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Inequalities for the Pion-Pion Partial Waves: General Considerations and New Inequalities*

A. P. Balachandran

Physics Department, Syracuse University, Syracuse, New York 13210†
and Istituto Nazionale di Fisica Nucleare-Sezione di Napoli,‡ *Napoli, Italy*

and

Maurice L. Blackmon

Physics Department, Syracuse University, Syracuse, New York 13210

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A class of inequalities for the pion-pion s and p waves has been discussed in a series of recent papers. The present work attempts to provide a systematic method for writing such inequalities. An infinite number of new inequalities for the pion-pion s and p waves are also derived.

I. INTRODUCTION

In previous work,[1-6] several inequalities were derived for the pion-pion s and p waves using the analyticity and positivity properties of the π-π scattering amplitude. These derivations were not all very systematic; in particular, no attempt was made in Refs. 1–3 to show that the inequalities were complete and independent or to suggest a methodical approach to the problem.[7] Their merit consisted in their simplicity. In the present work, we attempt to develop a general framework for a systematic derivation of all the independent inequalities. The point of view we adopt is in a certain sense complementary to that of Pennington.[6] While we find many useful results, we also feel that they are far from complete.

In Sec. II, we recall some of the positivity properties of the π-π partial waves proved by Martin,[8] Common,[9] and Yndurain.[10] The use of these positivity properties in conjunction with the crossing symmetry of the system leads to the partial-wave inequalities of our interest. The general discussion of these inequalities is facilitated by the two-variable expansion of the pion-pion amplitude introduced by Balachandran and Nuyts.[11] This expansion and some of its important properties are also summarized in this section.

In Sec. III, we introduce two classes $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(c)}$ of functions of s and t, one each for $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$. For each member $H^{(i)}(s,t) \in \mathcal{C}^{(i)}$ of either of these classes, there is associated an inequality involving the π-π s and p waves. The functions $H^{(0)}(s,t)$ in $\mathcal{C}^{(0)}$ are required to be antisymmetric in s and t while the functions $H^{(c)}(s,t)$ in $\mathcal{C}^{(c)}$ are required to be linear in s. The partial waves $h_l^{(i)}(s)$ of $H^{(i)}(s,t)$ are also required to satisfy certain positivity properties when $l=2, 4, 6, \ldots$.

In Sec. IV, a well-known theorem on the representation of polynomials of one variable which are non-negative on the interval $[0, 1]$ is stated. This theorem is useful for the imposition of the positivity requirements on $h_l^{(i)}$.

Section V studies the constraints on $h_l^{(0)}$ due to the antisymmetry of $H^{(0)}$. When there is a general scheme available for the imposition of positivity requirements on $h_l^{(0)}$ (perhaps a scheme of the sort suggested in Sec. IV), this section provides a systematic (if clumsy) method for treating the antisymmetry constraints and for writing the $\pi^0\pi^0 \to \pi^0\pi^0$ s-wave inequalities. The somewhat complementary nature of our approach and that of Pennington[6] is also pointed out.

Section VI illustrates the previous considerations in specific examples. In particular some inequalities due to Balachandran and Blackmon[1,2] and Pennington[6] are derived in a simple way.

In Sec. VII, we present some new inequalities for the pion-pion s and p waves.

II. RESUMÉ OF PREVIOUS WORK

A. Some Results of Martin, Common, and Yndurain

Let $A^{(0)}(s,t)$ and $A^{(c)}(s,t)$ denote the scattering amplitudes which in the s channel describe the reactions $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$, respectively. Their partial-wave expansions are

$$A^{(i)}(s,t) = \sum_{l=0}^{\infty} (2l+1) a_l^{(i)}(s) P_l(z_s) \tag{2.1}$$

$$= \sum_{l=0}^{\infty} (2l+1) b_l^{(i)}(t) P_l(z_t), \tag{2.2}$$

where $a_l^{(i)}(s) = 0$ if l is odd and $a_l^{(0)}(s) = b_l^{(0)}(s)$. The pion mass will be taken to be $\frac{1}{2}$ so that $s+t+u=1$. With this choice of units, the variables z_s and z_t are given by $z_s = 1+2t/(s-1)$ and $z_t = 1+2s/(t-1)$.

The partial waves $a_l^{(i)}(s)$ fulfill the Froissart-Gribov representation[12]

$$a_l^{(i)}(s) = \frac{4}{\pi(1-s)} \int_1^{\infty} dt' A_t^{(i)}(s,t') Q_l\left(\frac{2t'}{1-s} - 1\right),$$

$$i=0, c; \quad l=2, 4, 6, \ldots ; \quad s \in [0, 1), \tag{2.3}$$

where the absorptive parts $A_t^{(i)}(s,t')$ have the positivity property[13]

$$A_t^{(i)}(s,t') \geq 0, \quad i=0, c; \quad t' \geq 1; \quad s \in [0, 1). \tag{2.4}$$

As a consequence, one may show that[14]

$$\text{(a)} \quad a_l^{(i)}(s) \geq 0, \quad l=2, 4, 6, \ldots ; \quad s \in [0, 1) \tag{2.5}$$

$$\text{(b)} \quad a_l^{(i)}(s) = \int_0^{1/r(s)} d\xi\, \xi^l \phi^{(i)}(s,\xi),$$

$$i=0, c; \quad l=2, 4, 6, \ldots ; \quad s \in [0, 1) \tag{2.6}$$

where

$$r(s) = \frac{1+s}{1-s} + \left[\left(\frac{1+s}{1-s}\right)^2 - 1\right]^{1/2} \tag{2.7}$$

and

$$\phi^{(i)}(s,\xi) \geq 0 \text{ for } i=0, c; \quad \xi \in \left[0, \frac{1}{r(s)}\right]; \quad s \in [0, 1). \tag{2.8}$$

Note that (2.6) implies (2.5).

