Theory of Higher-Order Weak Interactions and *CP*-Invariance Violation. I. Leptonic Processes*

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(Received 2 December 1971)

In order to cure the divergence difficulties of higher-order weak interactions, we propose a theory in which the usual current× current Lagrangian is modified by inserting the nonpolynomial factor $\exp[f\,\varphi^\dagger\varphi(x)]$, where φ is a charged scalar field. We show that, as a result, the usual first-order weak amplitudes are unchanged, while second-order leptonic scattering amplitudes are finite. Explicit formulas are given for lepton-lepton scattering and the logarithmically divergent lepton self-mass. A discussion of Nth order is presented, in which it is shown that the naive degree of divergence is independent of order, and the superpropagator to all orders is derived. The groundwork is laid for the discussion in a following paper of hadronic processes and the appearance of *CP* violation as a higher-order weak effect.

I. INTRODUCTION

Of the many unsolved problems in weak interactions, two stand out as having no natural explanation within any of the usual formulations of the theory. The first of these is how to cure the divergence difficulties encountered in higher orders of perturbation theory; the second is how to understand the origin of *CP*-invariance violation as observed in neutral-*K* decay.

In the past, attempts to solve these problems have generally treated them as unrelated. The various approaches taken in the case of divergences have been summarized by Segrè[1] and by Gell-Mann *et al.*[2] One may identify two main categories of proposed solutions: (i) those in which the usual current×current theory is considered as a low-energy approximation to a more complicated theory in which various numbers and kinds of intermediate particles are exchanged,[3] and (ii) those in which selected sets of divergent Feynman diagrams are summed to all orders in the weak coupling constant.[4] In addition, there are the approaches of Lee and Wick,[5] who introduce an indefinite metric, and of Gell-Mann *et al.*,[2] in which the divergence problems are contained only in the so-called diagonal interactions which respect all strong-interaction symmetries.

In the case of *CP*-invariance violation, the number and variety of explanations has been so large that we cannot hope to review them all here. The source of this effect has been placed anywhere from the electromagnetic interactions, to the weak interactions themselves, to a new, "superweak" interaction that only shows up in the neutral-*K* system.[6] That such a wide range of theoretical possibilities exists is certainly a reflection of the fact that the effect has so far been observed only in decays of the long-lived K^0.

In this series of papers, we construct a theory of the weak interactions whose primary aim is the removal of divergences in perturbation theory, but which also has the virtue that *CP*-invariance violation emerges naturally as a higher-order weak effect. Furthermore, the magnitude of the *CP*-nonconserving effect in any process is intimately related to the degree of divergence of that process in the usual current×current theory. This type of mechanism for *CP*-invariance violation was first described phenomenologically by Wolfenstein.[7] Here we present a theory which not only provides a concrete realization of his suggestion, but which may also cure the divergence problem of higher-order weak interactions as well.

To accomplish these ends, we shall use the techniques of nonpolynomial Lagrangian field theory (NLFT) of type II, as described in a previous paper.[8-10] The first application of NLFT to weak-interaction theory was made by Fivel and Mitter.[11] While we have benefited from their work in a variety of ways, our approach is fundamentally different from theirs in a number of respects. They used a nonpolynomial of type I, which is more cumbersome and more ambiguous than our type II, and they chose to begin with an intermediate vector boson theory, so that the second-order weak corrections (in the Fermi constant, $G \approx 1.02 \times 10^{-5}\, M_{\text{proton}}^{-2}$) to the usual amplitudes are represented by complicated fourth-order diagrams. We avoid this difficulty by beginning with the Fermi current×current formulation. Furthermore, Fivel and Mitter made no attempt to relate *CP*-invariance violation to the divergence problem, whereas this relationship is one of the interesting consequences of our approach.

The general question of the application of NLFT to physical problems has aroused increasing interest during the last few years. At a fundamental level, it has been proved that certain types of NLFT are mathematically at least as well defined as the usual polynomial theories.[12] On the other hand, it has been stressed, especially by Salam and co-workers,[13] that quantum gravity necessarily introduces nonpolynomial effects, and that gravity-induced NLFT can be used to provide reasonable cutoffs for the divergent quantities in quantum electrodynamics. Our point of view is neither to worry overly much about mathematical rigor nor to speculate at length on the fundamental physical significance of the nonpolynomial that we introduce. Rather, we modify the weak interactions in a way that is as simple and as natural as possible, consistent with the objectives we hope to achieve, namely, to generate an S matrix that is unitary and which provides unambiguous expressions for the relevant amplitudes. At this level, the following two papers are relatively self-contained; the interested reader is referred to our previous paper [8] and the general literature for further details.[14]

The remainder of this first paper is divided into two parts. Section II contains a general description of our theory, and presents an overview of some of the important features of the second-order amplitudes. It is shown that the usual first-order amplitudes remain unchanged, and guidelines for estimating the magnitude of second-order corrections are laid down.

Sections III and IV are devoted to higher-order purely leptonic processes. We given explicit rules for calculating the physical amplitudes, and show that those describing lepton-lepton scattering are finite in second order. A brief treatment is given of the lepton self-mass, and we sketch the procedure for removing the infinities. We then go on to give the basic formulas needed for calculating amplitudes in higher orders, and prove that the naive degree of divergence of a diagram is independent of order (as it is in quantum electrodynamics). The general problem of proving renormalizability is not attempted here, although it is under investigation at the present time.

In paper II, we focus on the neutral K-meson system, and derive expressions for the usual CP-violating parameters in terms of the fundamental quantities of our theory. The discussion is largely only qualitative, because of our ignorance of the short-distance behavior of hadronic current matrix elements; however, we are able to use the free-quark model as a guide in providing reasonable estimates for the size of CP-invariance-violating effects. In particular, we find that the results $|\epsilon| \approx 10^{-3}$ and $\eta_{+-} \approx \eta_{00}$ fit comfortably within the framework of our theory, although we cannot derive them unambiguously. We give a brief discussion of the induced neutron dipole moment, and estimate its magnitude to be of order 10^{-27} e cm.

We devote one section of II to the discussion of the production of the particles associated with our nonpolynomial coupling, and find that the cross section is probably too small to have any immediate experimental consequences. In the last section of II we present some conclusions.

II. GENERAL DISCUSSION

The remarkable successes of the usual form of the weak Lagrangian, when one calculates only to order G, place severe restrictions on any attempted modification. On the other hand, the even worse divergences encountered in higher orders of perturbation theory suggest that some fundamental alteration must indeed be made.

We shall show that a suitably chosen nonpolynomial modification of the current$\times$current weak Lagrangian fulfills these requirements, in that no change whatever is wrought on first-order amplitudes, and second-order leptonic scattering and decay amplitudes are all finite. The degree to which our objectives have been achieved is less clear for second-order amplitudes involving hadrons, because of our ignorance of the short-distance behavior of products of hadronic weak currents. This problem will be discussed at length in II. As far as third and higher orders in G are concerned, we present some of the relevant formulas in Sec. IV. The question of whether the theory is fully renormalizable and the proof of unitarity to all orders in G are currently under investigation by us.

In the remainder of this section, we outline the general features of our proposed modification, up to order G^2. Our starting point is to multiply each of the positively charged vector–axial-vector weak currents, $j_\lambda^{(a)}(x)$, by a nonpolynomial function of a charged scalar field $\varphi(x)$. Here

$a = e$, μ for leptonic currents;

$a = 0$ for $\Delta S = 0$

and

$a = 1$ for $\Delta S = +1$ hadronic currents.

Specifically,[15]

$$j_\lambda^{(l)}(x) \equiv \bar{\nu}_l(x)\gamma_\lambda(1-\gamma_5)\, l(x) \to j_\lambda^{(l)}(x)\exp[\tfrac{1}{2}f_l\varphi^\dagger(x)\varphi(x)] \quad (l=e,\ \mu),$$
$$j_\lambda^{(0)}(x) \equiv J_\lambda^{(1+i2)}(x) - J_{5\lambda}^{(1+i2)}(x) \to j_\lambda^{(0)}(x)\exp[\tfrac{1}{2}f_0\varphi^\dagger(x)\varphi(x)], \tag{2.1}$$
$$j_\lambda^{(1)}(x) \equiv J_\lambda^{(4+i5)}(x) - J_{5\lambda}^{(4+i5)}(x) \to j_\lambda^{(1)}(x)\exp[\tfrac{1}{2}f_1\varphi^\dagger(x)\varphi(x)].$$

The particle represented by the field $\varphi(x)$ is assumed not to interact strongly with the known particles. The four minor coupling constants f_a are *a priori* all different and complex; the f_a and the mass M of the φ particles are to be determined by experiment.

Denoting by $j_\lambda^{(a)\prime}$ the modified currents in Eq. (2.1), we then form the (symmetrized) current $\times$current Lagrangian

$$\mathcal{L}_W(x) = 2^{-1/2}G : \sum_{a,b}[j_\lambda^{(a)\prime}(x)\, j_\lambda^{(b)\prime\dagger}(x)]_s :$$
$$= \sum_{a,b}\mathcal{L}_{ab}(x) : \exp[f_{ab}\varphi^\dagger(x)\varphi(x)] : , \tag{2.2}$$

where

$$\mathcal{L}_{ab}(x) = 2^{-1/2}G[j_\lambda^{(a)}(x)\, j_\lambda^{(b)\dagger}(x)]_s\,. \tag{2.3}$$

In Eqs. (2.2) and (2.3), $[AB]_s = \frac{1}{2}(AB+BA)$ and

$$f_{ab} = \tfrac{1}{2}(f_a + f_b^*) = f_{ba}^*\,. \tag{2.4}$$

Also, the nonpolynomial function of $\varphi^\dagger\varphi$ is understood to mean

$$:\exp[f\varphi^\dagger(x)\varphi(x)]: = \sum_{n=0}^{\infty}\frac{f^n}{n!} : [\varphi^\dagger(x)\varphi(x)]^n : . \tag{2.5}$$

It is at once clear that any first-order weak process not involving the emission or absorption of φ particles is unchanged, since the relevant matrix elements will be of the form

$$\sum_{a,b}\langle A|\mathcal{L}_{ab} : \exp[f_{ab}\varphi^\dagger\varphi] : |B\rangle, \tag{2.6}$$

and if $|A\rangle$ and $|B\rangle$ contain no φ particles, then the change in (2.6) is the factor

$$\langle 0| : \exp[f_{ab}\varphi^\dagger\varphi] : |0\rangle = 1. \tag{2.7}$$

Also, from formula (2.5), we see that if, for example, the state $|A\rangle$ contains $n\,(\varphi^+\varphi^-)$ pairs,

$$A = C + n\,(\varphi^+\varphi^-),$$

then the first-order amplitude takes the form

$$\sum_{a,b}\langle C|\mathcal{L}_{ab}|B\rangle\,\langle n(\varphi^+\varphi^-)| : \exp[f_{ab}\varphi^\dagger\varphi] : |0\rangle$$
$$= \sum_{a,b}(f_{ab})^n\langle C|\mathcal{L}_{ab}(x)|B\rangle e^{iP\cdot x}, \tag{2.8}$$

where P = total momentum of the n pairs. We note that the φ's are always produced in pairs, and that moreover we have made no provision for the decay of individual φ particles. A more detailed discussion of the production of φ particles will be given in paper II.

