Exact Solutions of the $\lambda\varphi^4$ Theory*

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Infinitely many sets of classical solutions of the field equation $\Box\varphi - \lambda\varphi^*\varphi\varphi = 0$ are given. Each set forms a nonlinear realization of the invariance group $SO_0(4,2)$ of the field equation. The problem of scattering is discussed.

I. INTRODUCTION

In this note we shall consider the classical field equation

$$\Box\varphi - \lambda\varphi^*\varphi\varphi = 0\,, \qquad \text{metric } {+}{+}{+}{-},\quad \lambda > 0. \tag{1}$$

This is one of the few physically interesting[1] wave equations on which several mathematical papers have been written.[2] Moreover there exists a large class of symmetry transformations under which Eq. (1) is invariant. These facts make a more detailed study of the solutions of Eq. (1) rewarding.

The action integral, from which Eq. (1) follows is not only invariant under the Poincaré group, but also under $SO_0(4,2)$. The SO(4)-invariant solutions are of special interest. In the case of the hydrogen atom they form the *ground state* (i.e., the state with lowest energy), and we expect something similar for this relativistic field theory. The solutions are physical (finite total energy-momentum), as they are specified by the eigenvalue 0 of the generators of the compact subgroup SO(4) (eigensolutions for eigenvalues $\neq 0$ are in general not solutions of the field equation).

We should like to make a further remark. Some physicists object to the $\lambda\varphi^4$ theory without a mass term $m^2\varphi$ because of the infrared divergences of the quantum field theory. However, this difficulty is perhaps of minor importance and of similar nature as the infinite total cross section for the scattering in the case of a $-1/r$ potential.

II. MANIFOLDS OF CLASSICAL SOLUTIONS

The real SO(4)-invariant solutions are given by

$$\varphi(y) = \frac{a(\xi,\xi^*)}{2\sqrt{\lambda}}\left[\frac{(\xi-\xi^*)^2}{(y-\xi)^2(y-\xi^*)^2}\right]^{1/2} \times \mathrm{cn}[(1+a^2)^{1/2}(\alpha-\alpha_0), k]\,,$$

where $a(\xi,\xi^*)$ is the real amplitude, cn the elliptical cosine,

$$k^2 = \frac{1}{2(1+1/a^2)}\,,$$

$$\alpha = \frac{1}{2i}\ln\frac{(y-\xi)^2}{(y-\xi^*)^2}\,, \quad \lambda > 0,$$

and ξ^μ a complex constant vector in the forward cone:

$$(\mathrm{Im}\,\xi^\mu)^2 < 0, \quad \mathrm{Im}\,\xi^4 > 0, \qquad -\infty < \mathrm{Im}\,\xi^i < \infty, \quad -\infty < \mathrm{Re}\,\xi^\mu < +\infty\,. \tag{2}$$

We shall, however, be interested in SO(4)-invariant solutions, which also form a representation of SO(2) of the maximal compact subgroup SO(4) $\times$SO(2) of the symmetry group $SO_0(4,2)$. This basic set of solutions is given by

$$\varphi_+ = u(\xi,\xi^*)2\left[\frac{j(j+1)}{\lambda}\right]^{1/2} \times \frac{[(\xi-\xi^*)^2]^{1/2}}{(y-\xi)^2}\left[\frac{(y-\xi^*)^2}{(y-\xi)^2}\right]^j\,, \tag{3}$$

$$\varphi_- = v(\xi,\xi^*)2\left[\frac{j(j+1)}{\lambda}\right]^{1/2} \times \frac{[(\xi-\xi^*)^2]^{1/2}}{(y-\xi^*)^2}\left[\frac{(y-\xi)^2}{(y-\xi^*)^2}\right]^j\,, \tag{4}$$

where $|u| = |v| = 1$. The eigenvalues of the generator of SO(2) are $-2j-1$ and $+2j+1$, $0 \le j < \infty$, respectively.