B. A Two-Variable Expansion $A^{(i)}(s,t)$

In previous work,[11] a two-variable expansion of the scattering amplitudes $A^{(i)}(s,t)$ was introduced. This expansion will play an important role in our later discussion. We shall, therefore, briefly summarize a few of its relevant properties here.

When s and z_s are restricted to the Mandelstam triangle $0 \leq s \leq 1$, $-1 \leq z_s \leq +1$, the amplitudes $A^{(i)}(s,t)$ are expanded in the series

$$A^{(i)}(s,t) = \sum_{\sigma=0}^{\infty} \sum_{l=0}^{\sigma} 2(\sigma+1)(2l+1) \alpha_{\sigma-l}^{(i)l} S_{\sigma-l}^{l}(s,t) . \tag{2.9}$$

The basis functions $S_{\sigma-l}^{l}$ are given by

$$S_{\sigma-l}^{l}(s,t) = (1-s)^l P_{\sigma-l}^{(2l+1,0)}(2s-1) P_l(z_s), \tag{2.10}$$

where $P_n^{(2l+1,0)}$ are Jacobi polynomials. We find from (2.9) that the partial waves have the expansion

$$a_l^{(i)}(s) = (1-s)^l \sum_{\sigma=l}^{\infty} 2(\sigma+1)\, \alpha_{\sigma-l}^{(i)l} \;\; P_{\sigma-l}^{(2l+1,0)}(2s-1). \tag{2.11}$$

Further, the orthogonality properties of the Jacobi and Legendre polynomials lead to the relation

$$\iint_\Delta ds\,dt\, S_n^l(s,t) S_N^L(s,t) \equiv \int_0^1 ds \int_0^{1-s} dt\, S_n^l(s,t) S_N^L(s,t) = \int_0^1 ds (1-s)\tfrac{1}{2}\int_{-1}^1 dz_s S_n^l(s,t) S_N^L(s,t) = [2(n+l+1)(2l+1)]^{-1}\delta_{lL}\delta_{nN}\,. \tag{2.12}$$

Here Δ is the Mandelstam triangle. It is easy to show, by using (2.12) for instance, that $\alpha_{\sigma-l}^{(i)l}$ can be expressed in terms of $a_l^{(i)}(s)$ in the form

$$\alpha_{\sigma-l}^{(i)l} = \int_0^1 ds (1-s)^{l+1} P_{\sigma-l}^{(2l+1,0)}(2s-1)\, a_l^{(i)}(s)\,. \tag{2.13}$$

The basis functions $S_{\sigma-l}^l(s,t)$ are diagonal in the s-channel angular momentum. We may also introduce the basis functions

$$T_{\sigma-l}^l(s,t) = (1-t)^l P_{\sigma-l}^{(2l+1,0)}(2t-1) P_l(z_t)\,,$$
$$U_{\sigma-l}^l(s,t) = (1-u)^l P_{\sigma-l}^{(2l+1,0)}(2u-1) P_l(z_u)\,, \tag{2.14}$$
$$l=0,1,2,\ldots,\sigma;\quad \sigma=0,1,2,\ldots$$

which are diagonal in the t- and u-channel angular momenta, respectively. [Here $z_u = 1+(2t)/(u-1)$.] The expansions of $A^{(i)}$ in these bases read

$$A^{(i)}(s,t) = \sum_{\sigma=0}^\infty \sum_{l=0}^\sigma 2(\sigma+1)(2l+1) \times \beta_{\sigma-l}^{(i)l} T_{\sigma-l}^l(s,t) = \sum_{\sigma=0}^\infty \sum_{l=0}^\sigma 2(\sigma+1)(2l+1) \times \beta_{\sigma-l}^{(i)l} U_{\sigma-l}^l(s,t)\,, \tag{2.15}$$

where s, t, u are constrained to be in the triangle Δ (which is invariant under permutations of s, t, u). We have used the fact that the t and u channels of $A^{(i)}(s,t)$ are identical in writing (2.15). The analogs of (2.12) are also valid for $T_{\sigma-l}^l$ and $U_{\sigma-l}^l$ due to the symmetry of Δ. The expansion coefficients in (2.15) are therefore given by

$$\beta_{\sigma-l}^{(i)l} = \int_0^1 dt (1-t)^{l+1} P_{\sigma-l}^{(2l+1,0)}(2t-1) b_l^{(i)}(t)\,. \tag{2.16}$$

Now the following identity may be proved between the three sets of basis functions[11]:

$$T_{\sigma-L}^L(s,t) = \sum_{l=0}^\sigma \frac{2l+1}{2L+1} X_L^l(\sigma) S_{\sigma-l}^l(s,t)\,, \tag{2.17}$$

$$U_{\sigma-L}^L(s,t) = \sum_{l=0}^\sigma \frac{2l+1}{2L+1} Y_L^l(\sigma) S_{\sigma-l}^l(s,t)\,. \tag{2.18}$$

The numbers $X_L^l(\sigma)$ and $Y_L^l(\sigma)$ are calculated in Ref. 11. It follows that

$$\alpha_{\sigma-l}^{(i)l} = \sum_{L=0}^\sigma X_L^l(\sigma)\beta_{\sigma-L}^{(i)L} \tag{2.19}$$

$$= \sum_{L=0}^\sigma Y_L^l(\sigma)\beta_{\sigma-L}^{(i)L}\,. \tag{2.20}$$

This is a system of crossing relations which connects a *finite* number of partial waves due to (2.13) and (2.16). Note that different values of σ are not connected by crossing in (2.19) and (2.20).

