Quasisecular Renormalization of the ϕ^4 Model of Quantum Field Theory

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The Heisenberg equations of the ϕ^4 model of quantum field theory in one space dimension are analyzed by a modified perturbation method. The modification focuses on terms with vanishing and small energy denominators. These terms give rise to operator-valued frequency (mass) renormalization and amplitude (wave-function) renormalization. These renormalization effects remain present in the limit of an infinite system because the number of terms with small energy denominators becomes infinite.

I. INTRODUCTION

The field equations of quantum field theories describing interacting particles are nonlinear partial differential equations whose solutions are operator-valued (i.e., q-number) functions of the independent (space-time) variables. Both the nonlinear and the operator aspects involve considerable complexity. Each by itself has been the object of considerable study.

The free-field quantum theory of noninteracting particles is described by a linear partial differential equation. The solution of the free-field problem is found through the introduction of the well-known creation- and destruction-operator formalism.[1] On the other hand, problems involving nonlinearity without the q-number complication (i.e., nonlinear c-number differential equations) are by themselves the object of study of a branch of mathematics.[2] The method of successive approximations (perturbation theory) has been found useful both in connection with problems of existence of solutions and in deriving approximations for solutions of differential equations whose nonlinearity is in some sense weak. The approximations obtained by straightforward perturbation theory are often qualitatively, as well as quantitatively, poor. For example, approximations to solutions of equations which can be proved to be periodic are found to be nonperiodic. Such problems are referred to as secular behavior of the approximation.[3] Significant progress has been made in modifying perturbation theory to eliminate these difficulties and obtain quantitatively meaningful solutions. It appears that the nonlinear aspect of quantum field theory may be a considerable part of the over-all problem. It is worthwhile noting in this respect that the principal method used to study the nonlinear field equations of quantum field theory is straightforward perturbation theory.[4]

It has been shown that secular behavior occurs in quantum field theory.[5] One of the ways that secular behavior can be avoided is through the introduction of adiabatic switching of the interaction.[6] This, however, may be an unfortunate way to handle the problem since the secular behavior of an approximation is an indication of physically interesting phenomena requiring detailed investigation. In particular, the switching must be applied selectively; the persistent self-interactions must not be switched. Adiabatic switching also precludes the occurrence of bound states.[7] The binding of particles like the self-interactions are persistent effects and should not be switched. This is the main objection to the use of adiabatic switching as a means of avoiding secular behavior. It may be useful to note that in renormalization theory adiabatic switching is modified[8] to account for self-interactions (mass renormalization), but not for binding essentially because the self-energies are infinite, whereas the binding energies are not. The appearance of infinities is apparently a more serious obstacle to the interpretation of quantum field theory than the failure to predict binding effects. The amazing success of quantum electrodynamics is in the calculation of the Lamb shift,[9] which is a small self-energy effect. However, the present knowledge of many new strongly interacting particles and resonances gives great importance to a theory allowing the possibility of bound states.

The presence of secular behavior is closely connected with the problem of vanishing denominators (hence the use of adiabatic switching which is designed to keep denominators from vanishing). In quantum field theory an additional problem arises, namely the existence of small (but nonvanishing) denominators. The presence of small denominators is in conflict with the basic rationale of perturbation theory, that higher-order corrections be small. We refer to terms with small denominators as quasisecular. It should be noted that even in those formulations of quantum field theory where the adiabatic hypothesis is avoided, such as the method of reduction formulas,[10] small denominators still occur when the fields are actually expanded via perturbation theory.

The object of this paper is to formulate a modified perturbation theory which avoids the occurrence of both vanishing and small denominators. We call this modification quasisecular perturbation theory.

Although the formulation is carried out only for the scalar, one-space-dimension model of quantum field theory[11] [i.e., the interaction Hamiltonian density $H_1 = -\frac{1}{4}\lambda\phi(x)^4$] it can be worked out for other boson field interactions and for the more interesting case of three space dimensions. The approximation developed in the paper is carried to the first nontrivial order, which for the ϕ^4 model is first order. The ϕ^4 interaction was chosen for this reason, in order to minimize calculational complications in presenting the essential ideas of the modified perturbation theory.

