Drell-Hearn Sum Rule from Light-Cone Current Commutators*

Duane A. Dicus and David R. Palmer
The Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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The Drell-Hearn sum rule for the nucleon anomalous magnetic moment is derived by using the light-cone commutator for the + components of the current. The $q^2 = 0$ limits of several other sum rules are also derived from the ++ commutator. These other sum rules have been previously derived but only by using a more model-dependent commutator.

In a recent paper[1] (hereafter referred to as DJT) it was shown that current commutators restricted to a lightlike surface can be used to derive fixed-mass sum rules without using the $p\to\infty$ technique necessary in the usual current-algebra approach. Several sum rules were derived in DJT but one that was conspicuously absent was the Drell-Hearn sum rule[2] for the nucleon anomalous magnetic moment. The usual current-algebra derivation[3] of this sum rule depends only on the commutator of the time components of two electromagnetic currents and therefore should be perfectly valid despite the $p\to\infty$ limit. Nevertheless it is disturbing that such a well-known sum rule has not been derived from light-cone commutators. A generalization of the Drell-Hearn sum rule for nonzero photon mass and momentum transfer has been derived[4] but only by considering the full problem of matrix elements away from the forward direction, and the zero-photon-mass and zero-momentum-transfer limits are not obvious. Here we present a simple derivation of the Drell-Hearn sum rule based on light-cone commutators. Our method of derivation assumes only the ++ commutator and, in addition to the Drell-Hearn, we are able to derive the $q^2=0$ limit of several of the sum rules of DJT which they derived by assuming the $+-$ or $+i$ commutator.

We define the first moment of a conserved vector current V_a^μ by the expression

$$D_a^\mu(x^+) = \int dx^- d^2x_\perp\, x^\mu V_a^+(x), \tag{1}$$

where, in this paper, μ and ν can be $-$ or i $(i=1,2)$ but never $+$. The commutator of two components of D_a^μ taken between nucleon states can be written as[5]

$$\langle ps|[D_a^\mu(x^+), D_b^\nu(x^+)]|p's\rangle_{\rm Born} + \langle ps|[D_a^\mu(x^+), D_b^\nu(x^+)]|p's\rangle_{\rm cont}$$
$$= -i(2\pi)^3 f_{abc}\langle ps|V_c^+(0)|ps\rangle e^{ix^+(p-p')^-}\partial^\mu_{(p)}\partial^\nu_{(p)}[\delta(p^+-p^{+\prime})\delta^{(2)}(p_\perp - p'_\perp)], \tag{2}$$

where we have used only the most reliable light-cone commutator

$$[V_a^+(x), V_b^+(y)]\delta(x^+-y^+) = if_{abc}V_c^+(x)\delta^{(4)}(x-y) \tag{3}$$

and have separated out the nucleon Born term from the continuum contribution.

The contribution from the continuum part of Eq. (2) can be evaluated by inserting a complete set of states and using the relation

$$\langle\alpha|D_a^\mu(x^+)|\beta\rangle = -i(2\pi)^3\frac{\delta(p_\alpha^+-p_\beta^+)\delta^{(2)}(p_{\alpha\perp}-p_{\beta\perp})}{p_\alpha^- - p_\beta^-}e^{ix^+(p_\alpha-p_\beta)^-}\langle\alpha|V_a^\mu(0)|\beta\rangle, \tag{4}$$

which follows from $\partial_+ D_a^\mu(x^+) = \int dx^- d^2x V_a^\mu(x)$ if $p_\alpha{}^2 \neq p_\beta{}^2$. Then, multiplying by $\int dq^-\delta(p_\alpha^- - p_\beta^- - q^-)$, the complete set of states can be removed and the continuum written as

$$(2\pi)^3\delta(p^+-p^{+\prime})\delta^{(2)}(p_\perp - p'_\perp)\int_{-\infty}^{\infty}\frac{dq^-}{(q^-)^2}C_{ab}^{\mu\nu}(p,q)\bigg|_{q^+=q_\perp=0}, \tag{5}$$

with

$$C_{ab}^{\mu\nu}(p,q) = \int\frac{d^4x}{2\pi}e^{iq\cdot x}\langle ps|[V_a^\mu(x), V_b^\nu(0)]|ps\rangle = [q^2p^\mu p^\nu - \nu(p^\mu q^\nu + p^\nu q^\mu) + \nu^2 g^{\mu\nu}]A_1^{ab}(\nu,q^2) + (q^\mu q^\nu - g^{\mu\nu}q^2)A_2^{ab}(\nu,q^2)$$
$$+ i\epsilon^{\mu\nu\alpha\beta}s_\alpha q_\beta W_3^{ab}(\nu,q^2) + iq\cdot s\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta W_4^{ab}(\nu,q^2), \tag{6}$$

where $\nu = p\cdot q$ and the nucleon polarization vector is given by $s_\mu = \bar{u}(ps)\gamma_\mu\gamma_5 u(ps)$.