We shall have occasion to consider certain functions $H^{(0)}(s,t)$ which are antisymmetric in s and t (and which enjoy some further properties as well). Let

$$H^{(0)}(s,t) = \sum_{\sigma=0}^\infty \sum_{l=0}^\sigma 2(\sigma+1)(2l+1)\chi_{\sigma-l}^l S_{\sigma-l}^l(s,t) \tag{2.21}$$

be the two-variable expansion of such a function. The analog of (2.13) for $\chi_{\sigma-l}^l$ is, therefore,

$$\chi_{\sigma-l}^l = \int_0^1 ds (1-s)^{l+1} P_{\sigma-l}^{(2l+1,0)}(2s-1)\, h_l^{(0)}(s)\,, \tag{2.22}$$

where $h_l^{(0)}(s)$ is the lth s-channel partial wave of $H^{(0)}(s,t)$.

III. THE AUXILIARY FUNCTIONS $H^{(0)}(s,t)$ AND $H^{(c)}(s,t)$ AND THE GENERAL FORM OF THE s- AND p-WAVE INEQUALITIES

We prove the following: *Let $H^{(i)}(s,t)$ $(i=0,c)$ be any function of s and t with the partial-wave expansions*[15]

$$H^{(i)}(s,t) = \sum_{l=0}^\infty (2l+1) h_l^{(i)}(s) P_l(z_s) \tag{3.1}$$

$$= \sum_{l=0}^\infty (2l+1) g_l^{(i)}(t) P_l(z_t)\,. \tag{3.2}$$

Further let $H^{(0)}(s,t)$ be antisymmetric in s and t so that $g_l^{(0)}(s) = -h_l^{(0)}(s)$, and let $H^{(c)}(s,t)$ be at most linear in s for fixed t so that $g_l^{(c)}(s)=0$ for $l \geq 2$. Then the inequality

$$\sum_{l=0}^1 (2l+1)\int_0^1 ds (1-s)[g_l^{(i)}(s) b_l^{(i)}(s) - h_l^{(i)}(s) a_l^{(i)}(s)] \geq 0 \tag{3.3}$$

is valid if any one of the following conditions is fulfilled by $H^{(i)}(s,t)$:

$$\text{(a)}\quad h_l^{(i)}(s) \geq 0, \qquad l = 2, 4, 6, \ldots;\quad 0 \leq s \leq 1 \tag{3.4}$$

$$\text{(b)}\quad \sum_{\substack{l \geq 2\\ l=\text{even}}} (2l+1) h_l^{(i)}(s)\xi^l \geq 0, \qquad 0 \leq \xi \leq 1/r(s);\quad 0 \leq s \leq 1 \tag{3.5}$$

$$\text{(c)}\quad \int_{-1}^{1} dy\, \frac{H^{(i)}(s, \tfrac{1}{2}(1-s)(1-y)) - h_0^{(i)}(s)}{x^2 - y^2} \geq 0, \qquad x \geq \frac{1+s}{1-s};\quad 0 \leq s \leq 1. \tag{3.6}$$

Note that for $i = 0$, only the $l = 0$ term survives in (3.3) and that since $g_0^{(0)}(s) = -h_0^{(0)}(s)$, the two terms in that equation are equivalent to one term.

For the proof, we note that

$$\iint_{\Delta} ds\,dt\, H^{(i)}(s, t) A^{(i)}(s, t) = \int_0^1 ds (1-s)\tfrac{1}{2} \int_{-1}^{1} dz_s H^{(i)}(s, t) A^{(i)}(s, t)$$

$$= \int_0^1 dt (1-t)\tfrac{1}{2} \int_{-1}^{1} dz_t H^{(i)}(s, t) A^{(i)}(s, t), \tag{3.7}$$

due to the symmetry properties of the Mandelstam triangle and of $ds\,dt$. The substitution of the partial-wave expansions of $H^{(i)}$ leads to

$$\sum_{l=0}^{\infty} (2l+1) \int_0^1 ds (1-s) h_l^{(i)}(s) a_l^{(i)}(s) = \sum_{l=0}^{\infty} (2l+1) \int_0^1 dt (1-t) g_l^{(i)}(t) b_l^{(i)}(t), \tag{3.8}$$

which may be written as

$$\sum_{l=0}^{1} (2l+1) \int_0^1 ds (1-s) [g_l^{(i)}(s) b_l^{(i)}(s) - h_l^{(i)}(s) a_l^{(i)}(s)] = \epsilon^{(i)} \sum_{\substack{l \geq 2\\ l=\text{even}}} (2l+1) \int_0^1 ds (1-s) h_l^{(i)}(s) a_l^{(i)}(s),$$

$$\epsilon^{(0)} = 2;\quad \epsilon^{(c)} = 1 \tag{3.9}$$

since $a_l^{(0)} = b_l^{(0)}$, $g_l^{(0)} = -h_l^{(0)}$, $a_l^{(c)} = 0$ if l is odd and $g_l^{(c)} = 0$ if $l \geq 2$. Now

