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Quasisecular Clothing Transformation for the ϕ^4 Model of Quantum Field Theory

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The first-order renormalized Heisenberg field for the ϕ^4 model of quantum field theory, calculated by the quasisecular perturbation method, is used to construct a clothing transformation. The clothing transformation gives the Fourier amplitude of the renormalized field from that of the bare field, except for the quasisecular terms. The properties of the clothing transformation are examined in the cases of one, two, and three space dimensions. The connection with the strange representations of the canonical commutation relations is discussed.

I. INTRODUCTION

Quantum field theories which attempt to describe interactions must deal with nonlinear partial differential equations whose solutions are operator-valued (q-number) functions of the space-time coordinates. The nonlinear terms involve the product of fields at the same space-time point (point interactions), and since the fields are taken to be (singular) operator-valued distributions this leads to a variety of difficulties.[1] The simplest of these difficulties is the infinite zero-point energy of the free-field Hamiltonian which is removed by the prescription of normal ordering. A more serious difficulty is that it becomes necessary to use one of the so-called strange representations of the commutation relations of the field operators[2,3] when interactions are present. The conclusion was drawn that standard perturbation theory is an inadequate technique for evaluating physical quantities in a mathematically logical manner avoiding infinities, because perturbation theory makes *a priori* use of the Fock representation.[4] Wightman[3] has conjectured that some significant part of the ultraviolet catastrophe is the price paid for ignoring the strange representation. On the other hand, in the branch of mathematics which treats nonlinear differential equations, the method of successive approximations (perturbation theory) has been found useful both in connection with problems of existence of solutions, and in deriving approximations to solutions.[5] However, there are many possible iteration schemes, only one of which is the standard perturbation theory that is familiar to physicists. It is important to investigate the possibility that some new (perturbative) iterative scheme might provide sensible results for the field equations. Dirac[6] has suggested that physicists working in quantum field theory may have to get results by developing a system of approximations similar to those used by engineers, because of the complexity of the problem. These approximations Dirac suggests would have to focus on the most important aspects of the problem and neglect many other aspects of lesser importance.

The approximations obtained by standard perturbation theory are often qualitatively, as well as quantitatively, poor.[7] For example, approximations to solutions of (classical) equations of motion which are known to be periodic are found to be nonperiodic. Such problems are referred to as secular behavior of the approximation.[7] In a previous paper[8] the importance of secular behavior in the perturbative solution of the ϕ^4 model of quantum field theory was elucidated, and the unphysical secular behavior was removed by a technique of frequency renormalization. A modified perturbation method was developed via this technique, which focused on terms in the approximate solution for the field involving small and vanishing energy denominators. The perturbative method was called quasisecular perturbation theory because terms with small energy denominators gave rise to behavior that was almost secular over the appropriate time scale. The frequency (mass) and amplitude (wave-function) renormalization were operator-valued and remained present in the limit of an infinite volume because the number of quasisecular terms is also infinite.

In this paper it is shown that quasisecular perturbation theory provides information about the appropriate (strange) representations which arise. The model chosen has been described previously[8]; it is the ϕ^4 model of quantum field theory.[9] The equation of motion for the case of one space dimension is

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)\phi=\lambda\phi^3\,, \tag{1.1}$$

and the real field ϕ is studied on the interval $-\frac{1}{2}L\leq x\leq+\frac{1}{2}L$ with periodic boundary conditions

($\hbar = c = 1$). The results obtained are valid in the case of two or three space dimensions with the appropriate natural notational changes. However, we choose to express most equations in one space dimension to simplify notation and avoid vector subscripts.

Using the Fourier decomposition,

$$\phi(t, x) = L^{-1/2} \sum_l a_l(t) e^{ilx} , \tag{1.2}$$

where the sum goes over $l = 2\pi n L^{-1}$, $n = 0, \pm 1, \pm 2, \ldots$, the field equations transform into an infinite array of nonlinearly coupled nonlinear equations.[8] The initial conditions on the field equations are given by the usual canonical commutation relations at $t = 0$. In the oscillator formulation these are

$$[a_k(0), \dot{a}_l(0)] = i\delta_{k,-l} , \tag{1.3}$$

$$[a_k(0), a_l(0)] = [\dot{a}_k(0), \dot{a}_l(0)] = 0 , \tag{1.4}$$

and $a_l^* = a_{-l}$ is the reality condition. The expansion ansatz is

$$a_l = \sum_{n \geq 0} \lambda^n a_l^{(n)} . \tag{1.5}$$

The initial conditions are given in terms of the expansion coefficients as

$$a_l^{(0)} = a_l(0), \quad \dot{a}_l^{(0)}(0) = \dot{a}_l(0) , \tag{1.6}$$

so that higher-order terms in Eq. (1.5) have the zero initial conditions. The zeroth-order solution of Eq. (1.1) is