For the Born contribution to the commutator (2) we write

$$\langle ps|[D_a^\mu(x^+), D_b^\nu(x^+)]|p's\rangle_{\text{Born}} = B_1 + B_2, \tag{7a}$$

$$B_1 = -(2\pi)^3 e^{ix^+(p-p')^-}\partial^\mu_{(p)}\partial^\nu_{(p)}[\delta(p^+-p'^+)\delta^{(2)}(p_\perp - p'_\perp)]\sum_r\langle ps|V_a^+(0)|p'r\rangle\frac{1}{2p'^+}\langle p'r|V_b^+(0)|p's\rangle - \begin{pmatrix}\mu\leftrightarrow\nu\\ a\leftrightarrow b\end{pmatrix}, \tag{7b}$$

$$B_2 = (2\pi)^3 e^{ix^+(p-p')^-}\partial^\mu_{(p)}[\delta(p^+-p'^+)\delta^{(2)}(p_\perp - p'_\perp)]\partial^\nu_{(k)}\left[\sum_r\langle ps|V_a^+(0)|kr\rangle\frac{1}{2k^+}\langle kr|V_b^+(0)|p's\rangle\right]\Bigg|_{k=p'} - \begin{pmatrix}\mu\leftrightarrow\nu\\ a\leftrightarrow b\end{pmatrix}. \tag{7c}$$

The vector-current nucleon vertex is $(q = p - p')$

$$\langle ps|V_a^\mu(0)|p's\rangle = 2\bar{u}(ps)\left[F_1^a(q^2)\gamma^\mu + \frac{F_2^a(q^2)}{2M}i\sigma^{\mu\lambda}q_\lambda\right]u(p's). \tag{8}$$

For the electromagnetic and isospin currents we have, respectively,

$$F_{1,2}^{\text{em}}(q^2) = \tfrac{1}{2}F_{1,2}^S(q^2) + \tfrac{1}{2}\tau_3 F_{1,2}^V(q^2)$$

and

$$F_{1,2}^a(q^2) = \tfrac{1}{2}\tau_a F_{1,2}^V(q^2)$$

so that $F_1^S(0) = F_1^V(0) = 1$, $F_2^S(0) = \frac{1}{2}(\kappa_p + \kappa_n)$, and $F_2^V(0) = \frac{1}{2}(\kappa_p - \kappa_n)$, where $\kappa_p(\kappa_n)$ is the anomalous magnetic moment of the proton (neutron) in nucleon magnetons. By substituting Eq. (8) into Eq. (7b), one immediately sees that the contribution to Eq. (2) from B_1 is exactly canceled by the right-hand side of Eq. (2).

Substituting Eq. (8) into Eq. (7c), on the other hand, we find

$$B_2 = \frac{(2\pi)^3}{M^2 p^+}\delta(p^+ - p'^+)\delta^{(2)}(p_\perp - p'_\perp)\xi^T\tilde{B}_2\xi, \tag{9a}$$

where ξ is the nucleon isospinor and

$$\begin{aligned}\tilde{B}_2 = {} & (p^+p^+g^{\mu\nu} - p^+p^\mu g^{+\nu} - p^+p^\nu g^{+\mu})\{[4M^2F_1^{a\prime}(0), F_1^b(0)] - \tfrac{1}{2}[F_2^a(0), F_2^b(0)]\} - M^2 g^{+\mu}g^{+\nu}[F_1^a(0) + \tfrac{1}{2}F_2^a(0), F_2^b(0)] \\ & + \tfrac{1}{2}i(g^{+\nu}\epsilon^{+\mu\alpha\beta} - g^{+\mu}\epsilon^{+\nu\alpha\beta})p_\alpha s_\beta\{F_1^a(0) + F_2^a(0), F_2^b(0)\} - \tfrac{1}{2}ip^+\epsilon^{+\mu\nu\alpha}s_\alpha\{F_2^a(0), F_2^b(0)\}.\end{aligned} \tag{9b}$$

Here $F_1^{a\prime}(0) = [dF_1^a(t)/dt]_{t=0}$. By substituting Eqs. (5)–(7) and (9) into Eq. (2) and equating tensor coefficients we obtain four sum rules:

$$\int_{\nu_T}^\infty d\nu A_1^{[ab]}(\nu, 0) = \xi^T\left\{\left[\frac{F_2^a(0)}{2M}, \frac{F_2^b(0)}{2M}\right] - 2[F_1^{a\prime}(0), F_1^b(0)]\right\}\xi, \tag{10a}$$

$$\int_{\nu_T}^\infty d\nu A_2^{[ab]}(\nu, 0) = \tfrac{1}{4}\xi^T[2F_1^a(0) + F_1^b(0), F_2^b(0)]\xi, \tag{10b}$$