$$\sum_{\substack{l \geq 2\\ l=\text{even}}} (2l+1) \int_0^1 ds (1-s) h_l^{(i)}(s) a_l^{(i)}(s)$$

$$= \int_0^1 ds (1-s) \int_0^{1/r(s)} d\xi \Big[\sum_{l \geq 2;\, l=\text{even}} (2l+1) h_l^{(i)}(s) \xi^l \Big] \phi^{(i)}(s, \xi) \tag{3.10}$$

$$= \frac{2}{\pi} \int_0^1 ds \int_1^{\infty} dt'\, A_t^{(i)}(s, t') \left(\frac{2t'}{1-s} - 1 \right) \int_{-1}^{1} dy\, \frac{H^{(i)}(s, \tfrac{1}{2}(1-s)(1-y)) - h_0^{(i)}(s)}{[2t'/(1-s) - 1]^2 - y^2}. \tag{3.11}$$

Here (3.10) is a consequence of (2.6) while (3.11) follows from (2.3) and the identities

$$\sum_{\substack{l \geq 2\\ l=\text{even}}} (2l+1) h_l^{(i)}(s) Q_l\left(\frac{2t'}{1-s} - 1 \right) = \tfrac{1}{2} \int_{-1}^{1} dy\, \frac{\sum_{l \geq 2;\, l=\text{even}} (2l+1) h_l^{(i)}(s) P_l(y)}{[2t'/(1-s) - 1] - y}$$

$$= \tfrac{1}{2} \int_{-1}^{1} dy\, \frac{\tfrac{1}{2}\{H^{(i)}(s, \tfrac{1}{2}(1-s)(1-y)) + H^{(i)}(s, \tfrac{1}{2}(1-s)(1+y))\} - h_0^{(i)}(s)}{[2t'/(1-s) - 1] - y}$$

$$= \frac{1}{2} \left(\frac{2t'}{1-s} - 1 \right) \int_{-1}^{1} dy\, \frac{H^{(i)}(s, \tfrac{1}{2}(1-s)(1-y)) - h_0^{(i)}(s)}{[2t'/(1-s) - 1]^2 - y^2}. \tag{3.12}$$

Equations (2.5), (3.4), and (3.9) imply (3.3). Similarly, (2.8), (3.5), and (3.10) as well as (2.4), (3.6), and (3.11) also imply (3.3). As regards the latter, note that $(2t')/(1-s) - 1 \geq (1+s)/(1-s) \geq 1$ for $t' \geq 1$ and $0 \leq s \leq 1$.

Among the three conditions (a), (b), and (c), (b) leads to better inequalities than (a) while (c) leads to the best possible results since (2.6)–(2.8) imply (2.5) and (2.3)–(2.4) imply both (2.5) and (2.6)–(2.8) while the converses are not true.[8–10] It is possible to obtain even more refined results by replacing (2.4) by more detailed unitarity properties of the imaginary parts of the t-channel partial waves in the physical region and correspondingly by modifying the positivity conditions on $H^{(i)}$. Such methods have been followed in Refs. 4 and 5 which may be consulted for further details. We should also mention here that the works of Griss[5] and Common and Pennington[6] utilize properties of $a_l^{(i)}$ [like those in (2.6)] which are implied by the Froissart-Gribov representation and which are more refined than (2.5).

The determination of the functions $H^{(i)}$ can be broken up into two steps: (1) Determine all those functions which fulfill one of the stated positivity properties. (2) Impose the antisymmetry condition on $H^{(0)}$ and the linearity condition on $H^{(c)}$. In Sec. IV, we briefly study the first problem, while in Sec. V, the implications of the antisymmetry condition on $H^{(0)}$ are analyzed. The linearity condition on $H^{(c)}$ however will not be discussed in any generality in this paper.

IV. A REPRESENTATION THEOREM FOR NON-NEGATIVE POLYNOMIALS

We shall restrict our considerations hereafter to functions $H^{(i)}(s,t)$ which are polynomials in s and t. This assumption seems permissible due to the completeness properties of polynomials in s and t for fairly wide classes of functions defined on the Mandelstam triangle. The partial waves $h_l^{(i)}(s)$ of such functions $H^{(i)}(s,t)$ are themselves polynomials in s which vanish like constant $\times(1-s)^l$ as $s\to 1$. The imposition of the positivity properties of the form (3.4) or (3.5) on these polynomials is greatly facilitated by the following well-known theorem[16]: *Any polynomial $R_n(x)$ of degree n which is non-negative in the interval $[0,1]$ can be represented in the form*

$$R_n(x)=x[A_m(x)]^2+(1-x)[B_m(x)]^2 \tag{4.1}$$

if $n=2m+1$ is odd, and in the form

$$R_n(x)=[C_m(x)]^2+x(1-x)[D_{m-1}(x)]^2 \tag{4.2}$$

if $n=2m$ is even. Here $A_m(x)$, $B_m(x)$, $C_m(x)$, $D_m(x)$ are polynomials of degree m in x with real coefficients.

This representation theorem has been previously used in the literature.[2,3,5,6] Its utility will become evident on consulting Sec. VI or the cited references.

V. SOME CONSIDERATIONS ON THE ANTISYMMETRY OF $H^{(0)}$

We write $H^{(0)}(s,t)$ as

$$H^{(0)}(s,t)=\lambda(s-t)+G(s,t), \tag{5.1}$$

where λ is an arbitrary constant and $G(s,t)$ is an antisymmetric function with a two-variable expansion [see Eq. (2.22)] with $\sigma\geq 2$. We write separately the only possible term with $\sigma\leq 2$, i.e., $\lambda(s-t)$, for convenience.