$$a_l^{(0)}(t) = (2\omega_l)^{-1/2} (\alpha_l e^{-i\omega_l t} + \alpha_{-l}^* e^{i\omega_l t}) , \tag{1.7}$$

where $\omega_l{}^2 = l^2 + m^2$ and

$$[\alpha_k, \alpha_l^*] = \delta_{k,l} , \tag{1.8}$$

$$[\alpha_k, \alpha_l] = [\alpha_k^*, \alpha_l^*] = 0 . \tag{1.9}$$

Note that the creation and destruction operators α_l and α_l^* are related to the exact Heisenberg field at time $t = 0$ by choice of the initial conditions. It is possible to construct an auxiliary Fock representation[10] using the α's and α^*'s. The representation consists of the unique vacuum Ψ_0 of the α's, $\alpha_l \Psi_0 = 0$ for all l, and the closure of the subspace spanned by applying polynomials in creation operators α_l^* to Ψ_0. It has the usual orthonormal basis in terms of the occupation-number representation $\Psi(\{n_k\})$, where $\alpha_l^* \alpha_l \Psi(\{n_k\}) = n_l \Psi(\{n_k\})$, $\forall l$, and $\{n_k\}$ is a sequence of non-negative integers only a finite number of which are nonzero.

The relationship between the auxiliary Fock representation and the appropriate representation of the α's and α^*'s determined by the first-order approximation to the Heisenberg field which is calculated by the quasisecular perturbation method is treated in Sec. VI. The renormalized Heisenberg field determines a Hilbert space which contains the physical vacuum and the physical particle states. This space may carry a representation of the commutation relations of the α's and α^*'s which does not contain a state Ψ such that $\alpha_l \Psi = 0$, $\forall l$ (i.e., a strange representation). This representation is unitarily equivalent in the case of one space dimension and is inequivalent (i.e., strange) in two dimensions and possibly does not exist in three space dimensions.

The viewpoint of this work is that the concept of quasisecularity focuses on the most important aspect of the nonlinearity of quantum field theory and may provide the type of approximation suggested by Dirac.[6] In the following it is shown that the quasisecular terms in the first-order approximation to the field $\phi(t, x)$ are qualitatively different from the other terms, in that the nonsecular terms are obtained from the zeroth-order (or bare) field by a (formally) unitary clothing transformation, whereas the quasisecular terms cannot. The interactions of the clothed particles are carried by the quasisecular terms.[8,11]

Section II is a recapitulation of the relevant properties of the first-order renormalized field. In Section III the generator of the transformation is displayed, which gives all but the quasisecular terms in the first-order approximation. The commutation relations of the renormalized field amplitudes are found in Sec. IV, and differ from those of the free field due to the presence of quasisecular terms. In Sec. V the generator of the clothing transformation is shown to be a (formally) anti-Hermitian transformation on the Fock representation for one space dimension; in this case the renormalization constants are finite.[12] In two and three space dimensions the transformation is not defined on the Fock-space basis. This is also the case as $L \to \infty$ in one dimension. In Sec. VI the properties of the (formally) unitary clothing transformation are examined. In one space dimension the clothing transformation is unitary within Fock space. In two and three dimensions it is interpreted as an improper unitary transformation,[13] which is an isometric isomorphism between Fock space and a new space containing the physical states.

II. FIRST-ORDER RENORMALIZED FIELD

In a previous paper[8] the method of quasisecular perturbation theory was introduced and applied to the ϕ^4 model of quantum field theory. The first-order renormalized solution for the amplitude $a_l(t)$ was found to be

$$a_l(t) = U_l e^{-i\Omega_l t} + e^{i\Omega_{-l}^* t} U_{-l}^* + W_l(t) . \tag{2.1}$$

The operator W_l has the property that its Fourier transform vanishes in intervals of width $2|\lambda|m^{-1}$ centered at $\pm\omega_l$. The renormalized frequency operator Ω_l is given by

$$\Omega_l = \omega_l - (2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l + 1)^{-1}V_l \ , \tag{2.2}$$

where

$$V_l = 3\lambda(2\omega_l L)^{-1}\sum_{p,q,r} D_{lpqr}(8\omega_p\omega_q\omega_r)^{-1/2}\alpha_{-p}^*\alpha_q\alpha_r \ . \tag{2.3}$$

The D function defines the region of quasisecularity around ω_l. Its value is 1 if both $l=p+q+r$ and $|\omega_l+\omega_p-\omega_q-\omega_r|\leq|\lambda|m^{-1}$ and zero otherwise.

