a boost along the $\mathfrak{z}$ axis it is readily seen that K is left unchanged and η is rescaled:

$$\eta \to e^{\omega}\eta , \tag{4}$$

where ω is the hyperbolic angle of the boost. One way to characterize the Feynman distribution is to say that in the region near $\eta = 0$ (where the partons are called "wee" by Feynman) the distribution is unchanged by boosting. From this it follows that by looking at the wee-parton sea it is not possible to distinguish hadrons of different momentum along the $\mathfrak{z}$ axis. This universality of the wee-parton sea is the single most important feature of the Feynman distribution, accounting not only for the properties of multiparticle final states but also for the constancy of high-energy cross sections.

It is very convenient to introduce a variable called rapidity in terms of which boosts act as translations and in which the wee-parton density is uniform. Equations (3) and (4) show that the required variable is

$$r = \ln\eta . \tag{5}$$

A second assumption which will be used involves the nature of the interactions between partons. We shall picture each parton as occupying a position on the η or r axis and we will furthermore assume a degree of locality of the interactions in this space. Specifically, we suppose that the interactions between partons extend only over a finite distance in rapidity so that a hadron can be considered to be a one-dimensional string whose parts interact only with neighboring parts. The one-dimensional axis used to label partons or string locations can be taken to be η, r, or any function of η.

To complete the description of the system's motion we introduce transverse degrees of freedom either in position or momentum space. The transverse location of partons with a given fraction η will be called $X(\eta)$ and the transverse momentum $K(\eta)$. The Hamiltonian of this system in the infinite-momentum frame is to be identified with $(\text{mass})^2$ as in Ref. 1 and is expected to have a part which can be described as attractive potential energy between partons of neighboring rapidity and a part describing the "nonrelativistic" kinetic energies of the partons. In Sec. II we will illustrate this viewpoint with the specific dual-parton model.

Consider next the collision of two such hadrons at large energy. One hadron called the right mover has a positive momentum p_R along the z axis and the other, the left mover, has momentum $-p_L$. Let us represent particle 1, the right mover, as a segment of the η axis extending from where the wee partons reside to p_R/L where L is the arbitrarily chosen momentum scale. Similarly particle 2, the left mover, is a segment of the η axis extending from 0 to $-p_L/L$.

Now according to Feynman the dynamics of such a collision initially affects only those partons in a bounded momentum interval. Let us call the characteristic size of the interval K. Then the partons which actually are affected by the collision are those in the interval

$$|\eta| \sim K/L .$$

Thus after collision we see a system of two strings each being perturbed, the perturbed section extending over an interval from $|\eta| = 0$ to $|\eta| = K/L$. Now, if we choose our scale L to be of order P_R or P_L then as P_R and P_L tend to infinity the perturbed segment becomes smaller and smaller, tending ultimately to a point at the $\eta = 0$ wee-parton point. The remainder of the partons are initially unaffected by the impulsive collision.

Now what becomes of the two systems after collision? If they were both ideal harmonic strings, the perturbed ends would act as a source of phonons or waves which would propagate back and forth on the string indefinitely. In this approximation each subsystem is described as a superposition of narrow resonances which make up the spectrum of a harmonic string. This, however, is not correct. We know that the actual collision products are not two excited stable states but rather a collection of secondary hadrons. Accordingly, we will assume that the excitation energy is quickly dissipated by the mechanism of fragmentation of the strings into segments near the wee ends. The process is depicted schematically in Fig. 1 where we show the evolution of the process as two initially free strings, a collision which quickly excites the wee ends and a final fragmented state.

The longitudinal momentum of a fragment of length $d\eta$ near the end $\eta = 0$ is given by the sum of longitudinal momenta carried by partons in that interval. This in turn is

$$L\eta dN \approx L\eta \frac{d\eta}{\eta} \sim d\eta\, L .$$

Thus near the wee end the length of a fragment $d\eta$ is directly measured by the longitudinal fraction of the fragment.

To obtain an idea about the pattern of fragmentation we shall rely on a symmetry argument.