Now suppose we specify partial waves $h_2^{(0)}(s)$, $h_4^{(0)}(s)$, $h_6^{(0)}(s)$, ... consistent with our positivity requirements. We then calculate, using (2.22), the coefficients $\chi_{\sigma-l}^l$ for $l=2,4,6,\ldots$; $\sigma\geq 2$. It is therefore of interest to know the answers to the following questions:

(i) Suppose we are given a set of constants $\chi_{\sigma-l}^l$ ($l=2,4,6,\ldots$; $\sigma\geq 2$) and suppose that we wish to find additional constants χ_σ^0, $\chi_{\sigma-l}^l$ ($l=1,3,5,\ldots$; $\sigma\geq 2$) such that

$$G(s,t)=\sum_{\sigma=2}^{\infty}\sum_{l=0}^{\sigma}2(\sigma+1)(2l+1)\chi_{\sigma-l}^l S_{\sigma-l}^l(s,t) \tag{5.2}$$

is antisymmetric in s and t. Under what conditions on the given set $\chi_{\sigma-l}^l$ is this possible?

(ii) To what extent does the imposition of s-t antisymmetry on $G(s,t)$ determine χ_σ^0 ($\sigma\geq 2$) in terms of $\chi_{\sigma-l}^l$ ($l=2,4,6,\ldots$; $\sigma\geq 2$)?

The answers to these questions are contained in the following results:

(a) *In order that there exist constants χ_σ^0, $\chi_{\sigma-l}^l$ (1, 3, 5, ... ; $\sigma\geq 2$) such that G(s,t) is antisymmetric in s and t, it is necessary and sufficient that the constants $\chi_{\sigma-l}^l$ ($l=2,4,6,\ldots$; $\sigma\geq 2$) fulfill a given set of m_σ linear equations which are linearly independent. They allow us to determine uniquely m_σ members of the set $\chi_{\sigma-l}^l$ ($l=2,4,6,\ldots$; $\sigma\geq 2$) in terms of the remainder.*

(b) *χ_σ^0 for $\sigma\geq 2$ is uniquely determined by $\chi_{\sigma-l}^l$ ($l=2,4,6,\ldots$; $\sigma\geq 2$).*

The definition of m_σ and the constraint equations mentioned in (a) will be given during the course of the proof.

Since an acceptable $H^{(0)}(s,t)$ is antisymmetric in s and t, then for *any* function $F^i(s,t)$ which is totally symmetric in s, t, u, we have

$$\iint_\Delta ds\,dt\; H^{(0)}(s,t)F^i(s,t)=0. \tag{5.3}$$

Expanding $F^i(s,t)$, we obtain

$$F^i(s,t)=\sum_{\substack{\sigma\\ l=\text{even}}}2(\sigma+1)(2l+1)f_{\sigma-l}^{l,i}S_{\sigma-l}^l(s,t), \tag{5.4}$$

where l is even due to the symmetry in t and u. Using (2.21) and (5.4) in (5.3), we find

$$\sum_{\substack{\sigma\geq 2\\ l=\text{even}}} \chi^{l}_{\sigma-l} f^{l,i}_{\sigma-l}=0\,. \tag{5.5}$$

Roskies[17] has shown that for each σ there are $m_\sigma+1$ totally symmetric linearly independent functions $F^i(s,t)$, where

$$m_\sigma+1=\text{number of integers in the closed interval } [\tfrac{1}{3}\sigma, \tfrac{1}{2}\sigma]\,. \tag{5.6}$$

Thus (5.5) is a set of $m_\sigma+1$ linearly independent equations. We can eliminate χ^0_σ from m_σ of these equations and obtain m_σ equations which are constraints which must be satisfied by the constants $\chi^{l}_{\sigma-l}$ ($l=2, 4, 6, \ldots$; $\sigma\geq 2$). The last equation then solves for χ^0_σ uniquely in terms of $\chi^{l}_{\sigma-l}$ ($l=2, 4, 6, \ldots$; $\sigma\geq 2$).

The method of construction of $H^{(0)}$ and the inequalities begins, therefore, by representing $h_2^{(0)}(s)$, $h_4^{(0)}(s)$, $h_6^{(0)}(s), \ldots$ according to the theorems of Sec. IV. This will give $\chi^{l}_{\sigma-l}$ ($l=2, 4, 6, \ldots$; $\sigma\geq 2$). The arbitrary constants defining $\chi^{l}_{\sigma-l}$ [coming from the polynomials A_m, B_m, C_m, and D_{m-1} defining $h_2^{(0)}(s)$] will be made to satisfy the m constraint equations of (a). Finally, χ^0_σ can be constructed using the last equation in (5.5). We have, therefore, an s wave

$$h_0^{(0)}(s)=\tfrac{1}{2}\lambda\,(3s-1)+\sum_{\sigma=2}^{\infty} 2(\sigma+1)\,\chi^0_\sigma P^{(1,0)}_\sigma(2s-1) \tag{5.7}$$

and the inequality (3.3) is determined. The term $\frac{1}{2}\lambda(3s-1)$ does not contribute due to the identity[18]

$$\iint_\Delta ds\,dt\,(s-t)\,A^{(0)}(s,t)=\tfrac{1}{2}\int_0^1 ds\,(1-s)(3s-1)a_0^{(0)}(s)=0\,. \tag{5.8}$$