The renormalized wave-function operator U_l is given by a rather long expression[14]

$$\begin{aligned} U_l = {} & (2\omega_l)^{-1/2}\alpha_l - \lambda(2\omega_l L)^{-1}\sum_{p,q,r}\delta_{l,p+q+r}(8\omega_p\omega_q\omega_r)^{-1/2} \\ & \times[(\omega_l-\omega_p-\omega_q-\omega_r)^{-1}\alpha_p\alpha_q\alpha_r + (\omega_l+\omega_p+\omega_q+\omega_r)^{-1}\alpha_{-p}^*\alpha_{-q}^*\alpha_{-r}^* \\ & + 3(\omega_l+\omega_p+\omega_q-\omega_r)^{-1}\alpha_{-p}^*\alpha_{-q}^*\alpha_r + 3(1-D_{lpqr})(\omega_l+\omega_p-\omega_q-\omega_r)^{-1}\alpha_{-p}^*\alpha_q\alpha_r] \\ & + 3\lambda(2\omega_l)^{-2}L^{-1}\sum_{p,q,r} D_{lpqr}(8\omega_p\omega_q\omega_r)^{-1/2}\alpha_{-p}^*\alpha_q\alpha_r \ . \end{aligned} \tag{2.4}$$

The factor $1-D_{lpqr}$, in the last term inside the square bracket of Eq. (2.4), serves to avoid the "dangerous denominator" which would arise when $\omega_l+\omega_p-\omega_q-\omega_r\approx 0$. The last term is the contribution to U_l from the quasisecular region. It is proportional to V_l which also enters Eq. (2.2) for the renormalized frequency. Every term of $U_l-(2\omega_l)^{-1/2}\alpha_l$ is proportional to λ in Eq. (2.4), which would not be the case if the dangerous energy denominators had not been eliminated via the D function. If the denominator were smaller than $|\lambda|m^{-1}$, the term would then contribute to zeroth order.

Treating the secular and quasisecular terms separately from the other terms, therefore, achieves two aims: (i) frequency renormalization (which is operator-valued, and hence different in states of different particle number); (ii) wave-function renormalization in which every term is proportional to λ. These desirable features of quasisecular perturbation theory make it possible to avoid many of the difficulties of standard perturbation theory.

III. CLOTHING TRANSFORMATION

It is shown in this section that all the terms of U_l in Eq. (2.4) except for the last are obtained via a transformation which is unitary in appearance. In Sec. V the clothing transformation is analyzed in detail; it is a proper unitary transformation for one space dimension, and an improper unitary transformation[13] in the case of two or three space dimensions. If the box is removed by letting $L\to\infty$ the clothing transformation is always improper, as expected from Haag's theorem.[15]

For convenience U_l is separated into two parts,

$$U_l = (2\omega_l)^{-1/2}\beta_l + (2\omega_l)^{-1}V_l \ , \tag{3.1}$$

where $(2\omega_l)^{-1/2}\beta_l$ includes all terms on the right-hand side of Eq. (2.4) except for the last. We show that $\beta_l = e^{\lambda S}\alpha_l e^{-\lambda S}$ to order λ, where $S^*=-S$ is (formally) anti-Hermitian and consequently $e^{\lambda S}$ is (formally) unitary. Expanding $e^{\lambda S}\alpha_l e^{-\lambda S}$ gives $\beta_l = \alpha_l + \lambda[S,\alpha_l] + O(\lambda^2)$, and therefore it is important that the dangerous denominators do not appear in Eq. (2.4), for otherwise the expansion would not be consistent to order λ.

At this point we proceed by writing down the generator of the clothing transformation

$$\begin{aligned} S = {} & (4L)^{-1}\sum_{k,p,q,r}\delta_{k,p+q+r}(\omega_k\omega_p\omega_q\omega_r)^{-1/2}[(\omega_k-\omega_p-\omega_q-\omega_r)^{-1}(\alpha_k^*\alpha_p\alpha_q\alpha_r - \alpha_r^*\alpha_q^*\alpha_p^*\alpha_k) \\ & + \tfrac{1}{4}(\omega_k+\omega_p+\omega_q+\omega_r)^{-1}(\alpha_k^*\alpha_{-p}^*\alpha_{-q}^*\alpha_{-r}^* - \alpha_{-r}\alpha_{-q}\alpha_{-p}\alpha_k)] \\ & + 3(8L)^{-1}\sum_{k,p,q,r}\delta_{k,p+q+r}(1-D_{kpqr})(\omega_k\omega_p\omega_q\omega_r)^{-1/2}(\omega_k+\omega_p-\omega_q-\omega_r)^{-1}\alpha_k^*\alpha_{-p}^*\alpha_q\alpha_r \ . \end{aligned} \tag{3.2}$$