In second order, a contribution to a typical amplitude with no external φ's will look like

$$T_{AB} = \int d^4x \sum_{\substack{a,b\\c,d}}\langle A|T\{:\mathcal{L}_{ab}(x)\exp[f_{ab}\varphi^\dagger(x)\varphi(x)]::\mathcal{L}_{cd}(0)\exp[f_{cd}\varphi^\dagger(0)\varphi(0)]:\}|B\rangle$$
$$= \int d^4x \sum_{\substack{a,b\\c,d}}\langle A|T\{\mathcal{L}_{ab}(x)\mathcal{L}_{cd}(0)\}|B\rangle\,\langle 0|T\{:\exp[f_{ab}\varphi^\dagger(x)\varphi(x)]::\exp[f_{cd}\varphi^\dagger(0)\varphi(0)]:\}|0\rangle$$
$$= \int d^4x \sum_{\substack{a,b\\c,d}}\langle A|T\{\mathcal{L}_{ab}(x)\mathcal{L}_{cd}(0)\}|B\rangle\,[1 - f_{ab}f_{cd}\Delta^2(x)]^{-1}. \tag{2.9}$$

In the second step of (2.9) we assumed that $|A\rangle$ and $|B\rangle$ contain no φ particles, and in the third we evaluated the φ "superpropagator" straightforwardly using Wick's theorem and summing the infinite series in $f_{ab}f_{cd}\Delta^2$. Here we see one of the basic features of NLFT; we sum to all orders in the minor coupling constants f, but do perturbation theory in the major coupling constant G. For more detail on the superpropagator and other two-point functions involving $\exp(f\varphi^\dagger\varphi)$, the reader is referred to Appendix A.

We remark that, according to the usual rules of NLFT,[8-10] we are adopting the Euclidicity postulate, whereby all momenta in the states $|A\rangle$ and $|B\rangle$ are assumed to first lie in the Euclidean or Symanzik region, so that (by Wick rotation) the integration over Minkowski space-time can be converted to integration over a Euclidean one. After performing all auxiliary integrations, amplitudes are then analytically continued to the physical region of the external momenta. The uninitiated reader should not be too distressed by this procedure. Computations in ordinary polynomial field theories may be carried out in the same way; for example, the renormalization program in quantum electrodynamics is most easily carried out with momenta in the Euclidean domain. This point is discussed further in Sec. III, where some explicit examples are given.

A particular consequence of this is that the propagator function Δ appearing in Eq. (2.9) assumes the form $\Delta(x;M)=MK_1(Mr)/4\pi^2 r$, where $r=(-x^2)^{1/2}>0$ is the Euclidean length and $K_1(Mr)$ is the usual modified Bessel function. We note that as $r\to 0$, $\Delta(r)\to 1/4\pi^2 r^2$, so that the superpropagator $1/[1-f^2\Delta^2(r)]$ provides a convergence factor of r^4 at the origin. This damping of ultraviolet divergences is the main motivation for introducing a nonpolynomial Lagrangian. Let us perform the Euclidean integration over angles in (2.9) defining

$$F_{AB}(r)\equiv\int d^3\Omega_x\langle A|T\{\mathcal{L}_{ab}(x)\mathcal{L}_{cd}(0)\}|B\rangle. \tag{2.10}$$

$F_{AB}(r)$ will be a function of the Euclidean length r and the various momenta and spins of the particles in states A and B. The amplitude in (2.9) is then

$$T_{AB}(f^2)=\int_0^\infty r^3 dr\,\frac{F_{AB}(r)}{1-f^2\Delta^2(r)}\,, \tag{2.11}$$

where we have abbreviated $f_{ab}f_{cd}=f^2$. We assume, of course, that the integral (2.11) exists. In the case of purely leptonic scattering amplitudes, where $F_{AB}(r)$ can be evaluated explicitly, we shall see in Sec. III that all such amplitudes are indeed finite. In the case of processes where hadronic currents are involved, additional assumptions must be made (see paper II).

Equation (2.11) displays $T_{AB}(f^2)$ as an analytic function in the f^2 plane, with a cut running from $f^2=0$ to infinity. Since we have assumed that the f_a are arbitrary complex numbers, in general the f^2 appearing in (2.11) will also be an arbitrary complex number, so that there is little more we can say. However, certain special cases are worthy of particular attention.

First, we consider the case of f^2 real (case I). This must necessarily happen in "diagonal" processes, where $b=c$ and $a=d$ in $f_{ab}f_{cd}$, and may happen in other processes as well. Then, for some value $r=r_0$, we must have $f^2\Delta^2(r_0)=1$ [recall that $\Delta(r)$ decreases monotonically from infinity to zero as r increases from zero to infinity], so that there is a singularity in the region of integration in (2.11). When this happens, then the integral *by definition* is taken to be the principal value. This is the same rule we employed in Ref. 8 as the simplest prescription consistent with unitarity. We show in Appendix A that, for $M=0$, the principal value also provides the best-behaved superpropagator in momentum space.[16]

The second case we condider (case II) occurs when f^2 has an infinitesimal (positive or negative) imaginary part. We are not forced to consider this case; however, at present there is no way of determining all nine parameters in our theory (the real and imaginary parts of the four f_a, and the mass M of the φ particle), and since the effects of the $\mathrm{Im}f_a$ do not go away as $\mathrm{Im}f_a\to 0$ (because of the cut in the f^2 plane) we find it convenient to assume that indeed each $\mathrm{Im}f_a$ is infinitesimal. We shall see that this causes us no trouble, and in fact introduces a certain elegance into the formulation of the theory. We bear in mind, however, that this assumption may have to be revised at a later date.

When f^2 has an infinitesimal imaginary part, we obtain a δ-function term in addition to the principal value of case I. Thus, we may combine cases I and II into a single formula:

$$T_{AB}(f^2)=T_{AB}^{(1)}(f^2)+T_{AB}^{(2)}(f^2), \tag{2.12}$$

with

$$T_{AB}^{(1)}(f^2)=\mathrm{P}\int_0^\infty r^3 dr\,\frac{F_{AB}(r)}{1-\mathrm{Re}f^2\Delta^2(r)} \tag{2.13}$$

and

$$T_{AB}^{(2)}(f^2)=i\pi\eta(f^2)\int_0^\infty\frac{r^3 dr}{\Delta^2(r)}F_{AB}(r)\delta\left(\frac{1}{\Delta^2(r)}-\mathrm{Re}f^2\right). \tag{2.14}$$

Here

$\eta(f^2)=0$ if f^2 is purely real

$=\mathrm{sgn}(\mathrm{Im}f^2)$ if f^2 has an infinitesimal imaginary part.

As we shall see in paper II, the occurrence of terms like $T_{AB}^{(2)}$ (and its analogs in higher orders) is responsible for the CP-violating effects in our theory.

Let us assume that $F_{AB}(r)$ behaves as c/r^n for small r. Without the superpropagator, of course, T_{AB} would be singular if $n\geq 4$. The degree of singularity could then be expressed in terms of a cutoff $1/\Lambda$ at the lower limit:

$$T_{AB}^{\text{cutoff}}\approx\frac{c}{n-4}\,\Lambda^{n-4}. \tag{2.15}$$

(For $n=4$ this should read $T_{AB}\approx c\ln\Lambda$.)

Replacing F_{AB} by c/r^n in (2.13) and (2.14), and making the further approximation for small r

$$\Delta(r)\approx\frac{1}{4\pi^2 r^2}\,, \tag{2.16}$$

we obtain

$$T_{AB}^{(1)}(f^2)\approx(-1)^{(n-1)/2}\tfrac{1}{2}c\pi(r_0)^{4-n} \tag{2.17a}$$

for n odd (the prinicpal value vanishes in our approximation for n even), and

$$T_{AB}^{(2)}(f^2) \approx \tfrac{1}{4} i c \pi (r_0)^{4-n}\,, \qquad (2.17b)$$

with

$$r_0^{\;4} \equiv \frac{\mathrm{Re} f^2}{16\pi^4}\,.$$

Of course, (2.17) holds only for $n \leq 7$, since the integral (2.13) diverges if $n \geq 8$. We see that the effect of the superpropagator is to regularize any diagram for which the ultraviolet behavior of the matrix element is no worse than r^{-7}. By comparing (2.17) with (2.15) we deduce that r_0^{-1} plays the role of a covariant cutoff,

$$r_0^{-1} \sim \Lambda\,.$$

From various estimates of the magnitude of weak-interaction cutoffs,[17] we should expect the parameters r_0 to be of order $\frac{1}{10}$ GeV^{-1}. Furthermore, since the small-r approximation (2.16) is valid only when $Mr \ll 1$, we find it convenient to assume that $Mr_0 \ll 1$, or, to be specific, that $M \approx 1$ GeV.

As a particular example of the estimates (2.17), consider a second-order amplitude with two intermediate leptons. Since each lepton propagator behaves as $1/r^3$ for small r, we expect $n=6$, so that

$$T_{AB} \approx G^2/r_0^{\;2} = G\,(G/r_0^{\;2}) \approx 10^{-3} G\,, \qquad (2.18)$$

the last approximate equality being valid when $r_0 = \frac{1}{10}$ GeV^{-1}. Thus many second-order amplitudes will be about a thousand times smaller than first order, and not down by a factor of 10^5 as a naive $O(G^2)$ argument might suggest. This enhancement is directly traceable to the divergence that T_{AB} would have without the superpropagator, and will have important consequences when we discuss CP-violation in paper II.