The approximate solution obtained is distinguished in a number of important ways: The solution exhibits amplitude (analogous to wave function) and frequency (analogous to mass) renormalizations; the renormalizations are given by q numbers rather than c numbers as in conventional perturbation theory; the renormalized amplitudes and frequencies are strongly dependent on the density of two-particle states in energy and momentum. This last result has as a consequence that field modes (hence particles) of approximately the same momentum can have an unexpectedly large coupling of a type somewhat similar to the pairing of opposite momentum electrons in the BCS theory of superconductivity.[12]

We close the introduction with an outline of the paper. In Sec. II the field equation and commutation relation are expressed in terms of the Fourier components of the field on the interval $-\frac{1}{2}L < x < \frac{1}{2}L$. The resulting array of interacting nonlinear quantum oscillators is used to set up the perturbation expansion. The first-order approximation is examined in detail in Sec. III. Quasisecular perturbation theory is introduced there. Particular attention is directed at the secular and quasisecular terms. The frequency renormalization operator arising from quasisecular perturbation is studied in Sec. IV. In Sec. V frequency renormalization is treated using an averaging or mean-value approximation. The result is expressed in terms of a density of (two-particle) states factor. The limit of a large system is investigated in Sec. VI. It is shown that the properties of the finite system described above are retained in that limit. This is an important distinction of quasisecular perturbation theory when compared with secular perturbation theory, since the number of exactly secular terms is (relatively) negligible in the limit of an infinite system, a consequence of which is that effects resulting from exact secularity vanish in that limit.[5] The density of states factor is studied in detail in Sec. VII. It is shown there that the density of states contributes an additional square root of λ (the ϕ^4 coupling constant) to the first-order correction to the renormalized frequencies. Thus the actual correction to the renormalized frequency varies as $\lambda^{3/2}$. Section VIII contains concluding remarks on further consequences of quasisecular perturbation theory.

II. COUPLED NONLINEAR FIELD OSCILLATORS

The field equation

$$(\Box + m^2)\phi = \lambda\phi^3\,, \tag{2.1}$$

where $\Box = \partial^2/\partial t^2 - \partial^2/\partial x^2$, is first studied on the interval $-\frac{1}{2}L \le x \le \frac{1}{2}L$. Using the Fourier decomposition

$$\phi(t, x) = L^{-1/2}\sum_l a_l(t)e^{ilx}\,, \tag{2.2}$$

where the sum goes over all numbers of the form $l = 2\pi n L^{-1}$, $n = 0, \pm1, \pm2, \ldots$, the field equation is transformed to an infinite array of nonlinearly coupled nonlinear oscillators. We examine these nonlinear equations using methods well known in the applied mathematics literature on nonlinear oscillator systems.[2]

The oscillator equations are given by

$$\ddot{a}_l + \omega_l^2 a_l = \lambda L^{-1} \sum_{l_1, l_2, l_3} a_{l_1}a_{l_2}a_{l_3}\delta_{l, l_1+l_2+l_3}\,, \tag{2.3}$$

where $\omega_l = (m^2 + l^2)^{1/2}$, $\ddot{a}_l = d^2a_l/dt^2$, and δ denotes the Kronecker function. The oscillator array is studied with the help of the expansion ansatz

$$a_l = \sum_{n\ge0} \lambda^n a_l^{(n)}\,. \tag{2.4}$$

With the aid of Eq. (2.4) each of the oscillator equations is converted to an infinite hierarchy of forced linear oscillator equations. Convergence considerations are not undertaken. We will in fact restrict attention to the first nontrivial order of the expansion and such terms of higher order in λ as are needed to give physically relevant approximations.

The field equation is further specified through the commutation relations

$$[\phi(0, x), \dot{\phi}(0, x')] = i\delta(x - x')\,, \tag{2.5a}$$

$$[\phi(0, x), \phi(0, x')] = [\dot{\phi}(0, x), \dot{\phi}(0, x')] = 0\,, \tag{2.5b}$$

and the reality condition

$$\phi(t, x) = \phi^*(t, x)\,. \tag{2.6}$$

Here * indicates the formal algebraic adjoint operation, which at this point is taken to be an antilinear involution. The adjoint will be further specified below when the space of states is considered.