The terms in the expression are grouped together so they are separately anti-Hermitian. We shall verify that the first term of S gives rise to two of the terms on the right-hand side of Eq. (2.4).

$$\begin{aligned} [\alpha_l, S_1] = {} & \left[\alpha_l, (4L)^{-1}\sum_{k,p,q,r}\delta_{k,p+q+r}(\omega_k\omega_p\omega_q\omega_r)^{-1/2}(\omega_k-\omega_p-\omega_q-\omega_r)^{-1}(\alpha_k^*\alpha_p\alpha_q\alpha_r - \alpha_r^*\alpha_q^*\alpha_p^*\alpha_k)\right] \\ = {} & (4L)^{-1}\sum_{p,q,r}\delta_{l,p+q+r}(\omega_l\omega_p\omega_q\omega_r)^{-1/2}(\omega_l-\omega_p-\omega_q-\omega_r)^{-1}\alpha_p\alpha_q\alpha_r \\ & - 3(4L)^{-1}\sum_{p,q,r}\delta_{r,p+q+l}(\omega_l\omega_p\omega_q\omega_r)^{-1/2}(\omega_r-\omega_p-\omega_q-\omega_l)^{-1}\alpha_q^*\alpha_p^*\alpha_r \ . \end{aligned} \tag{3.3}$$

The factor of 3 in the second term comes from the presence of the three creation operators in the previous line which play symmetric roles. The expression $\lambda(2\omega_l)^{-1/2}[S_1, \alpha_l]$ therefore takes care of the first and third terms in the brackets of Eq. (2.4). In a similar manner the other parts of S in Eq. (3.2) generate the remaining terms inside the brackets in Eq. (2.4).

It remains to demonstrate that the last term (i.e., the quasisecular term) of Eq. (2.4) cannot be generated by an anti-Hermitian S' (which is independent of l, and thus works for all l). Suppose on the contrary that

$$[\alpha_l, S'] = -3(2\omega_l)^{-3/2}L^{-1}\sum_{p,q,r} D_{lpqr}(8\omega_p\omega_q\omega_r)^{-1/2}\alpha^*_{-p}\alpha_q\alpha_r \,. \tag{3.4}$$

Then the terms of S' must contain two creation and two destruction operators (all adding up to momentum zero). Let

$$S' = L^{-1}\sum_{k,p,q,r}\delta_{k+p,q+r}f(k,p,q,r)\alpha^*_k\alpha^*_p\alpha_q\alpha_r \,, \tag{3.5}$$

where $(S')^* = -S'$ implies

$$f^*(r,q,p,k) = -f(k,p,q,r)\,. \tag{3.6}$$

Note that f can be taken symmetric in k and p and also in q and r since only the symmetric part contributes in Eq. (3.5). Then

$$[\alpha_l, S'] = 2L^{-1}\sum_{p,q,r}\delta_{l+p,q+r}f(l,p,q,r)\alpha^*_p\alpha_q\alpha_r \,, \tag{3.7}$$

and comparing with Eq. (3.4) gives

$$2f(l,p,q,r) = -3(2\omega_l)^{-3/2}(8\omega_p\omega_q\omega_r)^{-1/2}D_{l-pqr}\,. \tag{3.8}$$

However, the right-hand side does not satisfy Eq. (3.6) and is not symmetric in l and p, which is a contradiction. We conclude that the quasisecular part of the renormalized wave function, and hence U_l itself, cannot be generated by a (formal) unitary transformation acting on α_l.

IV. COMMUTATION RELATIONS OF THE U_l

The clothing transformation which relates α_l and β_l is convenient for calculating the commutation relations of the U_l. First note that an alternative expression for U_l is

$$U_l = e^{\lambda S}[(2\omega_l)^{-1/2}\alpha_l + (2\omega_l)^{-1}V_l]e^{-\lambda S}\,, \tag{4.1}$$

because the difference between Eq. (3.1) and Eq. (4.1) is of order λ^2. Since S is (formally) anti-Hermitian the commutator algebra of the U_l is taken to be the same as that of $(2\omega_l)^{-1/2}\alpha_l + (2\omega_l)^{-1}V_l$. Consider first the commutator

$$[\alpha_l + (2\omega_l)^{-1/2}V_l, \alpha_k + (2\omega_k)^{-1/2}V_k] = (2\omega_k)^{-1/2}[\alpha_l, V_k] + (2\omega_l)^{-1/2}[V_l, \alpha_k] + O(\lambda^2)\,. \tag{4.2}$$