Notice that $h_0^{(0)}(s)$ depends only on $h_l^{(0)}(s)$, l even [see Eq. (5.5)]. We might have anticipated that the odd-l partial waves would not play any role in the following way. The function

$$\mathcal{H}^{(0)}(s,t)\equiv\tfrac{1}{2}[H^{(0)}(s,t)+H^{(0)}(s,u)]$$

has the same even s-channel partial waves as $H^{(0)}(s,t)$ but has no odd s-channel partial waves. Moreover,

$$\iint_\Delta \mathcal{H}^{(0)}(s,t)A^{(0)}(s,t)=0\,,$$

which leads to the same inequality (3.3) as given by $H^{(0)}(s,t)$. One can obtain our results using functions with the mixed symmetry of $\mathcal{H}^{(0)}(s,t)$, but we preferred the $H^{(0)}$'s due to their simplicity.[19]

The relationship of our approach to the s-wave inequalities and to that of Pennington[6] may be pointed out here. Pennington starts with the general representation of functions with the symmetry of $\mathcal{H}^{(0)}$ and attempts to impose systematically the positivity constraints of the type (3.4) on its partial waves. We, however, start with functions with the right positivity properties and impose the symmetry properties as the second step. The two methods may thus be regarded as complementary.

VI. DERIVATION OF THE INEQUALITIES DUE TO BALACHANDRAN AND BLACKMON AND TO PENNINGTON

In this section, the considerations of the last two sections will be illustrated by a simple example.

We shall assume that $H^{(0)}(s,t)$ is a polynomial in s and t. The series (2.21), therefore, terminates at $\sigma=$ some σ_0 and the coefficients $\chi^{l}_{\sigma-l}$ vanish when σ exceeds σ_0. From (2.22) and the orthogonality relations

$$\int_0^1 ds(1-s)^{2l+1}s^\nu P^{(2l+1,0)}_{\sigma-l}(2s-1)=0,\quad \nu=0,1,2,\ldots,\sigma-l-1 \tag{6.1}$$

it is readily seen that the termination of the series at $\sigma=\sigma_0$ is equivalent to the following form for the partial waves:

$$\begin{aligned} h_l^{(0)}(s)&=(1-s)^l\times\text{polynomial of degree }(\sigma_0-l),\quad l\leq\sigma_0\\ &\equiv 0,\quad l>\sigma_0\,. \end{aligned} \tag{6.2}$$

The number of constraints m_σ on $\chi^{l}_{\sigma-l}$ ($l=2, 4, 6, \ldots$) due to the antisymmetry of $H^{(0)}$ is zero for $\sigma\leq 5$, one for $\sigma=6$, zero for $\sigma=7$, and so on. For algebraic simplicity, we shall consider only such $H^{(0)}$ for which $\chi^{l}_{\sigma-l}=0$ for $\sigma\geq 4$. Then the antisymmetry of $H^{(0)}$ imposes no constraints on $h_l^{(0)}$ ($l=2, 4, 6, \ldots$). The positivity conditions we shall impose on $h_l^{(0)}$ will be those stated in (3.4).

When $\chi^{l}_{\sigma-l}=0$, for $\sigma\geq 4$, the general form of $h_2^{(0)}$ consistent with positivity is [cf. Sec. IV and Eq. (6.2)]

$$\begin{aligned} h_2^{(0)}(s)&=(1-s)^2[\xi(1-s)+\eta s],\quad \xi,\eta\geq 0\\ &=(\xi-\eta)(1-s)^3+\eta(1-s)^2,\quad \xi,\eta\geq 0 \end{aligned} \tag{6.3}$$

while $h_4^{(0)}=h_6^{(0)}=h_8^{(0)}=\cdots=0$. Now we know that there exists an $H^{(0)}(s,t)$ antisymmetric in s and t with d and higher waves as given above. Further, two such $H^{(0)}$ will lead to inequalities which are equivalent modulo the sum rule (5.8). It is thus sufficient to find one such $H^{(0)}$. We shall now indicate a method for writing down such an $H^{(0)}$ by direct inspection.

The following formula can be derived by elementary manipulations[20]:

$$\tfrac{1}{2}\int_{-1}^{1} dz_s\, t^\nu P_l(z_s) = (1-s)^\nu N_l^\nu\,, \qquad N_l^\nu = \frac{(-1)^l(\nu\,!)^2}{(\nu+l+1)\,!\,(\nu-l)\,!}\,. \tag{6.4}$$

So let

$$H^{(0)}(s,t) = \frac{(\xi-\eta)}{N_2^3}(t^3-s^3) + \frac{\eta}{N_2^2}(t^2-s^2)\,. \tag{6.5}$$

This $H^{(0)}(s,t)$ is antisymmetric in s and t, has an $h_2^{(0)}$ as given in (6.3), and has $h_l^{(0)}=0$ for $l\geq 4$. Thus (6.5) is an acceptable $H^{(0)}$. Its s wave is

$$h_0^{(0)}(s) = \frac{\xi}{N_2^3}[N_0^3(1-s)^3 - s^3] + \eta\left[\frac{1}{N_2^2}\{N_0^2(1-s)^2 - s^2\} - \frac{1}{N_2^3}\{N_0^3(1-s)^3 - s^3\}\right], \quad \xi,\eta\geq 0\,. \tag{6.6}$$

Thus, we find the two independent inequalities[1,2,6]

$$-\int_0^1 ds(1-s)[N_0^3(1-s)^3 - s^3]\,a_0^{(0)}(s) \geq 0\,, \tag{6.7}$$

$$-\int_0^1 ds(1-s)\left[\frac{1}{N_2^2}\{N_0^2(1-s)^2 - s^2\} - \frac{1}{N_2^3}\{N_0^3(1-s)^3 - s^3\}\right] \geq 0\,. \tag{6.8}$$

The inequalities for the case $\chi_{\sigma-l}^l = 0$ for $\sigma\geq 5$ are given in Eqs. (18)–(21) of Ref. 1., (A6)–(A9) of Ref. 2, and by Pennington.[6] Although Refs. 1 and 2 do not give our derivation of the inequalities, we will not do so here in view of the fact that Common and Pennington[6] have already given a similar derivation.