In the oscillator formulation the equivalent commutation relations are given by

$$[a_k(0), \dot{a}_l(0)] = i\delta_{k,-l}\,, \tag{2.7a}$$

$$[a_k(0), a_l(0)] = [\dot{a}_k(0), \dot{a}_l(0)] = 0\,. \tag{2.7b}$$

The commutation relations take the role of initial conditions for the oscillator equations of motion. The reality condition is restated as

$$a_k(t)^* = a_{-k}(t)\,. \tag{2.8}$$

The equation for $a_l^{(0)}$,

$$\ddot{a}_l^{(0)} + \omega_l^{\,2} a_l^{(0)} = 0\,, \tag{2.9}$$

has the solution

$$a_l^{(0)}(t) = (2\omega_l)^{-1/2}(\alpha_l e^{-i\omega_l t} + \alpha^*_{-l} e^{i\omega_l t})\,, \tag{2.10}$$

where

$$[\alpha_k, \alpha_l^*] = \delta_{k,l}\,, \tag{2.11a}$$

$$[\alpha_k, \alpha_l] = [\alpha_k^*, \alpha_l^*] = 0\,. \tag{2.11b}$$

The commutation relations are chosen to satisfy the initial conditions Eqs. (2.7). The initial conditions for $a_l^{(n)}$, $n \geq 1$, are somewhat simpler:

$$a_l^{(n)}(0) = \dot{a}_l^{(n)}(0) = 0\,. \tag{2.12}$$

The equations they satisfy are, however, somewhat more involved.

$$\begin{aligned}\ddot{a}_l^{(1)} + \omega_l^{\,2} a_l^{(1)} &= L^{-1} \sum_{l_1 l_2 l_3} \delta_{l,l_1+l_2+l_3} a_{l_1}^{(0)} a_{l_2}^{(0)} a_{l_3}^{(0)} \\ &= L^{-1} \sum_{l_1 l_2 l_3} \delta_{l,l_1+l_2+l_3} (2\omega_{l_1} 2\omega_{l_2} 2\omega_{l_3})^{-1/2} \big[\alpha_{l_1}\alpha_{l_2}\alpha_{l_3} e^{-i(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})t} + \alpha^*_{-l_1}\alpha^*_{-l_2}\alpha^*_{-l_3} e^{i(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})t} \\ &\qquad + 3\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3} e^{i(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})t} + 3\alpha^*_{-l_3}\alpha^*_{-l_2}\alpha_{l_1} e^{i(\omega_{l_3}+\omega_{l_2}-\omega_{l_1})t}\big] \\ &\quad + 3(2L)^{-1} \sum_{l_1} \omega_{l_1}^{\,-1} a_l^{(0)}\,.\end{aligned} \tag{2.13}$$

The solutions of this equation are studied in Sec. III. In closing this section we note that the last term in Eq. (2.13) includes the series $\sum_{l_1} \omega_{l_1}^{\,-1}$ which is divergent. The term arises from the ϕ^3 term in Eq. (2.1) when the products of creation and destruction operators are reexpressed in normal order (creation operators to the left of destruction operators) as in Eq. (2.13). If ϕ^3 is defined in the first place as being in normal order the infinity is avoided (i.e., ϕ^3 is replaced by $:\phi^3:$). Alternatively, the last term in Eq. (2.13) can be treated as an infinite (c-number) mass renormalization term.

This divergent series may also be related to the zero-point energy $E_0 = \frac{1}{2}\sum_{l_1}\omega_{l_1}$ of the system, since

$$\sum_{l_1} \omega_{l_1}^{\,-1} = 2m^{-1}\frac{\partial E_0}{\partial m}\,. \tag{2.14}$$

It is assumed here that it makes sense to differentiate a divergent series term by term. It appears from Eq. (2.14) that the justification for dropping the last term of Eq. (2.13) involves essentially the same reasoning that permits the (more rapidly divergent series for the) zero-point energy to be set equal to zero. In the following the last term is dropped.

III. INTEGRATION OF THE FIRST-ORDER EQUATIONS

The terms on the right-hand side of Eq. (2.13) which are solutions of the homogeneous equation require special consideration. The remaining terms are easily treated. We first consider the solution of the equation

$$\ddot{b} + \omega_l^{\,2} b = c e^{i\omega t}\,, \tag{3.1}$$

when $|\omega| \neq \omega_l$. In accordance with Eq. (2.12) the initial conditions are taken as $b(0) = \dot{b}(0) = 0$. The solution is given by

$$b_1(t) = c(\omega_l^{\,2} - \omega^2)^{-1}[e^{i\omega t} - \tfrac{1}{2}(1 + \omega\omega_l^{\,-1})e^{i\omega_l t} - \tfrac{1}{2}(1 - \omega\omega_l^{\,-1})e^{-i\omega_l t}] \qquad (|\omega| \neq \omega_l)\,. \tag{3.2}$$