From Eq. (2.3),

$$[\alpha_l, V_k] = 3\lambda(2\omega_k L)^{-1}\sum_{q,r}D_{k-lqr}(8\omega_l\omega_q\omega_r)^{-1/2}\alpha_q\alpha_r \,,$$

so that the right-hand side of Eq. (4.2) becomes

$$3\lambda(8L)^{-1}(\omega_k{}^{-1} - \omega_l{}^{-1})\sum_{q,r}D_{k-lqr}(\omega_k\omega_l\omega_q\omega_r)^{-1/2}\alpha_q\alpha_r \,.$$

Since we may replace $\lambda e^{\lambda S}\alpha_q\alpha_r e^{-\lambda S}$ on the right by $\lambda U_q U_r$, up to corrections of $O(\lambda^2)$, we obtain

$$[U_l, U_k] = 3\lambda(8L)^{-1}\omega_l{}^{-1}\omega_k{}^{-1}(\omega_k{}^{-1} - \omega_l{}^{-1})\sum_{q,r}D_{k-lqr}U_qU_r \,. \tag{4.3}$$

The noncommutativity is a manifestation of the quasisecular terms.

The commutator $[U^*_l, U^*_k]$ may be obtained from Eq. (4.3) by taking the adjoint. The remaining commutator of interest is

$$[\alpha_l + (2\omega_l)^{-1/2}V_l, \alpha^*_k + (2\omega_k)^{-1/2}V^*_k] = \delta_{l,k} + 3\lambda(2\omega_k)^{-1/2}(\omega_k L)^{-1}\sum_{p,q}D_{kpql}(8\omega_p\omega_q\omega_l)^{-1/2}\alpha^*_q\alpha_{-p}$$
$$+ 3\lambda(2\omega_l)^{-1/2}(\omega_l L)^{-1}\sum_{p,q}D_{lpqk}(8\omega_p\omega_q\omega_k)^{-1/2}\alpha^*_{-p}\alpha_q + O(\lambda^2)\,. \tag{4.4}$$

In the same manner as before we write

$$[U_l, U_k^*] = (2\omega_k)^{-1}\delta_{l,k} + 3\lambda(4L)^{-1}(\omega_k\omega_l)^{-1}(\omega_k^{-1}+\omega_l^{-1})\sum_{p,q} D_{kpql}U_q^*U_{-p}\,. \tag{4.5}$$

The quasisecularity is again evident as the source of the second term on the right-hand side of Eq. (4.5).

V. PROPERTIES OF THE GENERATOR OF THE CLOTHING TRANSFORMATION

In this section the mathematical properties of the formally anti-Hermitian generator of the clothing transformation are investigated. In most cases the generator cannot be interpreted as a mapping of the auxiliary Fock representation space into itself. We show that there are two sources of this difficulty which are easily distinguished[16]: (i) infinite volume ($L\to\infty$), (ii) ultraviolet divergences. The first difficulty is always present and easily identified. It is related to Haag's theorem.[15] The second difficulty appears in a box (finite L) and is model-dependent. Van Hove[17] showed that for scalar mesons interacting with fixed (pointlike) nucleons the Hilbert space which contains the stationary states is "orthogonal" to Fock space in the case of three space dimensions, but not for one and two. Since the ϕ^4 interaction is more singular, the "orthogonality" occurs for two and three space dimensions, but not for one.[9,18]

We discuss the properties of the generator S by calculating upper and lower bounds for the norms $\|S\Psi(\{n_k\})\|$, where $\Psi(\{n_k\})$ is an element of the occupation number basis of the auxiliary Fock space. When the norm is infinite the states which are finite linear combinations of this basis are mapped out of the space by S. In Sec. VI it is shown that in this case $(\Psi(\{m_k\}), e^{\lambda S}\Psi(\{n_k\}))$ also vanishes $\forall\{n_k\}$ and $\{m_k\}$ so that the images of these states under $e^{\lambda S}$ have no component in the auxiliary Fock space.