$$\int_{\nu_T}^\infty \frac{d\nu}{\nu}W_3^{(ab)}(\nu, 0) = \frac{1}{4M^2}\xi^T\{F_1^a(0), F_2^b(0)\}\xi, \tag{10c}$$

$$\int_{\nu_T}^\infty d\nu W_4^{(ab)}(\nu, 0) = -\frac{1}{4M^2}\xi^T\{F_1^a(0) + F_2^a(0), F_2^b(0)\}\xi, \tag{10d}$$

where we have decomposed the invariant amplitudes into isospin symmetric (ab) and antisymmetric $[ab]$ parts and used their known crossing properties. Equations (10b) and (10d) are the $q^2 = 0$ versions of sum rules derived in DJT. Equation (10a), which is the Cabibbo-Radicati sum rule,[6] follows from the Dashen–Fubini–Gell-Mann sum rule[7] which was also derived in DJT. Equation (10c), however, is a new sum rule. The sum of Eqs. (10c) and (10d) is the Drell-Hearn sum rule more conventionally written (for protons) as

$$\int_{\nu_T}^\infty \frac{d\nu}{\nu}[\sigma_p(\nu) - \sigma_A(\nu)] = \frac{2\pi^2\alpha}{M^2}\kappa_p^{\,2}, \tag{11}$$

where $\sigma_{p,A}$ are total absorption cross sections for scattering of photons with spin parallel and antiparallel to the proton spin.

The key to the derivation of (10c) is the use of (4) which introduces factors of ν in the denominator of the sum rule. This technique only allows the derivation of sum rules at $q^2 = 0$ however, so we cannot, for example, derive the Drell-Hearn sum rule for $q^2 \neq 0$ which is needed for calculations of hyperfine splitting.

The method of derivation of (10b) and (10d) used in DJT consisted of integrating (6) over dq^- with q^+ set equal to zero. The integration variable was then changed to ν with $q^2 = -\vec{q}^{\,2}$ held constant. The sum rules (10b) and (10d) then emerged for all (spacelike) q^2 by setting $\mu = +$, $\nu = -$ $(\nu = i)$, i.e., from the commutator $[V^+, V^-]$ (or $[V^+, V^i]$). Here, however, we are able to derive these two sum rules by using only the commutator $[V^+, V^+]$ but only at $q^2=0$. The fact that a different method allows us to extract some of the same information from a different commutator is not too surprising since the commutators are connected by Lorentz invariance[8] and by current conservation:

$$[V^+(x),\ \partial_+ V^+(y) + \partial_- V^-(y) + \partial_i V^i(y)]\delta(x^+ - y^+) = 0.$$

If we entertain the possibility that the commutator Eq. (3) contains additional terms, then the sum rule Eq. (11) may be modified. By considering bilocal operators of various rank, it is not difficult to construct terms which, when added to the right-hand side of Eq. (3), do not violate any covariance requirements. One such term, of the form

$$\partial_\mu^{(x)}\partial_\nu^{(y)}[\epsilon^{+\mu\nu\lambda}a_\lambda(x|y)\delta(x^- - y^-)\delta^{(2)}(x_\perp - y_\perp)], \qquad (12)$$

is particularly interesting because it appears in the canonical commutator if we assume that the partons (quarks) possess an anomalous magnetic moment. The current then has an elementary Pauli term

$$V_a^\mu \sim \bar\psi\gamma^\mu \tfrac{1}{2}\lambda_a\psi + \frac{i\kappa^a}{2M}\partial_\lambda(\bar\psi\sigma^{\lambda\mu}\tfrac{1}{2}\lambda_a\psi); \qquad (13)$$

with this Pauli term, the right-hand side of Eq. (11) is modified to include terms proportional to $(\kappa^a)^2$. In the absence of strong interactions the sum rule then indicates that the nucleon anomalous moment is equal to the parton moment. The term (12) would lead to a violation of scaling for the spin-dependent structure functions, however, because it introduces a dimensional parameter κ^a into the theory. Also this type of theory does not automatically provide a solution to the problem[9] of a violation of the Drell-Hearn sum rule for t-channel isospin equal to one.

Finally additional sum rules can be derived by considering $[D_a^\mu(x^+),\ V_b^-(x)]$ but these are again $q^2=0$ versions of sum rules derived in DJT. No sum rules can be derived from $[D_a^\mu(x^+),\ V_b^+(x)]$.

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[5]We will use the following notation: Our metric has signature (+ − − −). Hadronic states are normalized according to $\langle ps|p's\rangle = 2p_0(2\pi)^3\delta^{(3)}(\vec p - \vec p\,')$ and spinors are normalized so that $\bar u(ps)u(ps) = M$. The components of a four-vector a in the light-cone basis are

$$a^\pm = a_\mp = 2^{-1/2}(a^0 \pm a^3) \text{ and } a_\perp = (a^1, a^2).$$

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