VII. DERIVATION OF SOME NEW INEQUALITIES

Here we derive some new s- and p-wave inequalities where we impose (3.5) rather than (3.4) as the positivity conditions on $h_l^{(i)}$. The method used will be one of direct inspection and will not utilize the general results of the previous sections. In connection with the work of this section, we would also like to refer to the papers of Griss[5] and of Common and Pennington.[6]

Let

$$N(s,t) = M(s,t) + \lambda_{pq}\overline{M}(s,t)\,, \tag{7.1}$$

where

$$M(s,t) = -\frac{t^p}{(p\,!)^2} = \sum_{l=0}^{p}(2l+1)m_l(s)P_l(z_s)\,, \tag{7.2}$$

$$\overline{M}(s,t) = \frac{t^q}{(q\,!)^2} = \sum_{l=0}^{q}(2l+1)\overline{m}_l(s)P_l(z_s)\,. \tag{7.3}$$

Here p and q are integers and

$$p > q \geq 2\,. \tag{7.4}$$

The dependence of M and $\overline{M}$ on p and q has been suppressed. Our method will consist in first adjusting the constant λ_{pq} such that

$$\sum_{\substack{l\geq 2\\ l=\text{even}}}(2l+1)[m_l(s)+\lambda_{pq}\overline{m}_l(s)]\,\xi^l \geq 0, \quad 0\leq s\leq 1;\quad 0\leq\xi\leq 1\,. \tag{7.5}$$

[Note that $r(s)$ in (3.5) is ≥ 1 for $0\leq s\leq 1$.] We shall then set

$$H^{(0)}(s,t) = sN(s,t) - tN(t,s)\,, \qquad H^{(c)}(s,t) = sN(s,t) \tag{7.6}$$

and

$$H^{(0)}(s,t) = (1-s)N(s,t) - (1-t)N(t,s)\,, \qquad H^{(c)}(s,t) = (1-s)N(s,t)\,. \tag{7.7}$$

The terms $-tN(t,s)$ and $-(1-t)N(t,s)$ in $H^{(0)}(s,t)$ do not contribute to $h_l^{(0)}(s)$ for $l\geq 2$. Thus, these functions $H^{(i)}$ fulfill all the necessary constraints and lead to the required inequalities.

The following conditions are sufficient for the validity of (7.5):

$$\sum_{\substack{l\geq 2\\ l=\text{even}}}^{(q)}(2l+1)[m_l(s)+\lambda_{pq}\overline{m}_l(s)] \geq \sum_{\substack{l\geq (q)+2\\ l=\text{even}}}(2l+1)[-m_l(s)], \quad 0\leq s\leq 1 \tag{7.8}$$

$$m_l(s) + \lambda_{pq}\overline{m}_l(s) \geq 0, \quad 0\leq s\leq 1\,;\quad l=2,4,6,\ldots,(q)\,. \tag{7.9}$$

We have denoted by (q) the largest even integer which does not exceed q. To prove the sufficiency of (7.8) and (7.9), note that $m_l(s) \leq 0$ for $l=$even and $0 \leq s \leq 1$ due to (6.4). Also since $\xi^\alpha \leq \xi^\beta$ for $\alpha \geq \beta \geq 0$ and $0 \leq \xi \leq 1$, we can write, for $0 \leq s \leq 1$ and $0 \leq \xi \leq 1$,

$$\sum_{\substack{l \geq (q)+2 \\ l=\text{even}}} (2l+1)[-m_l(s)]\xi^l$$

$$\leq \xi^{(q)+2} \sum_{\substack{l \geq (q)+2 \\ l=\text{even}}} (2l+1)[-m_l(s)]$$

$$\leq \xi^{(q)+2} \sum_{\substack{l \geq 2 \\ l=\text{even}}}^{(q)} (2l+1)[m_l(s)+\lambda_{pq}\bar{m}_l(s)], \quad \text{by (7.8)}$$

$$\leq \sum_{\substack{l \geq 2 \\ l=\text{even}}}^{(q)} (2l+1)[m_l(s)+\lambda_{pq}\bar{m}_l(s)]\xi^l, \quad \text{by (7.9)} \tag{7.10}$$

which proves the required result.

