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Chiral Symmetry Breaking and the Dual Pion Model*

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It has been previously shown that a zero-slope limit of the dual pion model with $m_\pi = 0$ is equivalent to the SU(2)⊗SU(2)-symmetric nonlinear σ model. We extend this result by showing that an appropriately defined zero-slope limit of the dual pion model with $m_\pi \neq 0$ exists and corresponds to the nonlinear σ model with $(\frac{1}{2},\frac{1}{2})$-symmetry breaking.

The purpose of this paper is to demonstrate that an appropriately defined zero-slope limit of the dual pion model[1-4] is equivalent to the nonlinear σ model.[5] This result has been anticipated by a number of authors[6,7] and in one case was essentially proved.[8] In the discussion below we emphasize some important points that we feel have not been fully exposed in previous discussions.

In Refs. 3 and 4 the dual pion model was formulated for arbitrary value of the pion mass. On-mass-shell N-pion amplitudes were obtained in the tree approximation in a fully factorizable scheme, which is probably ghost-free only in the unrealistic case $\alpha' m_\pi^2 = -\frac{1}{2}$. Furthermore, the amplitudes were shown to possess Adler zeros in the case $m_\pi = 0$, from which one may infer that in the zero-slope limit the dual pion amplitudes are the same as the ones obtained (in tree approximation) from the nonlinear SU(2)⊗SU(2)-symmetric pion Lagrangian.[9]

Following Scherk,[10] it is natural to inquire if the dual pion amplitudes tend to a finite limit when $\alpha' \to 0$ with m_π fixed and nonzero, and if so, to find the algebraic nature of the symmetry-breaking term in the corresponding Lagrangian. We prove below that this limit does indeed exist and that the symmetry-breaking term in the equivalent Lagrangian belongs to a $(\frac{1}{2}, \frac{1}{2})$ representation, as in the nonlinear σ model, for example. We emphasize that our theorem makes reference only to on-mass-shell quantities. However, it is a useful technical convenience in carrying out the proof to consider off-mass-shell extrapolations. The reason for this is the existence of a theorem, due to Osborn and Ellis,[11] which enables one to recognize the nonlinear σ model from formulas for amplitudes. It states that a set of off-mass-shell N-pion amplitudes is uniquely those of the nonlinear SU(2)⊗SU(2) σ model with $(\frac{1}{2}, \frac{1}{2})$ symmetry breaking if the amplitudes satisfy the following conditions. (1) They can be written in Chan-Paton form

$$T^{[N]} = \sum_P A_P^{[N]}(\{s_{ij}\}) \, \mathrm{Tr}(\tau_1 \tau_2 \cdots \tau_N), \quad (1)$$

where the sum extends over all permutations P that are inequivalent under cyclic permutations of the N external legs. For each permutation, $A_P^{[N]}$ is a cyclically symmetric function depending explicitly only on the Mandelstam variables

$$s_{ij} = (k_i + k_{i+1} + \cdots + k_j)^2$$

associated with sets of adjacent lines, and independent of the continued squared masses k_i^2. The τ_i's are Pauli matrices describing the charge state of the external pions. (2) The amplitudes are factorizable at each of the pion poles in terms of the analogously constructed amplitudes of lower order. Also, they may contain contact terms (terms without poles) that are at most linear in the appropriate Mandelstam invariants. (3) Each amplitude $A_P^{[N]}$ satisfies the Adler condition, i.e., it vanishes when any one of the pion momenta k_i vanishes and the remaining pions are on the mass shell.

To apply this theorem we first need to define an off-mass-shell continuation of the dual pion model amplitudes. It is a very difficult matter to formulate such a continuation in a way which is compatible with the factorization and duality properties of the on-mass-shell amplitudes without introducing extraneous and physically unallowable singularities.[12] Fortunately, it is permissible to use an off-shell extrapolation, which we would not propose seriously in its own right, but which satisfies a limited set of criteria adequate for application of the Osborn-Ellis theorem. The requirements are that the off-mass-shell amplitude corresponding to a particular cyclic permutation should (a) reduce to the correct on-mass-shell amplitude when all external momenta satisfy $k_i^2 = m_\pi^2$, (b) be cyclically symmetric in the N external lines, (c) factorize at the pion poles into lower-order expressions of the same type, and (d) satisfy the Adler condition. If we can find an off-mass-shell extrapolation satisfying these requirements, then it will suffice to show (e) that it has a finite limit when $\alpha' \to 0$ with m_π^2 fixed and (f) that any contact terms are at most linear in the Mandelstam invariants.