Suppose that the generator S of the clothing transformation acts on the vacuum Ψ_0:

$$S\Psi_0 = (16L)^{-1}\sum_{l,p,q,r}\delta_{l,p+q+r}(\omega_l\omega_p\omega_q\omega_r)^{-1/2} \times(\omega_l+\omega_p+\omega_q+\omega_r)^{-1}\alpha_l^*\alpha_{-p}^*\alpha_{-q}^*\alpha_{-r}^*\Psi_0\,. \tag{5.1}$$

The norm $\|S\Psi_0\|$ is bounded by

$$B\le\|S\Psi_0\|^2\le 24B\,, \tag{5.2a}$$

where

$$B=(16L)^{-2}\sum_{l,p,q,r}\delta_{l,p+q+r}(\omega_l\omega_p\omega_q\omega_r)^{-1} \times(\omega_l+\omega_p+\omega_q+\omega_r)^{-2}\,. \tag{5.2b}$$

For L large, B is given by the integral

$$B=2^{-11}\pi^{-3}L\int dp\,dq\,dr\,(\omega_{p+q+r}\omega_p\omega_q\omega_r)^{-1} \times(\omega_{p+q+r}+\omega_p+\omega_q+\omega_r)^{-2}\,. \tag{5.3}$$

The infinite-volume difficulty is immediately evident since $\|S\Psi_0\|$ is proportional to $L^{1/2}$, for large L. In addition, the integral may itself diverge; since the integrand is bounded the divergence is associated with the behavior at large momenta (i.e., ultraviolet divergence).

In two and three space dimensions the integrals giving B are obtained by replacing $(2\pi)^{-1}dp$ by $(2\pi)^{-2}d^2p$ and $(2\pi)^{-3}d^3p$, and likewise for dq and dr, and reinterpreting L as the area and volume, respectively. For one space dimension B can be bounded. Using $\omega_{p+q+r}^{-1}\le m^{-1}$ and $\omega_{p+q+r}+\omega_p+\omega_q+\omega_r\ge(\omega_p\omega_q\omega_r)^{1/3}$,

$$B\le 2^{-11}\pi^{-3}Lm^{-1}\int dp\,dq\,dr\,(\omega_p\omega_q\omega_r)^{-5/3}\,, \tag{5.4}$$

and each integral converges because the asymptotic behavior is $|p|^{-5/3}$. In this case $S\Psi_0$ is a (normalizable) state in the auxiliary Fock space. A similar argument proves the boundedness of $\|S\Psi(\{n_k\})\|$, because the finite number of particles which are present in the state $\Psi(\{n_k\})$ do not interfere with the four added by S [as in Eq. (5.1)] in the region of large momenta. All that changes is the numerical factor in the upper bound of Eq. (5.2a).

In the case of two dimensions

$$B=2^{-14}\pi^{-6}L\int d^2p\,d^2q\,d^2r\,(\omega_{p+q+r}\omega_p\omega_q\omega_r)^{-1} \times(\omega_{p+q+r}+\omega_p+\omega_q+\omega_r)^{-2}\,, \tag{5.5}$$

where the subscripts are all two-dimensional vectors. Using $\omega_{p+q+r}^2<3(\omega_p^2+\omega_q^2+\omega_r^2)$ and $(\omega_p+\omega_q+\omega_r)^2\le 3(\omega_p^2+\omega_q^2+\omega_r^2)$ a lower bound for B is obtained:

$$B\ge 2^{-16}\pi^{-3}L3^{-3/2} \times\int d^2p\,d^2q\,d^2r\,(\omega_p\omega_q\omega_r)^{-1}(\omega_p^2+\omega_q^2+\omega_r^2)^{-3/2}\,. \tag{5.6}$$

The integral can be transformed by the change of variables $|p|, |q|, |r|\to\omega_p, \omega_q, \omega_r$ into

$$8(2\pi)^3\int_m^\infty\int_m^\infty\int_m^\infty d\omega_p\,d\omega_q\,d\omega_r\,(\omega_p^2+\omega_q^2+\omega_r^2)^{-3/2}\,,$$

which diverges logarithmically when the radial variable $\rho=(\omega_p{}^2+\omega_q{}^2+\omega_r{}^2)^{1/2}$ is introduced.

It should be noted that the clothing transformation is closely related to the wave-function renormalization constant of standard field theory, and this is known to be infinite for two space dimensions.[18] The divergence of B is related to this infinity. Clearly for three space dimensions the divergence of B is even worse than in two. This indicates that S is not definable as a linear transformation on the auxiliary Fock space. We note that in Glimm's work[18] on ϕ^4 in two dimensions, clothing transformations similar to $e^{\lambda S}$ are introduced. It is found there that the part involving four creation operators α^* is the most singular part as is evident in the calculations given above. If S were definable on the auxiliary Fock space then the (first-order) clothing transformation could be used to prove that the Hamiltonian is defined as a (self-adjoint) operator on this space, in the first-order approximation of quasisecular perturbation theory. It is known that the (exact) Hamiltonian cannot be defined on this space.[18]