When $\omega = \pm\omega_l$ the solution of Eq. (3.1) takes on a very different form:

$$b_2(t) = \tfrac{1}{2}c\omega^{-1}[(2\omega_l)^{-1}e^{i\omega_l t} - (2\omega_l)^{-1}e^{-i\omega_l t} - ite^{i\omega t}] \qquad (\omega = \pm\omega_l)\,. \tag{3.3}$$

The magnitude of b_2 grows linearly with t for large t. While this (secular) behavior is clearly unphysical, the integration is nevertheless correct. In the standard perturbation theory the use of adiabatic switching circumvents this problem. However, it is the view of the authors that adiabatic switching is itself drastically unphysical and that the secular terms require very careful study.

In the limit of increasingly large L there also occur terms on the right-hand side of Eq. (2.13) for which ω comes very near to either ω_l or $-\omega_l$. This does not give rise to secularity, but it does result in small denominators in Eq. (3.2).

One of the main justifications for the use of perturbation procedures is that the higher-order contributions are small. This, however, is not the case for either large values of the time t in the secular terms or when small denominators are encountered. In the following we refer to the terms giving small denominators as quasisecular terms. (The justification for this terminology will be given below.) In closing this discussion of quasisecular terms we note that $a_l^{(1)}$ enters a_l [the solution of Eq. (2.3)] multiplied by λ. When the small denominators in the quasisecular terms are of magnitude λ, or smaller, we encounter a first-order correction which instead contributes to a_l as if it were of zeroth order. In the case where $\omega=\omega_l+\epsilon$, $|\epsilon|<|\lambda|/\omega_l$, we find

$$b_1(t)=-c\epsilon^{-1}(2\omega_l+\epsilon)^{-1}e^{i\omega_l t}\{e^{i\epsilon t}-\tfrac{1}{2}[1+(\omega_l+\epsilon)\omega_l^{-1}]-\tfrac{1}{2}[1-(\omega_l+\epsilon)\omega_l^{-1}]e^{-2i\omega_l t}\} \tag{3.4}$$

$$=b_2^+(t)+\epsilon\tilde{b}(t,\epsilon), \tag{3.5}$$

where $b_2^+(t)$ is the expression in Eq. (3.3) for $\omega=\omega_l$ and the error term $\tilde{b}$ is given by

$$\tilde{b}(t,\epsilon)=c(2\omega_l)^{-2}(it+\omega_l t^2)e^{i\omega_l t}-c(2\omega_l)^{-3}e^{i\omega_l t}+c(2\omega_l)^{-3}e^{-i\omega_l t}+O(\epsilon). \tag{3.6}$$

We see that the quasisecular solutions are approximately given by the secular solutions b_2 so long as t is sufficiently small [i.e., $\epsilon t(i+\omega_l t)$ must be small].

Summarizing to this point we find that the secular and quasisecular solutions are physically reasonable only for sufficiently small values of the time t and for such small values of t the quasisecular solutions are well approximated by the (exactly) secular solutions Eq. (3.3).

All of the terms whose time dependence is of the form $e^{\pm i(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})t}$ are nonsecular and have for their integral Eq. (3.2). We write the results of integrating these terms as

$$A_l^-(t)=L^{-1}\sum_{l_1l_2l_3}\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}[\omega_l^2-(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})^2]^{-1}\alpha_{l_1}\alpha_{l_2}\alpha_{l_3}$$
$$\times\{e^{-i(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})t}-\tfrac{1}{2}[1-(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})\omega_l^{-1}]e^{i\omega_l t}-\tfrac{1}{2}[1+(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})\omega_l^{-1}]e^{-i\omega_l t}\}, \tag{3.7a}$$

$$A_l^+(t)=L^{-1}\sum_{l_1l_2l_3}\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}[\omega_l^2-(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})^2]^{-1}\alpha^*_{-l_1}\alpha^*_{-l_2}\alpha^*_{-l_3}$$
$$\times\{e^{i(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})t}-\tfrac{1}{2}[1+(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})\omega_l^{-1}]e^{i\omega_l t}-\tfrac{1}{2}[1-(\omega_{l_1}+\omega_{l_2}+\omega_{l_3})\omega_l^{-1}]e^{-i\omega_l t}\}. \tag{3.7b}$$