We have already stated that the wee-parton sea is expected to be universal so that hadrons of different momentum cannot be distinguished by the properties of the low-η distribution. Therefore we expect that the pattern of fragmentation near $\eta = 0$ will be independent of the momentum P_R, say. Now since a longitudinal boost is simply a rescaling of the η axis, it follows that the fragmentation near the $\eta = 0$ end should not know about the length of the η axis and should be scale invariant under a

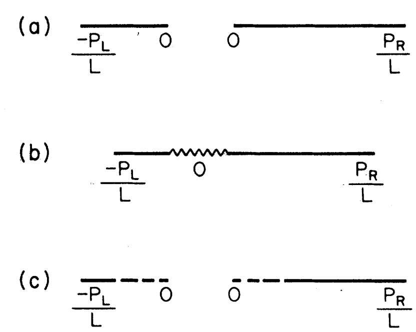

FIG. 1. The collision process of two high-energy hadrons depicted schematically as the evolution of the process of (a) two initially free strings, (b) a collision which quickly excited the wee ends, and (c) a final fragmented state.

rescaling of η. This requires the mean number of fragments per unit η, $dN/d\eta$, to be scale-invariant, which in turn means $dN/d\eta \sim 1/\eta$. Another way to say the same thing is that the mean number of partons on a fragment just after collision is the same for all fragments of low η.

Our third main assumption involves the way the conserved quantities are carried by the partons and what happens when fragmentation occurs. We will assume that the transverse momentum, charge, isospin, hypercharge, and baryon number are carried by the partons in such a way that operator-valued densities $\rho_i(\eta)$ or $\rho_i(r)$ exist in the η or rapidity space. By definition

$$\rho_i(\eta)d\eta = \rho_i(r)dr\,, \tag{6}$$

and each equals the amount of the ith conserved quantity carried by partons in the interval $d\eta$. We also expect to be able to define currents $\mathcal{J}_i(\eta)$ so that local conservation can be defined on the η axis.

Furthermore, we will assume that when fragmentation occurs the individual segments carry the same value of the conserved quantities that they had just before collision. This means that the collision is so impulsive and the fragmentation so quick that the charges do not have time to redistribute themselves before fragmentation sets in.

The power of these assumptions lies in the fact that they directly relate the current and momentum distributions within a single free hadron to the corresponding distributions in a high-energy collision. This allows us to relate different models of particles structure directly to predictions of multiparticle phenomena.

II. DUAL-PARTON MODEL

The dual-parton model[3] represents a hadron as a string of partons parametrized by a parameter θ running from $\theta=0$ to $\theta=\pi$. The density of partons is assumed to fluctuate about a mean given by

$$\frac{dN}{d\theta} \sim \frac{1}{\sin\theta}\,. \tag{7}$$

It is also assumed that the momentum density $P_\mu(\theta)$ ($\mu=1,2,3,4$) is given by

$$P_\mu(\theta) = \frac{P}{\pi} + P_{\text{fluct}}\,, \tag{8}$$

where P/π is the total hadron momentum divided by the width of the strip and P_{fluct} represents fluctuations which are mathematically describable by normal modes. If we allow P to increase to ∞ along the $\mathfrak{z}$ axis and divide $P_\mu(\theta)$ by $P_\mathfrak{z}$ to define the η density we get

$$\frac{\Delta\eta}{\Delta\theta} = \frac{1}{\pi} + \frac{P_{\text{fluct}}}{P}\,. \tag{9}$$

We have analyzed the size of P_{fluct} and have found that this part of $P_\mathfrak{z}(\theta)$ does not grow with increasing $P_\mathfrak{z}$ so that in the infinite-momentum limit $\Delta\eta/\Delta\theta = 1/\pi$.