III. LEPTONIC PROCESSES

As a first application of the formalism developed in Sec. II, we investigate purely leptonic processes in order G^2. For simplicity, we shall ignore the contributions to such processes from intermediate hadron states due to the repeated action of the semileptonic weak interaction.

The major portion of this section is devoted to calculating the scattering amplitude for $\nu_\mu + e^- \to \nu_\mu + e^-$. This prototype calculation will be carried out in detail so that we may achieve the following objectives: (i) to show that all leptonic scattering and decay amplitudes are finite in $O(G^2)$ and that they have the analytic structure (in energy-momentum variables) dictated by unitarity, (ii) to illustrate the calculation procedure which is imposed on us by NLFT and which is embodied in the Euclidicity postulate, and (iii) to take advantage of an instance in which we can compute exactly (since only lepton currents are involved) in order to develop intuition about second-order weak amplitudes modified by our superpropagator. This intuition will serve us well when we study processes involving hadronic currents in which exact calculation is impossible.

Following this study, we present a brief discussion of the electron self-energy. This quantity is logarithmically divergent in our theory, and we indicate how it might be separated into infinite renormalizations and finite propagator corrections.

The scattering $\nu_\mu + e^- \to \nu_\mu + e^-$ first occurs in $O(G^2)$ and is represented in our model by the Feynman diagram of Fig. 1. The amplitude describing this process is

$$\langle \nu_\mu(p_3) e^-(p_4) | \mathcal{T}^{(2)} | \nu_\mu(p_1) e^-(p_2)\rangle = i \int d^4x \langle \nu_\mu(p_3) e^-(p_4) | T\{\mathfrak{L}_{e\mu}(x)\mathfrak{L}_{\mu e}(0)\} | \nu_\mu(p_1) e^-(p_2)\rangle$$
$$\times \langle 0 | T\{ :\exp[f_{e\mu}\varphi^\dagger(x)\varphi(x)]: :\exp[f_{\mu e}\varphi^\dagger(0)\varphi(0)]: \} | 0\rangle\,. \qquad (3.1)$$

At this point, all momenta and the variable x in Eq. (3.1) are assumed to lie in the Euclidean region. Accordingly, the T instruction in this equation is really irrelevant. [If we were to take Eq. (3.1) seriously for momenta in the physical region and derive a Low equation from it by insertion of a complete set of states, the result would be badly divergent. This shows that the Euclidicity postulate is a necessity, not just a convenience.] This means that, in carrying out the contractions of lepton fields in Eq. (3.1), we may effectively use expressions of the form

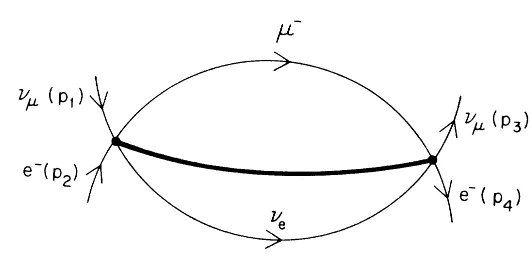

FIG. 1. Elastic $\nu_\mu e^-$ scattering in second order. Heavy line indicates the superpropagator.

$$\gamma_\lambda(1-\gamma_5)\langle 0|T\{l(x)\bar{l}(0)\}|0\rangle\gamma_\mu(1-\gamma_5)$$
$$=2i\,\gamma_\lambda(\gamma\cdot\partial)\Delta(x;m_l)\gamma_\mu(1-\gamma_5)$$
$$=-2i\,\gamma_\lambda\gamma^\rho\gamma_\mu(1-\gamma_5)\frac{x_\rho}{r}\Delta_1(r;m_l), \tag{3.2a}$$

where $r=(-x^2)^{1/2}>0$ is the Euclidean length and

$$\Delta(r;m_l)=\frac{m_l}{4\pi^2 r}K_1(m_l r),$$
$$\Delta_1(r;m_l)\equiv\frac{d}{dr}\Delta(r;m_l)$$
$$=-m_l^{\,2}K_2(m_l r)/4\pi^2 r$$
$$\to -(2\pi^2 r^3)^{-1}\ \text{for}\ m_l\to 0. \tag{3.2b}$$

Remembering also that one of the rules of our theory is that the principal-value integral is understood whenever the product $f_{ab}f_{cd}$ of minor coupling constants is *a priori* real, we find

$$\langle\nu_\mu e^-|\mathfrak{T}^{(2)}|\nu_\mu e^-\rangle=4(G/\sqrt{2})^2M^{\rho\sigma}F_{\rho\sigma}(p_3+p_4,f_{e\mu}f_{\mu e}), \tag{3.3}$$

where

$$M^{\rho\sigma}=\bar{u}_e(p_4)\gamma^\lambda\gamma^\rho\gamma^\mu(1-\gamma_5)u_e(p_2)$$
$$\times\bar{u}_\nu(p_3)\gamma_\lambda\gamma^\sigma\gamma_\mu(1-\gamma_5)u_\nu(p_1) \tag{3.4}$$

and, with $p_\lambda=(p_3+p_4)_\lambda$,

$$F_{\rho\sigma}(p;f_{e\mu}f_{\mu e})=i^3\,\mathrm{P}\int d^4x\,e^{ip\cdot x}\times\frac{x_\rho x_\sigma}{r^2}\frac{\Delta_1(r;m_\mu)\Delta_1(r;0)}{1-f_{e\mu}f_{\mu e}\Delta^2(r;m)}. \tag{3.5}$$

All $O(G^2)$ leptonic scattering or decay amplitudes can be written in the form of Eq. (3.3), with integrals having the same structure as $F_{\rho\sigma}$. Thus, the basic problem of the theory is the definition and calculation of such integrals. Part of this defining process is embodied in the principal-value prescription already imposed. The rest of the definition is based on the Euclidicity postulate, according to which we assume $p^2<0$ in Eq. (3.5) and then cast $F_{\rho\sigma}$ in a form suitable for continuation to the physical region (in the present case, $p^2>m_e^{\,2}$).

To this end, we first introduce the Euclidean momentum $q=(-p^2)^{1/2}=-ip>0$ and write $F_{\rho\sigma}$ in the form

$$F_{\rho\sigma}=\left[g_{\rho\sigma}\frac{1}{p}\frac{d}{dp}+p_\rho p_\sigma\left(\frac{1}{p}\frac{d}{dp}\right)^2\right]F(q^2=-p^2;f_{e\mu}f_{\mu e})$$
$$=\left[-g_{\rho\sigma}\frac{1}{q}\frac{d}{dq}+p_\rho p_\sigma\left(\frac{1}{q}\frac{d}{dq}\right)^2\right]F(q^2=-p^2;f_{e\mu}f_{\mu e})$$
$$=g_{\rho\sigma}F_1(q^2;f_{e\mu}f_{\mu e})+p_\rho p_\sigma F_2(q^2;f_{e\mu}f_{\mu e}). \tag{3.6}$$

Formally, $F(q^2)$ is given by

$$F(q^2)\equiv i\,\mathrm{P}\int d^4x\,e^{ip\cdot x}\frac{1}{r^2}\frac{\Delta_1(r;m_\mu)\Delta_1(r;0)}{1-f_{e\mu}f_{\mu e}\Delta^2(r)}$$
$$=\frac{4\pi^2}{q}\mathrm{P}\int_0^\infty dr\,J_1(qr)\frac{\Delta_1(r;m_\mu)\Delta_1(r;0)}{1-f_{e\mu}f_{\mu e}\Delta^2(r)}, \tag{3.7}$$

where the integration over Euclidean angles has been performed in the second step of Eq. (3.7). While $F(q^2)$ diverges, the integrals of physical interest, $F_{1,2}(q^2)$, do not. They are given by

$$F_1(q^2)\equiv-\frac{1}{q}\frac{d}{dq}F(q^2)$$
$$=\frac{4\pi^2}{q^2}\mathrm{P}\int_0^\infty dr\,r\,J_2(qr)\frac{\Delta_1(r;m_\mu)\Delta_1(r;0)}{1-f_{e\mu}f_{\mu e}\Delta^2(r)} \tag{3.8}$$

and

$$F_2(q^2)\equiv\left(\frac{1}{q}\frac{d}{dq}\right)^2F(q^2)$$
$$=\frac{4\pi^2}{q^3}\mathrm{P}\int_0^\infty dr\,r^2J_3(qr)\frac{\Delta_1(r;m_\mu)\Delta_1(r;0)}{1-f_{e\mu}f_{\mu e}\Delta^2(r)}. \tag{3.9}$$

It is easy to see that the form factors $F_{1,2}$ are finite for all $q^2>0$. For small r, the integrals behave as[18]

$$F_1(q^2)\sim\frac{1}{8\pi^2}\mathrm{P}\int_{r\to 0}dr\frac{r}{r^4-r_{e\mu}^{\,4}}\sim\frac{1}{r_{e\mu}^{\,2}} \tag{3.10a}$$

and

$$F_2(q^2)\sim\frac{1}{48\pi^2}\mathrm{P}\int_{r\to 0}dr\frac{r^3}{r^4-r_{e\mu}^{\,4}}\sim\ln r_{e\mu}, \tag{3.10b}$$

where, assuming that $M^2|f_{e\mu}|\ll 1$, $r_{e\mu}$ is given by

$$r_{e\mu}^{\,4}\cong|f_{e\mu}|^2/16\pi^4>0. \tag{3.11}$$

The leading dependence of F_1 and F_2 on $r_{e\mu}$ shown in Eqs. (3.10) is a reflection of the fact that these integrals are, respectively, quadratically and logarithmically divergent in the absence of the superpropagator. Thus, the length $r_{e\mu}$ acts as the reciprocal of a cutoff mass. In terms of the analytic structure of F_1 and F_2 in the complex variable $f_{e\mu}f_{\mu e}$, this dependence indicates the degree of the singularity at the branch point $f_{e\mu}f_{\mu e}=0$.

For $r\to\infty$, the integrands in Eqs. (3.8) and (3.9) behave as $q^{-2}J_2(qr)e^{-m_\mu r}$ and $q^{-3}J_3(qr)e^{-m_\mu r}$, respectively, thereby concluding the proof that F_1 and F_2 are finite for all $p^2=-q^2<0$.