The terms with time dependence given by $e^{\pm i(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})t}$ may or may not be secular. The quasisecular and exactly secular terms will be treated as secular terms and are integrated using Eq. (3.3). As noted above, this is reasonable so long as t is sufficiently small. But as also noted above the secular terms are physically unreasonable for $t>|\lambda|^{-1}\omega_l$. We fix the definition of the quasisecular terms in such a way that for $t\le|\lambda|^{-1}\omega_l$ the quasisecular terms are well approximated by the integral of the exactly secular terms Eq. (3.3). If we let $\omega=-(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})$ and $\epsilon=\omega-\omega_l$, as above, the condition for the validity of the approximation is given by

$$|\lambda\epsilon\omega_l^{-2}(it_M+\omega_l t_M^2)|\le 1, \tag{3.8}$$

where $t_M=|\lambda|^{-1}m$ is the maximum time set by the inequality $|t|\le|\lambda|^{-1}\omega_l$. We assume in addition that $|\lambda|\ll m^{1/2}$, which implies that $\omega_l t_M\gg 1$. With this the inequality can be restated as

$$|\omega_{l_2}+\omega_{l_3}-\omega_{l_1}-\omega_l|=|\epsilon|\le|\lambda'|=m^{-1}|\lambda|. \tag{3.9}$$

This inequality provides the criterion for the quasisecular terms. It is convenient to define $\lambda'=m^{-1}\lambda$.

The integrals of the nonsecular terms in Eq. (2.13) whose time dependence is given by $e^{\pm i(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})t}$ are

$$B_l^-(t)=3L^{-1}\sum_{l_1l_2l_3}{}'\,\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}[\omega_l^2-(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})^2]^{-1}\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3}$$
$$\times\{e^{i(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})t}-\tfrac{1}{2}[1+(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})\omega_l^{-1}]e^{i\omega_l t}-\tfrac{1}{2}[1-(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})\omega_l^{-1}]e^{-i\omega_l t}\}, \tag{3.10a}$$

$$B_l^+(t)=3L^{-1}\sum_{l_1l_2l_3}{}' \delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}[\omega_l^2-(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})^2]^{-1}\alpha^*_{-l_3}\alpha^*_{-l_2}\alpha_{l_1}$$
$$\times\{e^{-i(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})t}-\tfrac{1}{2}[1-(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})\omega_l^{-1}]e^{i\omega_l t}-\tfrac{1}{2}[1+(\omega_{l_1}-\omega_{l_2}-\omega_{l_3})\omega_l^{-1}]e^{-i\omega_l t}\}. \quad (3.10b)$$

The prime on the summation signs indicates that the sums are restricted to those values l_1, l_2, and l_3 for which the inequality Eq. (3.9) is *not* satisfied.

The quasisecular (including secular) terms have as their integrals

$$C_l^-(t)=-\tfrac{3}{2}(\omega_l L)^{-1}\sum_{l_1l_2l_3}{}'' \delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3}[(2\omega_l)^{-1}e^{i\omega_l t}-(2\omega_l)^{-1}e^{-i\omega_l t}-ite^{-i\omega_l t}], \quad (3.11a)$$

$$C_l^+(t)=\tfrac{3}{2}(\omega_l L)^{-1}\sum_{l_1l_2l_3}{}'' \delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha^*_{-l_3}\alpha^*_{-l_2}\alpha_{l_1}[(2\omega_l)^{-1}e^{i\omega_l t}-(2\omega_l)^{-1}e^{-i\omega_l t}-ite^{i\omega_l t}]. \quad (3.11b)$$

The double prime on the summation signs indicates that the sums are over the values of l_1, l_2, and l_3 for which the inequality (3.9) is satisfied.

The solution of Eq. (2.13) subject to the initial conditions Eq. (2.12) is then

$$a_l^{(1)}=A_l^-+A_l^++B_l^-+B_l^++C_l^-+C_l^+. \quad (3.12)$$

The first-order approximation to a_l is given by $a_l\approx a_l^{(0)}+\lambda a_l^{(1)}$. The approximation we have obtained for a_l is studied in detail below. Particular attention will be directed to the limit $L\to\infty$.