The η of a parton at position θ may be estimated by dividing the total η in an interval $\Delta\theta$ by the number of partons in that interval. The result is that partons at θ carry an η given by

$$\eta(\theta) \sim \sin\theta\,. \tag{10}$$

Thus the η axis and θ interval are really the same. A slight difference occurs because θ is not quite defined by η. For each η there are two positions on the string, but near the ends $\theta=0$ and $\theta=\pi$ there is a one to one correspondence between θ and η. Since we will be interested in the fragmentation near one end, say $\theta=0$, the wee partons at the other end are not too important and we may identify η with θ.

Since the string density near the ends varies as $(\sin\theta)^{-1}$ the density in η space also tends to the Feynman form $d\eta/\eta$.

The remaining degrees of freedom describing a parton at point θ are its two-dimensional transverse position $X_i(\theta)$ and various discrete quantities.

We shall assume that each parton couples to its nearest neighbor in θ or η space with attractive forces which for small separations are approximately harmonic-oscillator wells. Probably a more realistic idea is to think of $X_i(\theta)$ as the coarse-grained average of positions of several partons in an interval of rapidity of undetermined size δr and the force as an effective attraction between such clusters.

The interval δr represents a limitation on our program since the averages of quantities such as charge and momentum densities are only defined for intervals larger than δr. It is only useful to conceive of a hadronic reaction as the fragmentation of two strings if the available rapidity axis is many times longer than δr. This in turn means $\ln s > \delta r$ where s is the incident center-of-mass Mandelstam variable.

Recalling the fundamental analogy between relativistic infinite momentum mechanics[1] and two-dimensional Galilean mechanics we identify the energy as the mass squared of the entire system and consider a many-particle Hamiltonian with kinetic and potential energies. The kinetic energy of a parton is

$$E_K = \frac{K_i^2}{2\eta_i}\,, \tag{11}$$

and the potential energy we take to be

$$E_P = g\frac{(X_i - X_{i+1})^2}{2\eta_i}\,, \tag{12}$$

where g is a coupling parameter. Equation (12) is chosen because it represents an attraction with a smooth behavior near the origin which is expected to be the right thing for the interaction of two clusters. The factors of $1/\eta$ in both Eqs. (11) and (12) are present in order to give the infinite momentum Hamiltonian correct boost properties along the z axis. We can understand these factors as a consequence of time dilation. If two pairs have longitudinal momentum in ratio η_1/η_2 then the motion of the faster pair is time dilated relative to the slower pair by the same factor. Thus all things being equal, the energy contained in the faster pair should be smaller by the inverse power.

If we now identify $\eta \sim \sin\theta$ and take the limit of a continuous string, the energy is

$$M^2 = \int_0^\pi d\theta \left[\left(\frac{\partial X}{\partial \tau}\right)^2 + \left(\frac{\partial X}{\partial \theta}\right)^2\right], \tag{13}$$

where $\partial X/\partial\tau$ is the "nonrelativistic" velocity and is related to the momentum in the transverse plane. The time τ is the usual infinite-momentum dilated time

$$\tau = t/P_z \tag{14}$$

conjugate to M^2.

Equation (2) will be recognized as the Hamiltonian for a nonrelativistic harmonic string. Accordingly, we can deal with it using a conventional normal-mode decomposition[4]:

$$X_i(\theta,\tau) = X_i^{\text{c.m.}}(0) + P_i^{\text{c.m.}}\tau + i\sqrt{2}\sum_{l=1}^{l_{\max}} \frac{a^+(l)e^{il\tau} - a^-(l)e^{-il\tau}}{\sqrt{l}} \cos l\theta. \tag{15}$$

The normal modes describing the point θ range from a zero-frequency mode describing the linear center-of-mass motion, $X^{\text{c.m.}}(0) + P^{\text{c.m.}}\tau$, to a normal mode $l_{\max}(\theta)$. The maximum normal mode[3] $l_{\max}(\theta)$ is chosen to reflect that fact that in a region of the θ axis where the spacing of partons is $\Delta\theta$ the shortest wavelength which can propagate has wavelength $\Delta\theta$. This means

$$l_{\max}(\theta) \sim \frac{1}{\sin\theta}\ . \tag{16}$$

Equations (13)–(16) then define the motion of the parton string in the transverse plane and Eq. (10) prescribes the longitudinal motion.