Turning to the continuation of $F_{1,2}$ to $q^2=-p^2<0$, we first remark that, since $J_n(qr)\equiv J_n(-ipr)=i^{-n}I_n(pr)$ and $I_n(pr)\sim e^{pr}$ as $r\to\infty$, continuation to positive $p^2<m_\mu^{\,2}$ is a trivial matter. At $p^2=m_\mu^{\,2}$, one encounters the branch point of a cut (extending

to infinity) in the p^2 plane.

Let us assume $|q^2| = |p^2| < m_\mu{}^2$. Then, following the procedure detailed in our earlier paper,[8] we write ($j = 1$ or 2)

$$F_j(p^2) \equiv \frac{4\pi^2}{p^{j+1}} \mathrm{P}\int_0^\infty dr\, r^j I_{j+1}(pr) \frac{\Delta_1(r; m_\mu)\Delta_1(r; 0)}{1 - f_{e\mu} f_{\mu e}\Delta^2(r)}$$
$$= \frac{4\pi^2}{p^{j+1}}\left(\mathrm{P}\int_0^a + \int_a^\infty\right) dr\, r^j I_{j+1}(pr) \times \frac{\Delta_1(r; m_\mu)\Delta_1(r; 0)}{1 - f_{e\mu} f_{\mu e}\Delta^2(r)}$$
$$\equiv A_j(p^2) + B_j(p^2). \quad (3.12)$$

In Eq. (3.12), a is any length $> r_{e\mu}$. For $r \geq a > r_{e\mu}$, $|f_{e\mu} f_{\mu e}\Delta^2(r; m)| < 1$, so that the principal-value instruction is needed only for the integral $A_j(p^2)$ on the interval $r = 0$ to a. This integral, A_j, is an *entire* function of p^2, and is automatically continuable as it stands.

The analytic structure due to unitarity of the scattering amplitude $\langle \nu_\mu, e^- | \mathcal{T}^{(2)} | \nu_\mu, e^- \rangle$ resides in the integrals $B_{1,2}(p^2)$. The rather involved process of extending them to $p^2 > m_\mu{}^2$ is deferred to Appendix B. Here we merely quote the results, which are

$$B_1(p^2, |f_{e\mu}|^2) = \frac{1}{\pi}\int_{m_\mu{}^2}^\infty \frac{dk^2}{k^4(k^2 - p^2)} \times \left[\int_{m_\mu{}^2}^{k^2} \tfrac{1}{2}\kappa^2 d\kappa^2\, \sigma(\kappa^2; |f_{e\mu}|^2)\, d_a\!\left(\kappa^2; \frac{\kappa^2 p^2}{k^2}\right)\right] \quad (3.13)$$

and

$$B_2(p^2; |f_{e\mu}|^2) \equiv \frac{1}{p}\frac{d}{dp} B_1(p^2; |f_{e\mu}|^2)$$
$$= -\frac{4}{p^2} B_1(p^2; |f_{e\mu}|^2) + \frac{1}{\pi p^2}\int_{m_\mu{}^2}^\infty \frac{dk^2}{k^2 - p^2}\sigma(k^2; |f_{e\mu}|^2)\, d_a(k^2, p^2). \quad (3.14)$$

In Eqs. (3.13) and (3.14)

$$\sigma(k^2; |f_{e\mu}|^2) = \pi\sum_{n=0}^\infty |f_{e\mu}|^{2n}\hat{\Omega}_{2n+2}(k^2), \quad (3.15)$$

where $\hat{\Omega}_{2n+2}(k^2)$ is essentially a phase-space integral for a muon, an electron-neutrino, and $2n$ of the superpropagating φ particles, in which this collection of $2n+2$ particles has total center-of-mass energy $k = (k^2)^{1/2}$:

$$\hat{\Omega}_{2n+2}(k^2) = \frac{1}{(2\pi)^{3(2n+1)}}\int \frac{d^3q_1 d^3q_2}{4E_1E_2}\left(\prod_{i=1}^{2n}\frac{d^3p_i}{2\omega_i}\right) \times q_1\cdot q_2\, \delta^4\!\left(q_1 + q_2 + \sum_1^{2n} p_i - k\right). \quad (3.16)$$

The function

$$d_a(k^2, p^2) = ka[K_1(ka)I_0(pa) + (k/p)K_0(ka)I_1(pa)] \quad (3.17)$$

has already been discussed in detail in Ref. 8. While

$$\sigma(k^2; |f_{e\mu}|^2) \sim e^{kr_{e\mu}} \quad \text{as } k^2 \to \infty, \quad (3.18a)$$

nevertheless the integrals in Eqs. (3.13) and (3.14) exist, since

$$d_a(k^2, p^2) \sim e^{-ka} \quad (a > r_{e\mu}) \quad \text{as } k^2 \to \infty. \quad (3.18b)$$

Equations (3.12), (3.13), and (3.14) establish that the form factors $F_j(p^2)$ are the sum of an entire function of p^2, A_j, and of a function B_j which is analytic in the p^2-plane cut from $m_\mu{}^2$ to ∞. The absorptive parts of F_1 and F_2 are given by

$$\mathrm{Abs}F_1(p^2) = \mathrm{Abs}B_1(p^2) = \frac{1}{2p^4}\int_{m_\mu{}^2}^{p^2} d\kappa^2\kappa^2\sigma(\kappa^2; |f_{e\mu}|^2) \quad (3.19)$$

and

$$\mathrm{Abs}F_2(p^2) = \mathrm{Abs}B_2(p^2) = -\frac{2}{p^6}\int_{m_\mu{}^2}^{p^2} d\kappa^2\kappa^2\sigma(\kappa^2; |f_{e\mu}|^2) + \frac{1}{p^2}\sigma(p^2; |f_{e\mu}|^2), \quad (3.20)$$

where we have used the Wronskian relation $d_a(p^2, p^2) = 1$. It is readily verified that these are identical with the absorptive parts calculated from

$$\mathrm{Abs}\langle \nu_\mu(p_3)e^-(p_4)|\mathcal{T}^{(2)}|\nu_\mu(p_1)e^-(p_2)\rangle = \tfrac{1}{2}(2\pi)^4\sum_n |\langle \nu_\mu e^-|\mathfrak{L}_{\mu e} : \exp[f_{\mu e}\varphi^\dagger\varphi] : |n(p_n)\rangle|^2\delta^4(p_1 + p_2 - p_n). \quad (3.21)$$

Before proceeding, we make two comments on the meaning of the integrals A_j, entire functions of p^2, and of their magnitude relative to the dispersion integrals B_j. First, it is clear that A_j is essentially the sum of an infinite number of subtractions which are needed to make the integrals over $\sigma(k^2)$ converge. We have no real choice in how these subtractions are made (apart from the principal-value prescriptions); they are naturally determined as a result of the fact that $[1 - |f_{e\mu}|^2\Delta^2(r)]^{-1}$ is not expandable for $r \leq r_{e\mu}$.

The second remark is based on the observation that $A_1 \sim r_{e\mu}^{-2}$ and $A_2 \sim \ln r_{e\mu}$, whereas B_1 and B_2 remain finite as $r_{e\mu} \to 0$. As we remarked in Sec. II, and as will be further discussed in our following paper, we expect $r_{e\mu}^{-1} \lesssim 10$ GeV, so that for energies and masses $\lesssim r_{e\mu}^{-1}$, we should have $|A_j| \gtrsim |B_j|$. It is easy to see from Eq. (3.12) that A_j is independent of the relevant energy variables whenever they are considerably less than the cutoff length $r_{e\mu}$. Further, it is difficult to escape the conclusion that the same is true for the corresponding subtraction integrals arising from hadronic contributions to lepton-lepton scattering (if only we could calculate them). Thus, while it should be quite permissible (at low energies) to ignore dispersion integrals over the hadronic continuum, there may be no real justification for assuming A_j(hadron) $\ll A_j$(lepton). This feature is, of course, to be expected in any theory with a reasonably large cutoff, whether put in by hand or built in automatically as in NLFT. It is just a consequence of the fact that the short-distance (or high-energy) behavior of quantum field theories is essentially mass-independent. The only conditions we can think of under which the hadronic subtraction integrals would be negligible are if (a) matrix elements of hadron currents, such as $\langle 0|T(j_\lambda^{(a)\dagger}(x) j_\mu^{(a)}(0))|0\rangle$, are less singular on the light cone than the corresponding ones involving lepton currents, or (b) if the relevant hadronic masses are very large – for example, quark masses $\gtrsim 10$ GeV.

We now turn to a brief discussion of the electron self-energy in $O(G^2)$. It is beyond the scope of this paper to give a detailed account of the renormalization of the electron propagator, since the presence of parity violation is an essential complication in the renormalization procedure.[19] We shall therefore be content to show that the effect of the superpropagator is to reduce a quartic divergence to a logarithmic one, and to separate the self-energy into finite and infinite parts. In a proper renormalization scheme, the infinities are to be absorbed by mass counterterms and by wave-function and coupling-constant renormalizations.

For simplicity, we consider the contribution to the electron self-energy, Σ, due to electrons and their neutrinos only (see Fig. 2). To $O(G^2)$, Σ is to be calculated from the equation

$$\bar{u}_e(p)\Sigma(p)u_e(p) = (\tfrac{1}{2}i)\,\mathrm{P}\int d^4x \frac{\langle e^-(p)|T\{\mathcal{L}_{ee}(x)\mathcal{L}_{ee}(0)\}|e^-(p)\rangle}{1-f^2\Delta^2(x;M)}, \tag{3.22}$$

where we have put $f = f_{ee}$. Proceeding exactly as was done above for ν_μ-e scattering, we find that Σ may be written in the form

$$\Sigma(p) = \not{p}\,\sigma(p^2)(1-\gamma_5), \tag{3.23}$$

where σ is formally given by

$$\sigma(p^2) = \frac{128\pi^2 G^2}{p^2}\,\mathrm{P}\int_0^\infty dr\, r^2 I_2(pr)\frac{[\Delta_1(r;0)]^2\Delta_1(r;m_e)}{1-f^2\Delta^2(r;M)}. \tag{3.24}$$

Our claim that the superpropagator $[1-f^2\Delta^2]^{-1}$ reduces the quartic divergence to be expected in the usual current×current theory to a logarithmic divergence is at once apparent from Eq. (3.24). While this equation is valid only for $p^2 \le m_e^2$, it may be continued to $p^2 > m_e^2$ just as was done for the integrals in Eq. (3.12). In any event, the logarithmic divergence resides in the "subtraction integrals" over the interval $r=0$ to $r=a>r_e$ [defined by $f_{ee}^2\Delta^2(r_e;M)=1$].