In connection with (c) it should be noted that we are requiring factorization only at the pion poles, which are the only poles remaining in this zero-slope limit. It is implicitly understood that no other pole should appear at the pion mass. The relevance of this remark is that in the $\mathfrak{F}_1$ formulation[2] of the dual pion model there is a spurious decoupled spin-one state degenerate with the pion. If the off-mass-shell extrapolations were to destroy the gauge condition responsible for this decoupling, then this state could give rise to new poles that might survive in the zero-slope limit and we would be in trouble. This difficulty can be circumvented by working in the $\mathfrak{F}_2$ formulation in which the pion "ancestor" never appears as a candidate. The price one pays for using $\mathfrak{F}_2$ is that cyclic symmetry is less manifest.

An appropriate off-mass-shell continuation is suggested by the Osborn-Ellis theorem. The procedure (noted by many previous authors) consists of expressing the dual N-pion amplitude $A_P^{[N]}$ entirely in terms of m_π^2 and the variables $\{s_{ij}\}$ appropriate to the permutation P, and using the resulting expression to define off-mass-shell extrapolations. All one must do is to replace $p_i\cdot p_j$ by Mandelstam invariants according to the rules

$$2p_i\cdot p_{i+1}=s_{i,i+1}-2m_\pi^2,$$

$$2p_i\cdot p_{i+2}=s_{i,i+2}-s_{i,i+1}-s_{i+1,i+2}+m_\pi^2,$$

$$2p_i\cdot p_j=s_{i,j}-s_{i,j-1}-s_{i+1,j}+s_{i+1,j-1}\quad\text{for } j>i+2.$$

Condition (a) is then clearly satisfied.

To prove condition (b), we note that by cyclic symmetry of the on-shell amplitude, the off-shell amplitude is cyclic symmetric up to some additive function of m_π^2 and the variables $\{s_{ij}\}$, which vanishes when the pions are on-mass-shell. Since the dual-model formalism does not depend on the dimensionality of space-time and since for a space-time dimension $D\geq N-1$ the $\frac{1}{2}N(N-3)$ variables $\{s_{ij}\}$ are independent, it follows that all such functions of $\{s_{ij}\}$ and m_π^2 must vanish identically. The same reasoning can be used to show that although the off-shell amplitudes are in general not expected to be factorizable, they are factorizable at the pion poles. The Adler condition may be established for the continued amplitudes by the same method used in Ref. 3, 4, and 6. If one supposes that all external lines except k_2, say, are on the mass shell, then the amplitude vanishes in the limit $k_2\to 0$. Note that in this limit the variable $2k_2\cdot k_3$ appearing in the on-shell amplitude approaches $-m_\pi^2$.

Finally, we show inductively that the requirements (e) and (f), on the existence and structure of the zero-slope limit, are satisfied. For the four-point function,

$$A_4=\frac{g^2}{2\alpha'}\frac{\Gamma(\alpha'(m_\pi^2-s_{12})+\frac{1}{2})\Gamma(\alpha'(m_\pi^2-s_{23})+\frac{1}{2})}{\Gamma(\alpha'(2m_\pi^2-s_{12}-s_{23}))},$$

the limit $\alpha'\to 0$ clearly exists and has the correct structure (and satisfies the Adler condition, of course).

The general N-point amplitude is obtained from the expression

$$A_P^{[N]}=\frac{g^{N-2}}{\alpha'}\int_0^1\prod_{i=2}^{N-2}(dx_i x_i^{\alpha'(m_\pi^2-s_{1i})-1})\prod_{2\leq i<j\leq N-1}(1-x_i\cdots x_{j-1})^{-2\alpha' k_i\cdot k_j}\times[\alpha'^{(N-2)/2}\langle 0|k_2\cdot Hx_2^R k_3\cdot Hx_3^R\cdots x_{N-2}^R k_{N-1}\cdot H|0\rangle],\qquad(2)$$

with the replacement

$$2\alpha' k_i\cdot k_j\to 2\alpha' k_i\cdot k_j+\delta_{i+1,j}(\alpha' m_\pi^2+\tfrac{1}{2}),$$