Mathematical comment: The full invariance group (which is continuously connected with the identity) of Eq. (1) is the group $SO_0(4,2)$. Correspondingly, the definition of the field $\varphi(y)$ can be extended to the appropriate conformal compactification of Minkowski space, i.e., to the space $SO_0(4,2)/(SO_0(3,1)\times D_0)\otimes T_4 \approx S_3\times S_1$. For each value of j and the sign (+) or (−) the parameter manifold is homeomorphic to the symmetric space

$$SO_0(4,2)/SO(4)\times SO(2)\,. \tag{5}$$

Therefore, each set of solutions forms a nonlinear realization of the group $SO_0(4,2)$. For integer and half-integer j, the center of $SO_0(4,2)$ is represented by -1 and $+1$, respectively. The space (2) is a particular realization of (5). For integer or half-integer j the solutions (3) and (4) are one-valued functions on $S_3\times S_1$. Note the homeomorphism

$$SO_0(4,2)/SO(4)\times SO(2) \approx [SO_0(3,1)\times D_0]⊠T_4/SO(3) \ . \tag{6}$$

The Fourier analysis of the solutions (2) shows that for j an integer, $\varphi_+(y,\xi,\xi^*)$ has support in momentum space only in the domain $m^2=0$, $p^4>0$, and $0<m^2<\infty$, $p^4>0$. The solutions $\varphi_-(y,\xi,\xi^*)$, correspondingly, have support in the negative cone. This result depends essentially on $\lambda>0$ and $j=1,2,3,\ldots$.

For an example we choose $j=1$ and $\xi^\mu=(0,0,0,i)$. The solution (3) has the following Fourier decomposition:

$$\varphi_+=\frac{2}{\pi}\left(\frac{2}{\lambda}\right)^{1/2}\int\theta(m^2)\theta(p^4)e^{-p^4}[(1-2p^4)\delta(p^2)+1]\,e^{ipy}d^4p\ ,$$
$$u=1\ .$$

III. SCATTERING AND ASYMPTOTIC DECAY

To every nonsingular solution[2] of (1) there exists a free-field solution $\varphi_0^{\rm in}(y)$ such that

$$\lim_{y^4\to-\infty}\int|\varphi-\varphi_0^{\rm in}|d^3y=0,\quad \Box\varphi_0^{\rm in}=0\ .$$

We calculate this incoming wave for the above example $j=1$, $\xi^\mu=(0,0,0,i)$ with the physical definition of the in-fields

$$\varphi_0^{\rm in}(y)=\varphi(y)-\lambda\int\Delta_{\rm ret}(y-y')\varphi^*\varphi\varphi(y')d^4y'\ , \tag{7}$$

$$\Box_y\Delta_{\rm ret}(y-y')=\delta^4(y-y')\ ,$$

where the retarded free Green's function is defined as usual. Solving (7), we obtain

$$\varphi=\varphi_0^{\rm in}+16\left(\frac{2}{\lambda}\right)^{1/2}\frac{1}{[(y^i)^2-(y^4-i)^2]^2}\ . \tag{8}$$

It is easy to see that there is no scattering in this special example

$$\varphi_0^{\rm in}=\varphi_0^{\rm out}\ .$$

However, this is not the case in general. The in- and out-wave, differs by a factor

$$e^{4i\pi j},$$

and as the parameter j represents the total energy per unit charge, it looks as if there is s-wave scattering. A more detailed analysis will be published elsewhere.

We should like to make a few comments concerning Eq. (8).

(i) It is quite suprising that

$$\varphi_0=4\left(\frac{2}{\lambda}\right)^{1/2}\left(1-2i\frac{\partial}{\partial y^4}\right)\frac{1}{(y^i)^2-(y^4-i)^2}$$

is not SO(4)-invariant, although the interacting solution and the field equation has this symmetry. This is due to the noninvariant asymptotic condition. Therefore the S matrix need not be conformal-invariant.