III. APPLICATION TO TRANSVERSE-MOMENTUM SPECTRUM

Let us consider the average transverse momentum of a secondary string fragment by identifying it with the fluctuation of transverse momentum on the corresponding string segment before collision.[3] A fragment with small longitudinal fraction η is expected to originate from a point $\theta = \eta c$ and have length $\Delta\theta = \eta$, where c is an experimentally determinable number having to do with the density of secondaries in rapidity space. To compute the momentum on such a fragment consider first the momentum density defined to be canonically conjugate to $X(\theta)$. For the transverse components, $P^{\text{c.m.}}$ vanishes so that the fluctuating normal modes give the whole answer

$$P_\perp(\theta) = \frac{1}{\sqrt{2}\,\pi}\sum_l [a^+(l) + a^-(l)]\sqrt{l}\,\cos l\theta. \tag{17}$$

The momentum carried by a segment from θ_1 to θ_2 is

$$\int_{\theta_1}^{\theta_2} P_\perp\, d\theta.$$

The integration will wipe out those contributions to Eq. (17) from modes with wavelength smaller than $\theta_1 - \theta_2$ so that a rough approximation is

$$\frac{1}{\sqrt{2}\,\pi}(\theta_1-\theta_2)\sum_{l=1}^{l=(\theta_1-\theta_2)^{-1}} \sqrt{l}\,(a^+ + a^-)\cos l\theta = \frac{1}{\sqrt{2}\,\pi}\,\eta\sum_1^{\pi/\eta}(a^+ + a^-)\cos l\theta, \tag{18}$$

where $\theta = \frac{1}{2}(\theta_1 + \theta_2)$. This is the momentum on the segment corresponding to the secondary fragment of longitudinal fraction η. Squaring, taking the expectation value, and using an average estimate of $\frac{1}{2}$ for $(\cos l\theta)^2$ gives[3]

$$\langle P_\perp^2\rangle = \frac{\eta^2}{\pi^2}\sum_{l=1}^{\pi/\eta} l(\cos l\theta)^2 \sim 0.25\ . \tag{19}$$

The estimate is independent of η as long as η is small.

Since for a harmonic system the ground-state probability distribution is Gaussian for any coordinate or momentum, we conclude that the transverse-momentum correlation function. Define $P(\eta)$ must be approximately

$$e^{-4p_\perp^2}\ .$$

The scale of momentum is of course 1 GeV, or, more precisely, the inverse slope of Regge trajectories. This conclusion is probably only sensitive to the postulated stringlike nature of the hadron and to the approximation of round potential wells, i.e., harmonic oscillators.

A similar computation can be done for the transverse momentum correlation function. Define $P(\eta)$ to be the transverse momentum of a fragment with fraction η. The correlation function measures the degree of coupling and correlation between fragments at different points on the string. We define the correlation to be

$$F(\eta_1, \eta_2) = \langle \vec{P}_\perp(\eta_1) \cdot \vec{P}_\perp(\eta_2) \rangle . \qquad (20)$$

Using the normal-mode expansion of $P_\perp$ we find that the correlation function for small η_1 and η_2 behaves like

$$F(\eta_1, \eta_2) \sim -\eta_1\eta_2 \left[\frac{1}{(\eta_1+\eta_2)^2} + \frac{1}{(\eta_1-\eta_2)^2}\right] , \qquad (21)$$

or in rapidity space

$$F(r_1, r_2) \sim \frac{e^{r_1-r_2}}{(1+e^{r_1-r_2})^2} - \frac{e^{r_1-r_2}}{(1-e^{r_1-r_2})^2} . \qquad (22)$$

From Eqs. (20) and (21) we see that F is a symmetric function of r_1 and r_2 and depends only on $r_1 - r_2$.