We follow the conventional procedure[20] of separating Σ into finite and infinite parts by expanding $\sigma(p^2)$ in powers of $(\not{p}-m_e)$. Thus we first write

$$\not{p}\,\sigma(p^2) = \not{p}\,\sigma(m_e^2) + \not{p}[\sigma(p^2)-\sigma(m_e^2)]. \tag{3.25}$$

The second term in Eq. (3.25) is finite, and it is given (for $p^2 \le m_e^2$) by

$$\sigma(p^2)-\sigma(m_e^2) \equiv (p^2-m_e^2)A(p^2) = 8\pi^2G^2(p^2-m_e^2)\,\mathrm{P}\int_0^\infty dr\,\frac{r^6[\Delta_1(r;0)]^2\Delta_1(r;m_e)}{1-f^2\Delta^2(r;M)}\sum_{j,k=0}^\infty \frac{(\frac{1}{2}pr)^{2j}(\frac{1}{2}m_e r)^{2k}}{(j+k+1)!\,(j+k+3)!}. \tag{3.26}$$

Next we write

$$A(p^2) = A(m_e^2) + [A(p^2)-A(m_e^2)] \equiv A(m_e^2) + (\not{p}-m_e)B(p) \tag{3.27}$$

so that, using $p^2 - m_e^2 = 2m_e(\not{p}-m_e) + (\not{p}-m_e)^2$, we finally get the self-energy in the form

$$\Sigma(p) = \not{p}\,\sigma(m_e^2)(1-\gamma_5) + (\not{p}-m_e)[2m_e^2A(m_e^2)](1-\gamma_5) + (\not{p}-m_e)^2[(\not{p}+2m_e)A(p^2)+2m_eB(p)](1-\gamma_5). \tag{3.28}$$

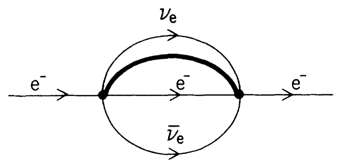

FIG. 2. Contribution of electron and two-neutrino intermediate state to the electron propagator. In addition, there are muonic and hadronic contributions in second order, which are neglected for simplicity. Heavy line indicates the superpropagator.

Part of the first, logarithmically divergent term on the right in Eq. (3.28) is cancelled by self-mass and kinetic-energy counterterms[19] in the "free" Lagrangian; the remainder is combined with the second (finite) term into wave-function and coupling-constant renormalization. The third term is the observable second-order weak correction to the electron propagator. A glance at Eq. (3.26) shows that the essential dependence of A and B on the length r_e is given by

$$A(p^2)=G^2\alpha(p^2)/r_e^{\,2}, \tag{3.29a}$$

$$B(p)=G^2\beta(p)\ln r_e\,. \tag{3.29b}$$

Before closing this section, a few remarks are in order on the possible consequences of assuming the minor coupling constants f_e and f_μ to be different in both magnitude and phase. The first of these is obviously a breakdown of μ-e universality. This difference between f_e and f_μ could be observed, in principle, in first-order weak φ production. More striking effects might be observed in second-order processes, however, if a special situation existed; for example, if $|f_{ee}|\ll|f_{\mu\mu}|$ or if $f_e\approx -f_\mu$, so that $|f_{e\mu}|\ll|f_{ee}|$ and $|f_{\mu\mu}|$, then certain amplitudes might be anomalously large because of the smallness of the relevant cutoff length.

Another interesting possibility is that of time-reversal noninvariance if $f_{e\mu}$ is not real. As mentioned in Sec. II, we expect violation of CP or T invariance in $O(G^2)$ whenever the pertinent coupling constants $f_{ab}f_{cd}$ are complex. In purely leptonic processes, this situation may arise in the second-order contributions to μ decay, $\nu_\mu+e^- \to \mu^-+\nu_e$, and so on. Defining cutoff lengths r_1 and r_2 by

$$\begin{aligned}&|f_{e\mu}f_{ee}\Delta^2(r_1\,;M)|=1,\\&|f_{e\mu}f_{\mu\mu}\Delta^2(r_2\,;M)|=1,\end{aligned} \tag{3.30}$$

the amplitudes for these processes would contain T-violating terms which are down from $O(G)$ by factors of $G/r_1^{\,2}$ and $G/r_2^{\,2}$. The techniques we have outlined in this section are easily extended to include the case in which the minor coupling constants are complex, so that calculation of the form of distributions characteristic of time-reversal noninvariance is a straightforward matter.

Unfortunately, at present we can take very little guidance from experiment as to the magnitudes and phase of f_e and f_μ. In the following paper on hadronic weak processes, it is shown that a consistent scheme of CP violation in the neutral K-meson system can be achieved by requiring that the minor coupling constants satisfy the following conditions [remember that $f_{ab}=\frac{1}{2}(f_a+f_b^*)=f_{ba}^*$]:

$$\begin{aligned}&f_{01}=f_{10}\,;\quad f_{0l}f_{1l}\neq f_{l0}f_{1l}\quad\text{for } l=e \text{ and/or } \mu;\\&|f_{0l}f_{l1}/16\pi^4|^{-1/4}\lesssim 10\text{ GeV}.\end{aligned} \tag{3.31}$$

These conditions will place rather broad restrictions on the magnitudes of f_e and f_μ, but say nothing about their relative phases.

IV. GENERAL REMARKS ON Nth-ORDER PERTURBATION THEORY

In the preceding section, we discussed the simplest nontrivial alteration of weak-interaction amplitudes, namely, that which is introduced by the basic superpropagator

$$\langle 0|T\{:\exp(f_1\varphi_1^\dagger\varphi_1)::\exp(f_2\varphi_2^\dagger\varphi_2):\}|0\rangle = [1-f_1f_2\Delta_{12}^{\,2}]^{-1}, \tag{4.1}$$

where $\varphi_i=\varphi(x_i)$, $\Delta_{12}=\Delta(x_1-x_2\,;M)$. We showed that lepton-lepton scattering was finite in $O(G^2)$, while lepton self-masses were logarithmically divergent. In this section, we shall generalize these considerations by writing down the superpropagators corresponding to emission and absorption of φ particles in $O(G^N)$, paying special attention to the divergence problem in the weak interactions of leptons.

Let us consider the weak process

$$A(l,\varphi)=A(l)+A(\varphi)\to B(l)+B(\varphi)=B(l,\varphi), \tag{4.2}$$

where $A(l)$ and $B(l)$ are the initial and final collections of leptons, a total of L leptons all together; $A(\varphi)$ and $B(\varphi)$ are the initial and final collections of φ particles, a total of $2m$ in all. The $O(G^N)$ contribution to the amplitude describing the process (4.2) is given by[21]

$$(2\pi)^4 i\,\delta(p_A-p_B)\langle B(l,\varphi)|\mathcal{T}^{(N)}|A(l,\varphi)\rangle = \frac{i^N}{N!}\int d^4x_1\cdots d^4x_N\left\langle B(l)\left|T\left[\prod_1^N \mathcal{L}_{a_ib_i}(x_i)\right]\right|A(l)\right\rangle\left\langle B(\varphi)\left|T\left[\prod_1^N :\exp(f_{a_ib_i}\varphi_i^\dagger\varphi_i):\right]\right|A(\varphi)\right\rangle. \tag{4.3}$$

The factor involving the φ particles in Eq. (4.3) is a sum over the functions $F^{(N)}_{(l_i m_i)}$ [for which $\sum_1^{(N)}(l_i+m_i)=2m$] defined in Appendix A by the equation[22]

$$T\left[\prod_1^N :\exp(f_i\varphi_i^\dagger\varphi_i):\right] \equiv \sum_{l_i,m_i=0}^{\infty} \delta_{\Sigma l_i,\Sigma m_i} F^{(N)}_{(l_i m_i)}(z_{ij}) : \prod_{k=1}^{N} \frac{\varphi_k{}^{l_k}\varphi_k^{\dagger m_k}}{l_k!\,m_k!} : . \tag{4.4}$$

Here, we have put $f_{a_i b_i}=f_i$ and introduced the (dimensionless) variables upon which $F^{(N)}_{(l_i m_i)}$ depends:

$$z_{ij}=z_{ji}=(f_i f_j)^{1/2}\Delta_{ij};\;\; z_{ii}\equiv 0 \quad (i,j=1,\ldots,N). \tag{4.5}$$

In Appendix A, the dependence on the z_{ij} of the Green's functions $F^{(N)}_{(l_i m_i)}$ is found to be given by an $(N-1)^2$-fold finite sum:

$$F^{(N)}_{(l_i m_i)}(z_{ij})=[\det(1-Z^{(N)})]^{-(\Sigma_1^N l_i+1)}\prod_{k=1}^{N}[l_k!\,m_k!\,f_k^{(m_k+l_k)/2}]\sum_{n_{ij}\geq 0}\left[\prod_{s=1}^{N}\delta_{l_s,\Sigma_k n_{ks}}\delta_{m_s,\Sigma_k n_{sk}}\right]\prod_{i,j=1}^{N}\frac{[\zeta_{ij}^{(N)}]^{n_{ij}}}{n_{ij}!}\,. \tag{4.6}$$