VI. PROPERTIES OF THE CLOTHING TRANSFORMATION

When S is not defined as an operator on the particle number basis of the auxiliary Fock space, it is expected that $e^{\lambda S}$ can still be interpreted as an improper unitary transformation. This can be made plausible by showing that $(\Phi, e^{\lambda S}\Psi)\to 0$ as $\|S\Psi\|\to\infty$, for Φ and Ψ in the number basis. Consequently, the state $e^{\lambda S}\Psi$ has no components in the auxiliary Fock space and is then interpreted as being the image of Ψ in a new space.[17]

To this end we calculate $(\Psi_0, e^{\lambda S}\Psi_0)$ via the Baker-Hausdorff approximation $e^{A+F}=e^{-[A,F]/2}e^Ae^F$. Let A be the part of λS in Eq. (3.2) with four creation operators and F be the remainder. Then $e^F\Psi_0=\Psi_0$ since F has always a destruction operator on the right. The commutator $[A,F]$ has one c-number term C and other terms which have creation operators on the left. The latter terms disappear when moved to the left side of the inner product. We obtain

$$\begin{aligned}(\Psi_0, e^{\lambda S}\Psi_0)&=(\Psi_0, e^{-C/2}e^A\Psi_0)\\&=e^{-C/2}(e^{A^*}\Psi_0, \Psi_0)\\&=e^{-C/2}\end{aligned}\tag{6.1}$$

and

$$C=-(16L)^{-2}\lambda^2\sum_{l,p,q,r}\delta_{l,p+q+r}(\omega_l\,\omega_p\,\omega_q\,\omega_r)^{-1}(\omega_l+\omega_p+\omega_q+\omega_r)^{-2}4\,![\alpha_l^*\alpha_{-p}^*\alpha_{-q}^*\alpha_{-r}^*,\alpha_{-r}\alpha_{-p}\alpha_{-q}\alpha_l]_c\,,\tag{6.2}$$

where $[\ ,\ ]_c$ denotes the c-number part of the commutator when the remaining terms are in normal order. This is just the vacuum expectation value of the commutator. We can bound C using B given in Eq. (5.2b):

$$24\lambda^2B\leq C\leq(24)^2\lambda^2B\,,\tag{6.3}$$

where the bounds are related to the minimum and maximum contributions from the four Bose operators.

When B is bounded the overlap $e^{-C/2}$ is nonzero, otherwise it is zero. The same arguments hold for the overlap $(\Phi, e^{\lambda S}\Psi)$, where Φ and Ψ are general elements of the number basis, since the bounds are modified only by finite numerical factors.

When the clothing transformation is not defined in Fock space, we interpret $e^{\lambda S}$ as an isometry from Fock space to a new Hilbert space which is orthogonal to Fock space, following van Hove's procedure.[17] The new Hilbert space is a copy of Fock space under the correspondence $\alpha_k\to\beta_k=e^{\lambda S}\alpha_k e^{-\lambda S}$ of Sec. III. The new vacuum state Ψ_0' is represented by $e^{\lambda S}\Psi_0$ because $\beta_k e^{\lambda S}\Psi_0=e^{\lambda S}\alpha_k\Psi_0=0\ \forall k$, and the new occupation number representation is obtained by applying products of β^*'s to Ψ_0'. By defining the inner product in the new Hilbert space via the new occupation number basis, it is trivial that $e^{\lambda S}$ represents an isometric isomorphism between Fock space and the new Hilbert space $\mathcal{H}'$. More important, the renormalized amplitudes U_l are well-defined operators in $\mathcal{H}'$ because $U_l=(2\omega_l)^{-1/2}\beta_l+2\omega_lV_l(\beta)$, where $V_l(\beta)$ is the expression in Eq. (2.3) with α's replaced by the corresponding β's.

We conclude by comparing the properties of the clothing transformation in cases of one, two, and three space dimensions.

In one space dimension the clothing transformation determined from first-order quasisecular perturbation theory was shown to exist as a mapping within the auxiliary Fock space. The unitarity of this operator is a difficult technical question which is related to the self-adjointness of the generator S.[19] The unitarity is strongly suggested by the work of Jaffe[9] and Glimm[18] on the ϕ^4 model in one space dimension. That work shows that the Hamiltonian and the Heisenberg fields are (densely defined) self-adjoint operators on the auxiliary Fock space so that the Van Hove difficulty does not arise as the cutoff is removed (no ultraviolet divergences).