Of course, the result should only be believed for $r_1 - r_2$ larger than the coarse grain averaging sizes which define limitations on the string model. Specifically, we believe that the correlation is a symmetric function of $r_1 - r_2$ which asymptotically behaves like[5]

$$F(r_1, r_2) \sim -e^{-|r_1-r_2|} . \qquad (23)$$

IV. CURRENTS OF CHARGES

So far we have considered the distribution of momentum in a hadron and in a collision process. Similar methods will now be applied to the study of the charge distribution. For definiteness, consider the electric charge which we assume to be distributed with a density $\rho(\eta)$. By definition $\rho(\eta)d\eta$ is the total charge carried by partons in the interval $d\eta$ centered around η.

Our method will be to relate $\rho(\eta)$ to the electromagnetic form factor[6] so as to establish a connection between the form factor and the distribution of charge among secondary fragments.

If we consider a fast-moving hadron moving along the z axis with an infinite momentum and localized at a transverse position which we take to be the origin of the transverse plane, then an observer looking down the z axis will see a charge distribution $\sigma(b)$ where b is the distance from the origin. The form factor of the hadron is the Fourier transform of $\sigma(b)$.

Let us consider the contribution to $\sigma(b)$ from partons in the interval $d\eta$ centered at η. The average charge of partons in this interval is defined as $e(\eta)$. We shall assume that the distribution of such partons in the transverse plane is governed by a density $s(\eta, b)$. The contribution to σ is then

$$\frac{d\eta}{\eta} e(\eta) s(\eta, b) = \sigma(\eta, b) d\eta . \qquad (24)$$

The factor $d\eta/\eta$ is just the number of partons in the interval, $e(\eta)$ is their average charge, and $s(\eta, b)$ is their density. Obviously, $e(\eta)/\eta = \rho(\eta)$.

Now in a harmonic system the distribution of any coordinate is always Gaussian, so that we will assume $s(\eta, b)$ is of the normalized form

$$s(\eta, b) = \frac{e^{-b^2/f(\eta)}}{f(\eta)} ,$$

where $f(\eta)$ is the mean square distance of partons at η from the center of mass. The result is

$$\sigma(b) = \int \sigma(\eta, b) d\eta = \int \frac{\rho(\eta)}{f(\eta)} e^{-b^2/f(\eta)} d\eta .$$

The Fourier transform of this is clearly given by

$$F(q^2) = \int \rho(\eta) \exp[-\tfrac{1}{4} q^2 f(\eta)] \, d\eta . \qquad (25)$$

Now $f(\eta)$ is the expectation value of the squared distance of partons at η to the center of mass. Using Eq. (14) and

$$l_{\max} \sim \frac{1}{\sin\theta}$$

we readily compute that $f(\eta) \sim -4 \ln\eta$, so that Eq. (24) becomes

$$F(q^2) \sim \int_0^1 \rho(\eta) \eta^{q^2} d\eta . \qquad (26)$$

With the form of Eq. (25) for $f(q^2)$ we can try to ask what different assumptions about $F(q^2)$ mean for $\rho(\eta)$ and vice versa. For example, if $\rho(\eta)$ is constant then the integral is

$$F(q^2) = \frac{1}{q^2+1} .$$

The pole, characteristic of a vector meson of $m^2 = 1$, is generated by the region of integration very near $\eta = 0$. The real electromagnetic form factor has poles at the position of the ρ-meson and ω-meson which for simplicity we take to be $m^2 = \frac{1}{2}$. This can be accomplished by allowing $\rho(\eta)$ to behave as $\eta^{-1/2}$ near $\eta = 0$. Since in the interval $\Delta\eta$ there will be $\Delta\eta/\eta$ secondary fragments, the mean charge of a secondary fragment in a high-energy collision is predicted[7] to behave like $\eta^{1/2}$ as $\eta \to 0$.

This correlation between the way the charge distribution penetrates into the wee-parton sea and the spectrum of mesons which couple to that charge is in our opinion very general and should be experimentally tested.