Here, $Z^{(N)}$ is the $N\times N$ symmetric matrix of the z_{ij}, and $\zeta_{ij}^{(N)}=\zeta_{ji}^{(N)}$ is the cofactor of $(1-Z^{(N)})_{ij}=\delta_{ij}-z_{ij}$. Two special cases of interest are (i) the Nth-order vacuum superpropagator $(l_i,m_i=0)$

$$\begin{aligned}F^{(N)}(z_{ij}) &\equiv \langle 0|T\left[\prod_1^N :\exp(f_i\varphi_i^\dagger\varphi_i):\right]|0\rangle \\ &= [\det(1-Z^{(N)})]^{-1}\end{aligned} \tag{4.7}$$

and (ii) the case $N=2$:

$$F^{(2)}_{l_1m_1;l_2m_2}=\frac{l_1!\,m_1!\,l_2!\,m_2!}{(1-z_{12}{}^2)^{l_1+l_2+1}}\,f_1{}^{(l_1+m_1)/2}f_2{}^{(l_2+m_2)/2}z_{12}{}^{l_1-m_1}\sum_{n=0}^{\min(m_1,l_2)} z_{12}{}^{2n}[n!\,(m_1-n)!\,(l_2-n)!\,(n+l_1-m_1)!]^{-1}$$

$$(\text{for } l_1-m_1\geq 0). \tag{4.8}$$

If $l_1-m_1<0$, simply use $F^{(2)}_{l_1m_1;l_2m_2}=F^{(2)}_{m_1l_1;m_2l_2}$ and interchange l_i and m_i in Eq. (4.8). The functions $F^{(2)}_{(l_i m_i)}$ are proportional to the hypergeometric functions[23]

$${}_2F_1(l_1+1,\, m_2+1,\, l_1-m_1+1; z_{12}{}^2).$$

The various factors in Eq. (4.6) have a simple physical interpretation. For example, each term in $\det(1-Z^{(N)})$ corresponds to one of the $N!$ ways of constructing a closed loop of propagating φ particles among any subset of the N space-time points x_i. For large distances $r_{ij}{}^2=-(x_i-x_j)^2>0$, $[\det(1-Z^{(N)})]^{-1}$ is expandable in a multiple power series in the z_{ij}, each term of which represents the circulation of an arbitrary number of φ particles around any collection of these closed loops, subject only to charge conservation. For short distances, this series is replaced by the continuation $[\det(1-Z^{(N)})]^{-1}$, which provides certain convergence properties on the "light cones" $r_{ij}=0$. The appearance of the cofactors $\zeta_{ij}^{(N)}$ whenever there are external φ particles at the points x_i and x_j corresponds to the ways of propagating particles from x_i to x_j; the Kronecker δ's in Eq. (4.6) ensure that the $\zeta_{ij}^{(N)}$ appear in a manner consistent with charge conservation.

There are two important questions to be answered before the formulas in Eqs. (4.3) and (4.6) can be used to compute higher-order weak amplitudes for leptonic processes. First, does a simple generalization of the principal-value prescription used in $O(G^2)$ suffice to handle the singularities of $[\det(1-Z^{(N)})]^{-1}$ encountered in the integrals of Eq. (4.3)? Moreover, can such a prescription be shown to respect unitarity in the sense that the absorptive part of a given amplitude can be expressed as products of appropriate lower-order amplitudes? The second question is whether or not we have actually constructed a renormalizable theory of weak interactions. Both these problems are rather involved and are now being investigated by us.

While we have no definite answers at this writing, one affirmative indication that the theory might be renormalizable is easily established. Let us define the superficial degree of divergence $D(N;L,2m)$ of an Nth-order diagram involving L external leptons and $2m$ external φ particles as follows: Such a diagram is represented by the amplitude in Eq. (4.3); $D(N;L,2m)$ is defined to be the power of $|x|^{-1}$ in the integrand of this amplitude when all $x_{ij}{}^2$ are assumed small.[24] We recall that lepton propagators $\sim x^{-3}$ as $x^2\to 0$; furthermore, inspection of Eq. (4.6) shows that

$$F^{(N)}_{(l_i m_i)}\sim[\Delta^{-N}]^{\Sigma l_i+1}[\Delta^{N-1}]^{\Sigma l_i}=\Delta^{-N-\Sigma l_i}\sim x^{2N+2m}$$

as $\Delta^{-1}\sim x^2\to 0$. Then

$$\begin{aligned}D(N;L,2m)&=4(1-N)+\tfrac{3}{2}(4N-L)-(2N+2m)\\&=4-\tfrac{3}{2}L-2m,\end{aligned} \tag{4.9}$$

where $4(1-N)$ comes from $N-1$ integrations, $3[\frac{1}{2}(4N-L)]$ from $\frac{1}{2}(4N-L)$ lepton propagators; and $-(2N+2m)$ from the superpropagator. The obvious point is that D is *independent* of the order N. Theories in which D increases with N are nonrenormalizable – a prime example being the usual current×current theory of weak interactions, for which $D(N;L)=4+2N-\frac{3}{2}L$. Thus the nonpolynomial modification we have introduced is just enough to remove this factor of $2N$. This, of course, is no proof of renormalizability, but it is a hopeful first sign.

Finally, we point out that Eq. (4.9) enables us to isolate the basic divergences in our theory. These occur whenever D is positive, with the value of D denoting the type of divergence. (Of course, in Nth order, a given graph may be more or less divergent than is superficially indicated by D, depending on the value of D for various subgraphs and on such technicial matters as symmetric integration.) These basic divergences – excluding vacuum graphs – are listed below; the corresponding $O(G^2)$ graphs are shown in Fig. 3.

$L=0$, $2m=2$: $D=2$ (φ self-mass, quadratically divergent);

$L=2$, $2m=0$: $D=1$ (lepton self-mass, linearly divergent – actually only logarithmically divergent); (4.10)

$L=0$, $2m=4$: $D=0$ (φ-φ scattering, logarithmically divergent).

This concludes the main body of this paper. To summarize, we have modified the weak current ×current Lagrangian with a nonpolynomial field function, $\exp[f\varphi^\dagger\varphi]$. In this paper we have concentrated on the divergence problem in leptonic weak interactions, finding that lepton-lepton scattering is finite, while the lepton self-energy is only logarithmically divergent in $O(G^2)$. On the basis of power-counting arguments, we expect this feature to persist to all orders in the major coupling constant G. Along the way, we have developed a lore for estimating the size of higher-order weak amplitudes based on the short-distance behavior of products of leptonic currents. This lore will be useful to us in paper II, where we discuss the mechanism of CP-invariance violation.

ACKNOWLEDGMENTS

We wish to thank Professor Henry Primakoff and Professor Michael Whippman for many useful and stimulating conversations during the course of this work. One of us (K. L.) wishes to acknowledge the hospitality of the Aspen Center for Theoretical Physics, where part of this work was carried out.

APPENDIX A: THE GENERAL Nth-ORDER SUPERPROPAGATOR

In order G^N, an amplitude describing a process with a given number of external leptons and hadrons and any (even) number of external φ particles is constructed just as in the usual Fermi theory, except that the matrix element of time-ordered currents is multiplied by certain combinations of Green's functions (superpropagators) $F^{(N)}_{l_1m_1,\ldots,l_Nm_N}$, defined as follows. Let $\varphi_k=\varphi(x_k)$, $k=1,\ldots,N$, and denote by $\mathfrak{F}^{(N)}$ the time-ordered product

$$\mathfrak{F}^{(N)} \equiv T\{:\exp(f_1\varphi_1^\dagger\varphi_1):\cdots:\exp(f_N\varphi_N^\dagger\varphi_N):\}$$
$$= \sum_{l_i,m_i=0}^{\infty}\delta_{\Sigma l_i,\Sigma m_i}F^{(N)}_{(l_im_i)}:\prod_{k=1}^{N}\frac{\varphi_k^{l_k}\varphi_k^{\dagger m_k}}{l_k!\,m_k!}:, \tag{A1}$$

in which the Kronecker $\delta_{\Sigma l_i,\Sigma m_i}$ ensures conservation of charge. We shall calculate $\mathfrak{F}^{(N)}$ and identify $F^{(N)}_{(l_im_i)}$ from Eq. (A1).

As in all theories involving nonpolynomial interactions, the calculation of $\mathfrak{F}^{(N)}$ is most easily carried out using Hori's expression[25] of Wick's theorem; in our case,

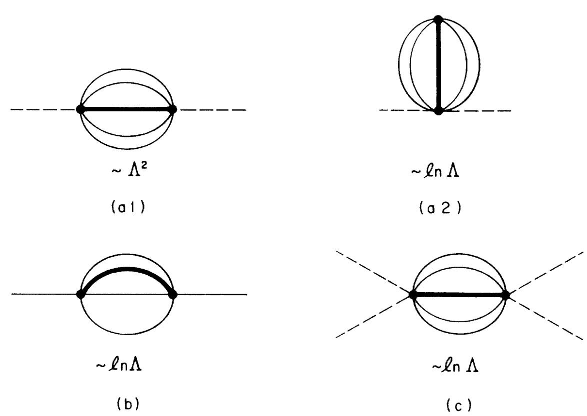

FIG. 3. Second-order divergent diagrams: (a) φ self-mass, (b) lepton self-mass, (c) φ-φ scattering. There are additional diagrams which contribute to φ-φ scattering, but they are finite. The degree of divergence as a function of a cutoff mass Λ is shown in each case. Heavy line indicates the superpropagator. Dotted lines are external φ particles. Solid lines are charged leptons or neutrinos.

$$\mathfrak{F}^{(N)} = :\exp\left[\tfrac{1}{2}\sum_{i\neq j}^{N}\sum_{b=1}^{2}\Delta_{ij}\frac{\partial^2}{\partial\varphi_i^{(b)}\partial\varphi_j^{(b)}}\right] \times\exp\left[\tfrac{1}{2}\sum_{k=1}^{N}\sum_{c=1}^{2}f_k\varphi_k^{(c)}\varphi_k^{(c)}\right]:. \tag{A2}$$

In Eq. (A2) we have introduced the Hermitian fields

$$\varphi^{(1)} = 2^{-1/2}(\varphi^\dagger + \varphi),\quad \varphi^{(2)} = i2^{-1/2}(\varphi^\dagger - \varphi); \tag{A3}$$

also,

$$\Delta_{ij} \equiv i\Delta_F(x_i - x_j; M) = \langle 0|T\{\varphi_i^\dagger\varphi_j\}|0\rangle = \langle 0|T\{\varphi_i^{(b)}\varphi_j^{(b)}\}|0\rangle \tag{A4a}$$

is the usual Feynman propagator. According to the Euclidicity postulate, we assume the coordinates x_i to lie in the Euclidean region, so that $r_{ij} = r_{ji} = [-(x_i - x_j)^2]^{1/2} > 0$, and

$$\Delta_{ij} = \Delta_{ji} = MK_1(Mr_{ij})/4\pi^2 r_{ij} \tag{A4b}$$

is a positive, decreasing function of r_{ij}.

To compute $\mathfrak{F}^{(N)}$ from Eq. (A2), we first replace $\varphi_i^{(b)}$ and Δ_{ij}, respectively, by the dimensionless quantities $f_i^{1/2}\varphi_i^{(b)}$ and

$$z_{ij} = z_{ji} = (f_i f_j)^{1/2}\Delta_{ij};\quad z_{ii} \equiv 0 \quad (i,j = 1,\ldots,N). \tag{A5}$$

For now, we shall assume the f_i to be *real* and *positive* (we can always continue in them later on). Then, Eq. (A4b) implies that the $N\times N$ matrix $Z^{(N)} = (z_{ij})$ is a real, symmetric matrix, and may be diagonalized by an orthogonal transformation, A (Ref. 26). Thus, with tilde denoting transpose,

$$Z^{(N)} = \tilde{A}\,\Lambda^{(N)}A, \tag{A6}$$

where $\Lambda^{(N)}$ is the diagonal matrix whose elements are the real eigenvalues $\lambda_1, \ldots, \lambda_N$ of $Z^{(N)}$.