as described in Refs. 3 and 4. Consider splitting the integral expression in Eq. (2) into terms with and without pion poles. In the pion pole terms, the lower-order amplitudes are, by the inductive hypothesis, linear in α' and $\{s_{ij},m_\pi^2\}$ in the zero-slope limit. The pion propagators in these terms necessarily are inversely proportional to α'. Since the number of propagators is one less than the number of vertices for tree diagrams, the terms with pion poles have an over-all factor α', cancelling the $1/\alpha'$ in Eq. (2) and leaving a finite limit. The terms in the integral without pion poles must reduce to a polynomial in $\{\alpha' s_{ij},\alpha' m_\pi^2\}$ as $\alpha'\to 0$ and only terms at most linear in α' will be relevant. Furthermore, the full amplitude is known to be finite at the Adler point and hence the possible constant term [which would give a singular limit because of the $1/\alpha'$ in Eq. (2)] must vanish.[8] Thus the contact terms are also linear in $\{\alpha' s_{ij}\}$ and $\alpha' m_\pi^2$ as $\alpha'\to 0$. $A_P^{[N]}$ is thus finite and has the correct structure to satisfy condition (f). This completes the proof of the applicability of the Osborn-Ellis theorem and establishes the equivalence of the zero-slope limit and the nonlinear σ model.

The explicit calculation of the limit of expression (2) is difficult in general. As a further check on the validity of the indirect logic of this paper we have calculated directly that

$$A_4 \sim g^2 k_1 \cdot k_3 ,$$

$$A_6 \sim g^4 \left(\frac{k_1 \cdot k_3 \, k_4 \cdot k_6}{m_\pi^2 - s_{13}} + \frac{k_2 \cdot k_4 \, k_5 \cdot k_1}{m_\pi^2 - s_{24}} + \frac{k_3 \cdot k_5 \, k_6 \cdot k_2}{m_\pi^2 - s_{35}} + \frac{1}{4} \sum_{i=1}^{6} k_i \cdot k_{i+2} \right).$$

(Indices in the sum are to be understood as modulo 6.) These results are in agreement with what one finds for on-mass-shell amplitudes from the nonlinear σ model.

We conclude with a somewhat philosophical comment about the possible relevance of considerations of the type presented in this paper. Progress in the development of dual resonance models has been so great as to justify some hope of ultimately constructing a Born term for a realistic theory of hadrons. However, we are beginning to get the impression that further progress may require a greater understanding of the field-theoretical foundations of dual models. One indication for this point of view is the tendency for ghost-free and renormalizable dual models to contain a massless vector meson. An intriguing possibility is that this restriction may ultimately be overcome by an analog of the Higgs-Kibble mechanism.[13] This appears to require a field-theoretical formulation.

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[1]A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971).

[2]A. Neveu, C. B. Thorn, and J. H. Schwarz, Phys. Letters 35B, 529 (1971).

[3]A. Neveu and C. B. Thorn, Phys. Rev. Letters 27, 1758 (1971).

[4]J. H. Schwarz, Phys. Rev. D 5, 886 (1972).

[5]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[6]R. C. Brower, Phys. Letters 34B, 143 (1971).

[7]G. Frye and L. Susskind, Phys. Rev. D 1, 1682 (1970); J. Ellis and B. Renner, Nucl. Phys. B21, 205 (1970).

[8]K. Kawarabayashi and S. Kitakado (unpublished).

[9]We recognize here the ambiguity that the Adler condition does not distinguish between SU(2)⊗SU(2), O(3,1) and E(3). (See J. Ellis, Ref. 11.) Examination of the 4-pion amplitude shows that SU(2)⊗SU(2) is the relevant group structure for the dual pion model.

[10]J. Scherk, Nucl. Phys. B31, 222 (1971).

[11]H. Osborn, Lett. Nuovo Cimento 2, 717 (1969); 3, 135(E) (1970); J. Ellis, Nucl. Phys. B21, 217 (1970).

[12]In the special case $\alpha' m_\pi^2 = -\frac{1}{2}$ one could probably apply the recently developed techniques of Neveu and Scherk [CERN Report No. CERN-TH-1499, 1972 (unpublished)]. However, this mass value is completely irrelevant for the limit we are interested in studying, and the Neveu-Scherk method does not seem applicable for other values of the pion mass. It is conceivable that no acceptable off-mass-shell amplitudes can be defined for dual models with ghosts.

[13]This possibility has also been noted by D. Gross and P. Ramond (private communications) and probably by others as well.