Although the wee-parton sea is expected to be electrically neutral on the average, the quantity of charge in a bin Δr very deep into the wee-parton sea will fluctuate. The fluctuations are of interest because they give insight into the dynamical laws governing the charge distribution. Since the same dynamical laws are responsible for the spectrum of particles we expect some relations between the charge, SU(3), and baryon-current fluctuations and the spectrum of particles.

Specifically we will consider the quantity of charge found in a bin of width Δr centered around rapidity r as $r \to -\infty$. The translation invariance of the rapidity axis due to longitudinal boost invariance should guarantee that, very far from the endpoint of the r axis, the properties of the charge fluctuations are independent of r. However, the properties may well depend on the bin size Δr. We shall therefore consider the average of the square of the total electric charge found in the interval Δr. This quantity, called $\langle Q^2(\Delta r)\rangle$, is directly measurable in very-high-energy production experiments in which the wee-parton tail has sufficient length to include several smearing lengths δr.

Consider first a model in which the rapidity axis is populated randomly with an equal number of positive and negative charges. In such a completely statistical model the average squared charge will grow linearly with the number of partons so that

$$\langle Q^2(\Delta r)\rangle \sim \Delta r$$

as Δr increases.

Now if the population is random we expect the mean charge of a hadron to be of order $\sqrt{n}$, where n is the number of partons. Another way to say this is that a random population is consistent only when the energy required to change the charge by an amount of order $\sqrt{n}$ is very small. Therefore, such a model would lead to the unphysical result of a spectrum of very-high-charge hadrons almost degenerate with the ground-state hadrons.

Let us consider the more physical possibility that high charges are not formed or more exactly that exotic states are strongly suppressed. A convenient picture which we have previously discussed[4] is that the matter contained between the two ends $\theta=0$ and $\theta=\pi$ is neutral (exactly) and that at each end a quark resides. The quantum numbers would then certainly be nonexotic. That this theory is untenable follows from our previous arguments which demonstrate that the charge must be smeared with a $(\sin\theta)^{-1/2}$ density across the θ axis.

Let us, however, suppose that the hadronic string is composed of $\bar{q}q$ pairs, each neutral, and an extra quark and antiquark at each end. Let us further suppose that the string is polarizable so that the existence of a charge at an end slightly pulls the opposite charges toward it and repels the like charges. Let us consider one of the ends near $\theta=0$ where we can assign each parton a rapidity $\ln\theta$. The positive charges shift an amount $\delta^+ r$ and the negative charges an amount $\delta^- r$. The net charge at position r then becomes

$$\rho(r)=\frac{\partial\delta^+}{\partial r}-\frac{\partial\delta^-}{\partial r}=\frac{\partial\delta}{\partial r}\ .$$

We shall assume that the charges shift just enough to cancel the extra "valence" charges at the end leaving a charge density near the end given by

$$\rho(r)=e^{-r/2}\ ,$$

which is equivalent to $\eta^{-1/2}=\rho(\eta)$.

Now let us consider a point r deep in the wee-parton sea and the charge contained in a bin $2\Delta r$ centered at that point. The total charge is

$$\int_{r-\Delta}^{r+\Delta}\rho\, dr=\delta(r+\Delta)-\delta(r-\Delta)$$

and the mean square charge is

$$\langle\delta(r+\Delta)^2+\delta(r-\Delta)^2-2\delta(R+\Delta)\delta(r-\Delta)\rangle\ .$$

Now invoking the uniformity of the rapidity axis we set the first two terms equal to $\langle\delta(-\infty)\rangle^2$. The last term represents a correlation between points separated by rapidity $2\Delta r$. If the chain is a near-neighbor coupled system we can suppose that the correlation goes to zero as Δr grows so that

$$\langle[Q(\Delta)]^2\rangle\sim 2\delta^2\ ,$$

where δ is independent of r.

Thus we are led to predict that $[Q(\Delta)]^2$ is independent of Δ for large Δ as well as being independent of the position of Δ.