Letting

$$\rho_k^{(b)} = \sum_{i=1}^{N} A_{ki}[f_i^{1/2}\varphi_i^{(b)}],$$

we obtain $\mathfrak{F}^{(N)}$ in the factorized form

$$\mathfrak{F}^{(N)} = :\prod_{k=1}^{N}\prod_{b=1}^{2}\exp\left[\tfrac{1}{2}\lambda_k\left(\frac{d}{d\rho_k^{(b)}}\right)^2\right]\exp[\tfrac{1}{2}(\rho_k^{(b)})^2]:. \tag{A7}$$

We now use the formula

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\sigma\,\exp[-\alpha^2\sigma^2 \pm \beta\sigma] = (2\alpha^2)^{-1/2}\exp(\beta^2/4\alpha^2),\quad \alpha > 0 \tag{A8}$$

to write

$$\exp(\tfrac{1}{2}\rho^2) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} d\sigma\,\exp[\sigma\rho - \tfrac{1}{2}\sigma^2] \tag{A9a}$$

and the operator equation

$$\exp\left[\tfrac{1}{2}\lambda\left(\frac{d}{d\rho}\right)^2\right] = (2\pi)^{-1/2}\int_{-\infty}^{\infty} d\tau\,\exp\left[\lambda^{1/2}\tau\frac{d}{d\rho} - \tfrac{1}{2}\tau^2\right]. \tag{A9b}$$

The application of the right-hand side of Eq. (A9b) to that of Eq. (A9a) translates the latter according to $\rho \to \rho + \lambda^{1/2}\tau$, with the result

$$\exp\left[\tfrac{1}{2}\lambda\left(\frac{d}{d\rho}\right)^2\right]\exp(\tfrac{1}{2}\rho^2) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} d\sigma\,\exp[\sigma\rho - \tfrac{1}{2}\sigma^2(1-\lambda)] = (1-\lambda)^{-1/2}\exp[\rho^2/2(1-\lambda)],\quad 1-\lambda > 0. \tag{A10}$$

The conditions $1-\lambda_k > 0$ ($k = 1, \ldots, N$) (i.e., that $1 - Z^{(N)}$ be a positive definite matrix), under which the integral in Eq. (A10) converges to the quoted result, are just those conditions under which the Wick series obtained from Eq. (A2) converges. Analytic continuation provides the definition of superpropagators outside this region.

From Eqs. (A7) and (A10) we get for $\mathfrak{F}^{(N)}$

$$\mathfrak{F}^{(N)} = [\det(1 - Z^{(N)})]^{-1} \times :\exp\left[\tfrac{1}{2}\sum_{k=1}^{N}\sum_{b=1}^{2}(\rho_k^{(b)})^2/(1-\lambda_k)\right]:. \tag{A11}$$

It now remains only to undo the transformation A and express everything in terms of the fields φ_i, $\varphi_i^\dagger$. This is straightforward, and we find for the Green's functions the following $(N-1)^2$-fold *finite* sum:

$$F_{(l_i m_i)}^{(N)}(z_{ij}) = [\det(1 - Z^{(N)})]^{-(\Sigma_1^N l_i + 1)}\prod_{k=1}^{N}[l_k!\,m_k!\,f_k^{(m_k+l_k)/2}]\sum_{n_{ij}\geq 0}\left[\prod_{s=1}^{N}\delta_{l_s,\Sigma_k n_{ks}}\delta_{m_s,\Sigma_k n_{sk}}\right]\prod_{i,j=1}^{N}\frac{[\zeta_{ij}^{(N)}]^{n_{ij}}}{n_{ij}!}, \tag{A12}$$

where $\zeta_{ij}^{(N)} = \zeta_{ji}^{(N)}$ is the cofactor of $(1 - Z^{(N)})_{ij}$. At this point, one may continue to complex values of the minor coupling constants f_i if desired.

We now turn our attention to the momentum-space behavior of the superpropagator. For the case of a massless φ particle, we can calculate the Fourier transform of the basic second-order function,

$$F(x) \equiv F^{(2)}[(f_1 f_2)^{1/2}\Delta(x)] = [1 - f_1 f_2 \Delta^2(x)]^{-1}, \tag{A13}$$

in closed form. Let us suppose that $f^2 = f_1 f_2 > 0$. As noted in Ref. 8, the definition of the Fourier transform of F is ambiguous in this case because of a singularity at $f^2\Delta^2 = 1$. For example, we can follow Eq. (2.5) of Ref. 8 by defining (for $p^2 = -q^2 < 0$)

$$\tilde{F}^{(\pm)}(p^2) \equiv i\int d^4x\, e^{ip\cdot x} F(f\Delta \pm i\epsilon)$$
$$= (2\pi)^4 i\delta^{(4)}(p) + \frac{4\pi^2}{q}\int_0^\infty dr\, r^2 J_1(qr)\, f^2\Delta^2(r)[1 - f^2\Delta^2(r) \mp i\epsilon]^{-1}. \tag{A14}$$

Then a class of Fourier transforms can be defined by writing, for any real constant b,

$$\tilde{F}(p^2; b) \equiv \tfrac{1}{2}[(1-ib)\tilde{F}^{(+)}(p^2) + (1+ib)\tilde{F}^{(-)}(p^2)]$$
$$= \frac{4\pi^2}{q}\int_0^\infty dr\, r^2 J_1(qr)\, f^2\Delta^2(r)\left[\frac{\mathrm{P}}{1 - f^2\Delta^2(r)} + \pi b\,\delta(f^2\Delta^2 - 1)\right], \tag{A15}$$

where we have ignored the contact term $(2\pi)^4 i\delta^{(4)}(p)$. We shall see that the "best" superpropagator of this class is obtained for $b = 0$, i.e., by defining the Fourier transform as the principal-value integral $\tilde{F}(p^2; 0)$ whenever $f^2 = f_1 f_2$ is positive.

Assuming $M = 0$, so that $\Delta(r) = (4\pi^2 r^2)^{-1}$, and defining the length $r_0 > 0$ by

$$r_0^2 = |f/4\pi^2|,$$

we get for Eq. (A15)

$$\tilde{F}(p^2; b) = \frac{\pi^3 b}{q} r_0^3 J_1(qr_0) + \tilde{F}(p^2; 0). \tag{A16}$$

The integral $\tilde{F}(p^2; 0)$ is treated in the following way: Write

$$\tilde{F}(p^2; 0) = \frac{2\pi^2 r_0^2}{q}\left[\mathrm{P}\int_0^\infty dr\, \frac{r^2 J_1(qr)}{r^2 - r_0^2} - \int_0^\infty dr\, \frac{r^2 J_1(qr)}{r^2 + r_0^2}\right]. \tag{A17}$$

Both integrals in Eq. (A17) may be calculated using the formula [27]

$$\int_0^\infty dr\, \frac{r^2 J_1(qr)}{r^2 + a^2} = aK_1(qa), \quad \mathrm{Re}\, a > 0.$$

With $\epsilon \to 0^+$ understood, we have

$$\mathrm{P}\int_0^\infty dr\, \frac{r^2 J_1(qr)}{r^2 - r_0^2} = \tfrac{1}{2}\{(\epsilon - ir_0)K_1(q(\epsilon - ir_0)) + (\epsilon + ir_0)K_1(q(\epsilon + ir_0))\}$$
$$= -\tfrac{1}{2}(\pi r_0)\, Y_1(qr_0), \tag{A18}$$

where we have used

$$K_1(e^{\pm i\pi/2}z) = -\frac{\pi}{2}H_1^{(2,1)}(z)$$
$$= -\frac{\pi}{2}[J_1(z) \mp iY_1(z)].$$

Thus, for $q^2 = -p^2 < 0$, we have

$$\tilde{F}(p^2; b) = (2\pi)^4 i\,\delta^{(4)}(p) + \frac{b\pi^3 r_0^3}{q} J_1(qr_0) - \frac{\pi^3 r_0^3}{q}\left[Y_1(qr_0) + \frac{2}{\pi}K_1(qr_0)\right]. \tag{A19}$$

As is required by unitarity, $\tilde{F}$ is real below the threshold at $p^2 = 0$, and $\tilde{F}$ dies as $Y_1(qr_0)/q$ as $p^2 \to -\infty$. The continuation into the timelike region is obtained by putting $q = -ip$:

$$\tilde{F}(p^2; b) = (2\pi)^4 i\,\delta^{(4)}(p) + i\frac{\pi^3 r_0^3}{p}[I_1(pr_0) + J_1(pr_0)] - \frac{\pi^3 r_0^3}{p}\left[Y_1(pr_0) + \frac{2}{\pi}K_1(pr_0) - bI_1(pr_0)\right]. \tag{A20}$$

The absorptive part of $\tilde{F}$ grows as e^{pr_0} as $p^2 \to +\infty$. It is also clear that the dispersive part will grow as e^{pr_0} *unless* we take $b = 0$, in which case it simply behaves as

$$\frac{1}{p}Y_1(pr_0) \sim p^{-3/2}\sin\left(pr_0 - \frac{3\pi}{4}\right) \text{ as } p^2 \to +\infty. \tag{A21}$$

Thus, from the point of view of growth of the superpropagator $\tilde{F}(p^2)$, the best prescription seems to be to take $b = 0$.