(ii) The quantum field theory which corresponds to (1) exhibits the dilatational symmetry of (1) as long as we restrict ourselves to treegraphs. However, as the free-field mass-zero Feynman propagator is not conformal-invariant, the corresponding dilatational-invariant S matrix is not conformal-invariant (except in 1st order). The situation is here similar as for the $m=0$ Bethe-Salpeter (B.S.) equation: Although the Wick-rotated B.S. equation is SO(4)-invariant the nonrotated B.S. equation is *not* $SO_0(3,1)$-invariant.

(iii) The result of the conformal-invariant scattering analysis[3] shows that for two incoming mass-0 particles there are no inelastic channels.

IV. THE QUANTIZATION OF THE FREE FIELD $\lambda=0$

To the sets of Eqs. (3) and (4) there corresponds in the free case $\lambda=0$ the particular solutions

$$\frac{1}{(2\pi)^2}\frac{1}{(y-\xi)^2}=\frac{1}{(2\pi)^3}\int d^4p\,e^{ip(y-\xi)}\delta(p^2)\theta(p^4)\ ,$$
$$\frac{1}{(2\pi)^2}\frac{1}{(y-\xi^*)^2}=\frac{1}{(2\pi)^2}\int d^4p\,e^{ip(y-\xi^*)}\delta(p^2)\theta(-p^4)\ . \tag{9}$$

The annihilation and creation operators in this wave-packet basis are defined by

$$a(\xi^*)=\frac{i}{(2\pi)^2}\int\frac{1}{(y-\xi^*)^2}\frac{\overleftrightarrow{\partial}}{\partial y^4}\varphi(y)d^3y\ ,$$
$$b^*(\xi)=\frac{i}{(2\pi)^2}\int\frac{1}{(y-\xi)^2}\frac{\overleftrightarrow{\partial}}{\partial y^4}\varphi(y)d^3y\ . \tag{10}$$

From Eq. (10) it follows that the operators $a(\xi^*)$ and $b^*(\xi)$ obey the equation

$$\Box_{\xi^*}a(\xi^*)=0,\quad \Box_\xi b^*(\xi)=0\ . \tag{11}$$

From Eq. (10) and the well-known commutation relations of the free field $\varphi(y)$ we can derive the nonvanishing commutation relations

$$[a(\xi^*), a^*(\xi')] = [b(\xi^*), b^*(\xi')] = \frac{1}{(2\pi)^2}\frac{1}{(\xi' - \xi^*)^2} \, . \qquad (12)$$

Note that the $SO_0(4,2)$ symmetry determines (12) up to a real factor. The Hilbert space is defined as usual $a(\xi^*)|0\rangle = 0$, $b(\xi^*)|0\rangle = 0$. In our further investigations we shall consider if it is possible to quantize the nonlinear field Eq. (1) along these lines.

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Twist Relation, Third Factorization, and the General Vertex in a Dual Multiparticle Theory with Nonlinear Trajectories*

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The following properties of the dual multiparticle theory with nonlinear trajectories are presented: (I) the twist relation, (II) the unsymmetric general vertex obtained directly from third factorization, and (III) the symmetric general vertex.

I. INTRODUCTION

In previous papers,[1] the N-point Born terms of a dual multiparticle theory with nonlinear trajectories[2] were factorized and the vertex involving two external particles with nonzero spins was obtained. In this paper, we will obtain the twisted propagator and the general vertex in the nonlinear model. These are the two basic ingredients in the Kikkawa-Sakita-Virasoro unitarization program.[3] However, further steps in this program have not yet been carried out for the nonlinear model.

In I, we found it convenient to introduce a six-dimensional formalism for purposes of proving factorization. In Sec. II of this paper, we develop this formalism further by introducing a special choice of kinematic variables which are particularly suited to the study of duality and factorization. Using the formalism developed in Sec. II, we go on to derive the twist relation in Sec. III and the unsymmetric general vertex and the symmetric vertex in Sec. IV.

II. SIX-DIMENSIONAL FORMALISM

In first factorization, we were naturally led to consider six-dimensional "vectors." These "6-vectors" neither satisfy additivity, nor do they scale. Therefore, they do not form a linear vec-