The main point we wish to emphasize is the close connection between statistical properties of the moments of ρ and the spectrum of hadrons. Since the difference between hadronic states involves the charge and quantum-number distributions, measurements of the moments of $\rho(r)$ directly probe the dynamics which define the spectrum of hadrons.

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Coherent Production, Low-Energy Theorems, and Anomalous Ward Identities

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The significance of two low-energy theorems relating $\pi\to 2\gamma$, $\gamma\to 3\pi$, and $2\gamma\to 3\pi$ is stated. The processes $\pi^{\pm}+(Z,A)\to(Z,A)+\pi^{\pm}+\pi^{0}$, $e^+e^-\to e^+e^-3\pi$, and $e^+e^-\to 3\pi$ are discussed as experiments providing a check on these theorems. Some rate estimates are given.

I. INTRODUCTION

Recently, Adler *et al.*,[1] Terentiev,[2] and others[3,4] derived two low-energy theorems relating $\pi\to 2\gamma$, $\gamma\to 3\pi$, and $2\gamma\to 3\pi$. Let the general amplitudes (not necessarily on shell) for $\pi^0\to 2\gamma$ and $\gamma\to\pi^+\pi^-\pi^0$ be denoted by

$$M(\gamma(k_1)+\gamma(k_2)\to\pi^0)=i\,|\epsilon_1\epsilon_2 k_1 k_2|\times F^{\pi}(k_1^2, k_2^2, (k_1+k_2)^2), \qquad (1)$$

$$M(\gamma(k)\to\pi^0(q_0)+\pi^+(q_+)+\pi^-(q_-)) =(-i)|\epsilon q_+ q_- q_0|\,F^{3\pi}((q_++q_0)^2, (q_++q_-)^2; q_+^2, q_-^2, q_0^2, (q_++q_-+q_0)^2)\,, \qquad (2)$$

where we have introduced the notation

$$|abcd|\equiv\epsilon_{\mu\nu\sigma\lambda}a^{\mu}b^{\nu}c^{\sigma}d^{\lambda}.$$

We define the "coupling constants" $F^{\pi}\equiv F^{\pi}(0,0,0)$ and $F^{3\pi}\equiv F^{3\pi}(0,0;0,0,0,0)$. Then the first low-energy theorem states that

$$eF^{3\pi}=F^{\pi}f_{\pi}^{-2} \qquad (3)$$

where f_{π} is the usual π_{l2} coupling constant. Experimentally, F^{π} and $F^{3\pi}$ are not accessible. Only an approximate version of Eq. (3), namely,

$$eF^{3\pi}(s,t;m_{\pi}^2,m_{\pi}^2,m_{\pi}^2,W^2)\simeq F^{\pi}(0,0,m_{\pi}^2)f_{\pi}^{-2}, \qquad (3')$$

with s, t, W^2 at most of the order of a few m_{π}^2, can be tested experimentally. That Eq. (3′) is a good approximation of Eq. (3) is the content of the hypothesis of the partial conservation of the axial-vector current (PCAC).

The second theorem states that the amplitudes for $\gamma+\gamma\to 3\pi^0$ and for $\gamma+\gamma\to\pi^+\pi^-\pi^0$ may be computed up to second order in momenta in terms of F^{π}, $F^{3\pi}$, and a parameter specifying the isospin structure of the chiral symmetry-breaking interaction (Fig. 1).

The purpose of this paper is to stress the unique theoretical significance of Eq. (3) and (3′) and to discuss some of the points that one encounters in planning experiments to measure $\gamma\to 3\pi$ and $2\gamma\to 3\pi$. A number of technical details are relegated to three appendixes.

II. THEORETICAL SIGNIFICANCE

The theorem in Eq. (3) and (3′) follows from (a) gauge invariance, (b) Gell-Mann's current algebra and PCAC, and (c) that the electromagnetic *current* commutes with the neutral axial *charge* at equal times. Now it is well known that a naive application of (a), (b), and (c) leads to the erroneous conclusion[4,5] that $\pi\to 2\gamma$ and $\gamma\to 3\pi$ are sup-