APPENDIX B: ANALYTIC CONTINUATION OF LEPTON-LEPTON SCATTERING AMPLITUDES

In the calculation of lepton-lepton scattering amplitudes in second order in G, it is necessary to continue integrals of the type

$$B_j(p^2)=\frac{4\pi^2}{p^{j+1}}\int_a^\infty dr\, r^j I_{j+1}(pr)\frac{\Delta_1(r;m)\Delta_1(r;0)}{1-f^2\Delta^2(r;M)}\ ,\quad j=1,2 \tag{B1}$$

from the Euclidean region, $p^2<0$, to $p^2>0$. In Eq. (B1) we may assume f^2 to be complex. The length a is chosen by

$$a>r_0\quad (|f^2\Delta^2(r_0;M)|=1),$$

so that, in the region $a\leqslant r\leqslant\infty$, $|f^2\Delta(r;M)|<1$ and we can write

$$B_j(p^2)=\sum_{n=0}^\infty f^{2n}\frac{4\pi^2}{p^{j+1}}\int_0^\infty dr\, r^j I_{j+1}(pr)\Delta_1(r;m)\Delta_1(r;0)[\Delta(r;M)]^{2n}. \tag{B2}$$

The next step is to represent the product $\Delta_1(r;m)\Delta_1(r;0)[\Delta(r;M)]^{2n}$ of Feynman propagators as an integral over a single propagator function. For this, we use the representations

$$\Delta(x;m)=\int\frac{d^3q}{(2\pi)^3 2E}[\theta(x_0)e^{-iq\cdot x}+\theta(-x_0)e^{iq\cdot x}] \tag{B3a}$$

and

$$\begin{aligned} i\gamma\cdot\partial\Delta(x;m)&=\int\frac{d^3q}{(2\pi)^3 2E}\gamma\cdot q[\theta(x_0)e^{-iq\cdot x}-\theta(-x_0)e^{iq\cdot x}]\\ &=-i\gamma\cdot x\frac{1}{r}\Delta_1(r;m)\ \text{ when } r^2=-x^2>0. \end{aligned} \tag{B3b}$$

From Eqs. (B3) we find

$$\begin{aligned}\Delta_1(r;m)\Delta_1(r;0)[\Delta(r;M)]^{2n}&=\tfrac{1}{4}\,\mathrm{Tr}\{[i\gamma\cdot\partial\Delta(x;m)][i\gamma\cdot\partial\Delta(x;0)]\}[\Delta(r;M)]^{2n}\\ &=\int_{(m+2nM)^2}^\infty d\kappa^2\int\frac{d^4k}{(2\pi)^3}\int\frac{d^3q_1d^3q_2}{(2\pi)^3 4E_1E_2}\prod_{i=1}^{2n}\frac{d^3p_i}{(2\pi)^3 2\omega_i}\,q_1\cdot q_2\\ &\qquad\times[\theta(x_0)e^{-ik\cdot x}+\theta(-x_0)e^{ik\cdot x}]\,\delta^{(4)}\Big(q_1+q_2+\sum_1^{2n}p_i-k\Big)\delta(k^2-\kappa^2)\\ &=\int_{(m+2nM)^2}^\infty d\kappa^2\,\hat\Omega_{2n+2}(\kappa^2)\Delta(r;\kappa),\end{aligned} \tag{B4}$$

where $\hat\Omega_{2n+2}$ is given by Eq. (3.16), and

$$\Delta(r;\kappa)=\kappa K_1(\kappa r)/4\pi^2 r. \tag{B5}$$

We now have $B_j(p^2)$ in the form

$$B_j(p^2)=\frac{1}{\pi}\int_{m^2}^\infty d\kappa^2\sigma(\kappa^2;f^2)\,b_a^{(j)}(\kappa^2,p^2), \tag{B6}$$

where

$$\sigma(\kappa^2;f^2)=\pi\sum_{n=0}^\infty f^{2n}\hat\Omega_{2n+2}(\kappa^2), \tag{B7}$$

and

$$b_a^{(j)}(\kappa^2,p^2)=\frac{4\pi^2}{p^{j+1}}\int_a^\infty dr\, r^j I_{j+1}(pr)\Delta(r;\kappa). \tag{B8}$$

Since

$$B_2(p^2)=\frac{1}{p}\frac{d}{dp}B_1(p^2),$$

it suffices to calculate $b_a^{(1)}(\kappa^2,p^2)$. We assume at first that $p^2<\kappa^2\leqslant m^2$ in Eq. (B8). Then, using Eq. (B5) and the series expansion for $I_2(pr)$, we have

$$b_a^{(1)}(\kappa^2,p^2)=\frac{1}{4\kappa^2}\sum_{n=0}^\infty\frac{(p/2\kappa)^{2n}}{n!(n+2)!}\int_{\kappa a}^\infty dx\,x^{2n+2}K_1(x).$$

Using (see p. 87 of Ref. 18)

$$\int_{\kappa a}^{\infty} dx\, x^{2n+2}K_1(x) = 2^{2n+1}n!\,(n+1)!\,\kappa a\left[K_1(\kappa a)\sum_{m=0}^{n}\frac{(\frac{1}{2}\kappa a)^{2m}}{(m!)^2} + K_0(\kappa a)\sum_{m=0}^{n}\frac{(\frac{1}{2}\kappa a)^{2m+1}}{m!\,(m+1)!}\right],$$

we find

$$b_a^{(1)}(\kappa^2,p^2) = \frac{\kappa a}{2\kappa^2}[K_1(\kappa a)\beta_0(\kappa^2,p^2) + K_0(\kappa a)\beta_1(\kappa^2,p^2)]. \tag{B9}$$

Here $\beta_l(\kappa^2,p^2)$ ($l=0$ or 1) is given by

$$\beta_l(\kappa^2,p^2) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\theta(n-m)\frac{(p/\kappa)^{2n}}{n+2}\,\frac{(\frac{1}{2}\kappa a)^{2m+l}}{m!\,(m+l)!}\,. \tag{B10}$$

Using

$$\frac{1}{n+2} = 2\int_0^{\infty} e^{-2(n+2)t}dt,$$

we may rewrite Eq. (B10) as

$$\begin{aligned} B_l &= 2\int_0^{\infty} dt\, e^{-4t}(\kappa/e^{-t}p)^l \sum_{n,m=0}^{\infty}\theta(n-m)(e^{-t}p/\kappa)^{2(n-m)}\frac{[\frac{1}{2}(e^{-t}pa)]^{2m+l}}{m!\,(m+l)!}\\ &= \kappa^2\left(\frac{\kappa}{p}\right)^l\int_0^{\infty}dt\, e^{-(1-l/2)t}\frac{I_l(e^{-t/2}pa)}{\kappa^2e^t-p^2}\\ &= \kappa^4\int_{\kappa^2}^{\infty}\frac{dk^2}{k^4(k^2-p^2)}\left(\frac{k}{p}\right)^l I_l(\kappa ap/k), \qquad l=0,1. \end{aligned} \tag{B11}$$

Putting Eqs. (B9) and (B11) into Eq. (B6), we finally obtain the desired (continuable) expression for $B_1(p^2)$:

$$B_1(p^2) = \frac{1}{\pi}\int_{m^2}^{\infty}\frac{dk^2}{k^4(k^2-p^2)}\int_{m^2}^{k^2}\tfrac{1}{2}(\kappa^2d\kappa^2)\,\sigma(\kappa^2;f^2)d_a(\kappa^2,\kappa^2p^2/k^2), \tag{B12}$$

where

$$d_a(\kappa^2,\ \kappa^2p^2/k^2) = \kappa a\left[K_1(\kappa a)I_0(\kappa ap/k) + \frac{k}{p}K_0(\kappa a)I_1(\kappa ap/k)\right]. \tag{B13}$$

For $B_2(p^2)$, we find

$$\begin{aligned} B_2(p^2) &= \frac{1}{p}\frac{d}{dp}B_1(p^2)\\ &= -\frac{4}{p^2}B_1(p^2) + \frac{1}{\pi p^2}\int_{m^2}^{\infty}\frac{dk^2}{k^2-p^2}\sigma(k^2;f^2)d_a(k^2,p^2). \end{aligned} \tag{B14}$$

Equations (B12) and (B14) are dispersion relations for the functions B_1 and B_2. The integrals certainly exist since, even though $\sigma(k^2;f^2)\sim e^{kr_0}$ as $k\to\infty$, the function $d_a\sim e^{-ka}$ in this limit and $a>r_0$. It is obvious that B_1 and B_2 have the analytic properties one would expect on the basis of unitarity.

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*Supported in part by the National Science Foundation.

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[14]We shall hereafter refer to the present paper as I and to the following one as II.

[15]For notational simplicity, we often use only subscripts for space-time indices $\lambda = 0, 1, 2, 3$, although the Bjorken-Drell metric is being used ($g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$); also the Cabibbo-angle factors $\sin\theta$ and $\cos\theta$ are included in the currents $j_\lambda^{(q)}$.

[16]The importance of this condition has been stressed by Lehmann and Pohlmeyer, Ref. 12.

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[21]The reader is reminded that Eq. (4.3) is a formal expression only. From it one is to construct the amplitude in the Euclidean region of momenta, perform all auxiliary integrations over Euclidean coordinates, and finally continue the amplitude to the appropriate physical momenta.

[22]The Kronecker $\delta_{\Sigma l_i, \Sigma m_i}$ expresses the conservation of charge of the φ particles.

[23]Our notation for the hypergeometric functions is that of Ref. 18, Chap. II.

[24]This definition of the degree of divergence is equivalent to the usual definition in momentum space. See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965); compare Eq. (4.9) of the present paper with Eq. (19.62) of this reference.

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[27]See Ref. 18, p. 96.

PHYSICAL REVIEW D VOLUME 6, NUMBER 2 15 JULY 1972

Theory of Higher-Order Weak Interactions and *CP*-Invariance Violation. II. The Neutral *K* System*

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(Received 2 December 1971)

Continuing our exposition of a nonpolynomial theory of higher-order weak interactions, we examine the neutral K system, with particular emphasis on the CP-invariance-violating amplitudes. These first appear in second order in the weak interactions. We use the free-quark model to estimate the short-distance singularity of products of hadronic currents, and we find that with appropriate choices of the minor coupling constants our theory is consistent with the known experimental results. In particular, we find $|\epsilon| \approx 10^{-3}$, and we give an argument leading to $\eta_{+-} \approx \eta_{00}$. The neutron dipole moment is a third-order weak effect in our theory and is estimated to be about 10^{-27} e cm. We calculate the production cross section for the superpropagating particles, and find it to be too small for the particles to have yet been observed.

I. INTRODUCTION

In the preceding paper[1] we have treated leptonic processes in higher-order weak interactions, using a particular nonpolynomial modification of the usual current×current interaction Lagrangian. In this paper, using the same Lagrangian as in I, we turn to two somewhat more speculative subjects: (i) the effect of our modification on hadronic weak processes (with special attention to CP noninvariance) and (ii) the production of the φ particles that are coupled nonpolynomially to the usual weak cur-