In two space dimensions the new space $\mathcal{H}'$ is distinct from the auxiliary Fock space. The operators α_l and α_l^* are defined on $\mathcal{H}'$ which then carries a strange representation of their canonical commutation relations. We note that this representation is strange since it does not contain a state which is destroyed by all of the α_l.

In three space dimensions the situation is somewhat more unusual. Again the new space $\mathcal{H}'$ is distinct from the auxiliary Fock space, but α_l and α_l^* are *not* defined as operators on $\mathcal{H}'$. Therefore, it makes no sense to introduce the concept of a strange representation of the α's on $\mathcal{H}'$. To see that the α's are not defined on $\mathcal{H}'$, we examine $\alpha_l = e^{-\lambda S}\beta_l e^{\lambda S}$ and obtain in first order

$$\alpha_l = \beta_l + \lambda(2\omega_l)^{-1/2}L^{-1}\sum_{p,q,r}\delta_{l,p+q+r}(8\omega_p\omega_q\omega_r)^{-1/2} \times[(\omega_l-\omega_p-\omega_q-\omega_r)^{-1}\beta_p\beta_q\beta_r+(\omega_l+\omega_p+\omega_q+\omega_r)^{-1}\beta^*_{-l}\beta^*_{-q}\beta^*_{-r} + 3(\omega_l+\omega_p+\omega_q-\omega_r)^{-1}\beta^*_{-p}\beta^*_{-q}\beta_r+3(1-D_{lpqr})(\omega_l+\omega_p-\omega_q-\omega_r)^{-1}\beta^*_{-p}\beta_q\beta_r] . \quad (6.4)$$

The term involving the creation of three β^*'s is not a well-defined operator on the number basis of the β's in $\mathcal{H}'$ for the same reasons that S was not a well-defined operator as discussed in Sec. V. In the case of three dimensions those boundedness arguments serve to prove that states are mapped out of $\mathcal{H}'$ when this term is applied to them. It follows that the domain of the α's does not include the number basis of the β's.

It is possible that the field amplitudes α_l, in three space dimensions, do not exist as operators in (the space of physical states) $\mathcal{H}'$. Nevertheless, suitable expectation values of the field and products of fields can exist. It would then appear that the field must be regarded as a bilinear form on $\mathcal{H}'$ rather than an operator.[20]

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[3]A. S. Wightman, in *High Energy Electromagnetic Interactions and Field Theory,* Cargèse Lectures, 1964, edited by M. Lévy (Gordon and Breach, New York, 1967), pp. 248, 260, 261.

[4]S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, Evanston, Ill., 1961), p. 416.

[5]N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Hindustani, Delhi, 1961); N. Minorsky, *Introduction to Nonlinear Mechanics* (Edwards, Ann Arbor, Mich., 1947); J. K. Hale, *Nonlinear Oscillations* (McGraw-Hill, New York, 1963).

[6]P. A. M. Dirac, *Lectures on Quantum Field Theory* (Yeshiva University, New York, 1966), p. 2.

[7]Bogoliubov and Mitropolsky, Ref. 5, p. 39.

[8]S. Aks, J. Sienicki, and B. Varga, preceding paper, Phys. Rev. D 6, 520 (1972). The basic method of quasisecular renormalization is developed. The method has been used to exhibit the existence of a two-particle bound state in the ϕ^4 model (S. Aks and B. Varga, unpublished).

[9]A. M. Jaffee, Rev. Mod. Phys. 41, 576 (1969).

[10]Reference 4, p. 134.

[11]J. Ford, J. Math. Phys. 2, 387 (1961). Ford shows that energy transfer between the modes of an array of weakly coupled classical nonlinear oscillators proceeds only through the quasisecular terms which arise in the equations of motion.

[12]Reference 3, p. 262.

[13]G. Barton, *Introduction to Advanced Field Theory* (Interscience, New York, 1963), p. 126. An improper unitary transformation is an isometry whose domain and range are two different Hilbert spaces.

[14]See Eq. (4.2) of Ref. 8. The fifth and seventh terms on the right-hand side of that equation are combined into one term in Eq. (2.4) of the present paper.

[15]Reference 3, pp. 249, 250.

[16]L. Van Hove, in *The Quantum Theory of Fields,* Ref. 1, pp. 171–3.

[17]L. Van Hove, Physica 18, 145 (1952).

[18]J. Glimm, Commun. Math. Phys. 10, 1 (1968).

[19]Reference 3, p. 263.

[20]S. G. Eckstein, B. Varga, and S. Aks, Nuovo Cimento 8B, 451 (1972). Examples from quantum mechanics are given to show that infinite matrices which do not represent operators must be handled with care to avoid paradoxes.