Now (7.8) is the same as

$$\tfrac{1}{2}[N(s,t)+N(s,u)]\,|_{z_s=+1} - [m_0(s)+\lambda_{pq}\bar{m}_0(s)] \geq 0, \quad 0 \leq s \leq 1\,. \tag{7.11}$$

This gives, on using (6.4),

$$\lambda_{pq}\frac{(q-1)}{q!\,(q+1)!} \geq (1-s)^{p-q}\frac{(p-1)}{p!\,(p+1)!}, \quad 0 \leq s \leq 1\,. \tag{7.12}$$

Since the maximum of the right-hand side is at $s=0$, we thus find

$$\lambda_{pq} \geq \frac{(p-1)}{(q-1)}\frac{q!\,(q+1)!}{p!\,(p+1)!}\,. \tag{7.13}$$

Similarly, (7.9) is equivalent to

$$\lambda_{pq} \geq \frac{(q+l+1)!\,(q-l)!}{(p+l+1)!\,(p-l)!}, \quad l=2,4,6,\ldots,(q)\,. \tag{7.14}$$

The maximum of the right-hand side is reached when $l=(q)$ since

$$\frac{\partial}{\partial l}\frac{(q+l+1)!\,(q-l)!}{(p+l+1)!\,(p-l)!} = \frac{\partial}{\partial l}\frac{1}{(q+l+2)(q+l+3)(q+l+4)\cdots(p+l+1)} \times \frac{1}{(q-l+1)(q-l+2)(q-l+3)\cdots(p-l)} > 0 \tag{7.15}$$

as a consequence of

$$\frac{\partial}{\partial l}\frac{1}{(q+l+1+\rho)(q-l+\rho)} > 0, \quad \rho=1,2,3,\ldots\,.$$

Thus (7.14) is the same as

$$\lambda_{pq} \geq \frac{[q+(q)+1]!\,[q-(q)]!}{[p+(q)+1]!\,[p-(q)]!}\,. \tag{7.16}$$

Putting (7.13) and (7.16) together, we finally have

$$\lambda_{pq} \geq \text{Max}\left\{\frac{(p-1)}{(q-1)}\frac{q!\,(q+1)!}{p!\,(p+1)!}, \frac{[q+(q)+1]!\,[q-(q)]!}{[p+(q)+1]!\,[p-(q)]!}\right\} \equiv \mu_{pq}\,. \tag{7.17}$$

It is sufficient to consider the equality sign here since the inequality when $\lambda_{pq}=\mu_{pq}+|\epsilon|$ is a linear combination with positive coefficients of the inequalities with $\lambda_{pq}=\mu_{pq}$ and one of the inequalities stated in Refs. 1 and 2.[21]

Finally with $\lambda_{pq}=\mu_{pq}$, where p and q are integers which fulfill (7.4), the inequalities may be written down using (6.4) to project the s and p waves from (7.6) and (7.7). We will leave it to the interested reader to write down the corresponding formulas.

The inequalities of this section can be generalized in at least two different ways. The first generalization is effective only for the inequalities involving $a_0^{(0)}$. It is sufficient to illustrate it here by an example. The principle of the method should then become clear if the discussion in Sec. IV of Ref. 2 is also consulted where such generalizations are studied in detail when the positivity condition is (3.4). Consider

$$H^{(0)}(s,t) = s^m\left[-\frac{t^p}{(p!)^2}+\lambda_{pq}\frac{t^q}{(q!)^2}\right] - t^m\left[-\frac{s^p}{(p!)^2}+\lambda_{pq}\frac{s^q}{(q!)^2}\right] + \frac{\lambda_{pq}}{(q!)^2}[t^m s^{q-1}-s^m t^{q-1}] + \frac{\lambda_{pq}}{(q!)^2}[st^{q-1}-ts^{q-1}]\,. \tag{7.18}$$

Here λ_{pq} is given by (7.17) and we shall assume for purposes of illustration that $q \geq 3$ and that m is an integer ≥ 2. This function is clearly antisymmetric in s and t. The contribution of the term in the first bracket to $h_l^{(0)}$ fulfills (3.5) by construction. The contribution of $t^m s^p/(p\,!)^2$ also fulfills (3.5) since in fact it fulfills (3.4) due to (6.4). The contribution of $-\lambda_{pq} t^m s^q/(q\,!)^2$ has the wrong sign by (6.4), but the sum

$$-\lambda_{pq}\frac{t^m s^q}{(q\,!)^2}+\lambda_{pq}\frac{t^m s^{q-1}}{(q\,!)^2}=\frac{\lambda_{pq}}{(q\,!)^2}\,t^m s^{q-1}(1-s) \tag{7.19}$$

clearly contributes with the correct sign. Similarly,

$$-\lambda_{pq}\frac{s^m t^{q-1}}{(q\,!)^2}+\lambda_{pq}\frac{s t^{q-1}}{(q\,!)^2}=\lambda_{pq}\frac{s(1-s^{m-1})t^{q-1}}{(q\,!)^2} \tag{7.20}$$

contributes with the right sign while the last term $-\lambda_{pq} t s^{q-1}/(q\,!)^2$ makes zero contribution to $h_l^{(0)}$ for $l \geq 2$. Thus (7.18) may be used to construct an inequality for $a_0^{(0)}$.

The second generalization of the considerations of this section is effective for both processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ and consists in replacing the discrete variables of the sort q, p, and m in (7.6), (7.7), and (7.18) by continuous variables restricted to suitable intervals. This method has been explained in previous work[2,3] and will not be pursued further here.

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*Supported in part by the U. S. Atomic Energy Commission.

†Permanent address.

‡Postal address: Istituto di Fisica Teorica, Mostra d'Oltremare, pad. 19-80125 Napoli, Italy.

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[20]See, for example, Ref. 2, footnote 9.

[21]Namely, one of the inequalities from Eqs. (13) and (14) in Ref. 1 or from Eqs. (A1) and (A2) in Ref. 2.