IV. FREQUENCY RENORMALIZATION OPERATOR

The terms in $a_l\approx a_l^{(0)}+\lambda a_l^{(1)}$ which vary as $e^{\pm i\omega_l t}$ and $te^{\pm i\omega_l t}$ are of particular interest, thus Eq. (3.12) is rewritten to make this dependence more explicit. We write

$$a_l(t)=U_l^-e^{-i\omega_l t}+U_l^+e^{i\omega_l t}+iV_l^-te^{-i\omega_l t}-iV_l^+te^{i\omega_l t}+W_l(t). \quad (4.1)$$

The quantities U_l^-, U_l^+, V_l^-, and V_l^+ are defined below:

$$U_l^-=(2\omega_l)^{-1/2}\alpha_l-\lambda(2\omega_l L)^{-1}\sum_{l_1l_2l_3}\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}(\omega_l-\omega_{l_1}-\omega_{l_2}-\omega_{l_3})^{-1}\alpha_{l_1}\alpha_{l_2}\alpha_{l_3}$$
$$-\lambda(2\omega_l L)^{-1}\sum_{l_1l_2l_3}\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}(\omega_l+\omega_{l_1}+\omega_{l_2}+\omega_{l_3})^{-1}\alpha^*_{-l_1}\alpha^*_{-l_2}\alpha^*_{-l_3}$$
$$-\lambda 3(2\omega_l L)^{-1}\sum_{l_1l_2l_3}{}'\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}(\omega_l+\omega_{l_1}-\omega_{l_2}-\omega_{l_3})^{-1}\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3}$$
$$-\lambda 3(2\omega_l L)^{-1}\sum_{l_1l_2l_3}{}'\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}(\omega_l-\omega_{l_1}+\omega_{l_2}+\omega_{l_3})^{-1}\alpha^*_{-l_3}\alpha^*_{-l_2}\alpha_{l_1}$$
$$+\lambda 3(2\omega_l)^{-2}L^{-1}\sum_{l_1l_2l_3}{}''\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3}$$
$$-\lambda 3(2\omega_l)^{-2}L^{-1}\sum_{l_1l_2l_3}{}''\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha^*_{-l_3}\alpha^*_{-l_2}\alpha_{l_1}, \quad (4.2)$$

$$U_l^+=[U_{-l}^-]^*, \quad (4.3)$$

$$V_l^-=3\lambda(2\omega_l L)^{-1}\sum_{l_1l_2l_3}{}''\delta_{l,l_1+l_2+l_3}(8\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha^*_{-l_1}\alpha_{l_2}\alpha_{l_3}, \quad (4.4)$$

$$V_l^+=[V_{-l}^-]^*. \quad (4.5)$$

Each of the terms in the remainder W_l contains exponential factors $\exp[\pm i(\omega_{l_1}\pm\omega_{l_2}\pm\omega_{l_3})t]$, where $||\omega_{l_1}\pm\omega_{l_2}\pm\omega_{l_3}|-\omega_l|>|\lambda'|$. Thus, there is a gap in the spectrum of Fourier frequencies of W_l on both sides of $\pm\omega_l$. The gap enables us to extract for more detailed study the terms whose time dependence is $e^{\pm i\omega_l t}$ (or $te^{\pm i\omega_l t}$). Since the gap width is independent of L, the problems connected with the accumulation of Fourier frequencies with increasing L are avoided. This will facilitate the study of the limit $L\to\infty$.

The coefficients $U_l^\pm$ are related to the wave-function renormalization constant occurring in the standard formulation of quantum field theory.[13] We note that in this treatment $U_l^\pm$ contain operators in contrast to the standard formulation where-in the wave-function renormalization constants are numerically valued.

It is also possible to express Eq. (4.1) with renormalized frequencies (essentially mass renormalization) and here again operator quantities appear. The operator U_l^-, like the operator α_l, does not have a left inverse (since α_l maps the vacuum state into zero in the standard particle-number representation), but along with α_l and U_l^- can have a right inverse. A right inverse exists for an operator which is onto (or surjective). To investigate the existence of the right inverse we write $U_l^- = (2\omega_l)^{-1/2}\alpha_l + \lambda F_l$. The right inverse of $(2\omega_l)^{-1/2}\alpha_l$ is $(2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1}$. We seek an operator $R_l^- = (2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1} + \lambda G_l$ such that $U_l^- R_l^- = I$. This condition is satisfied (to order λ inclusive) by solving for G_l in

$$(2\omega_l)^{-1/2}\alpha_l G_l + F_l(2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1} = 0. \tag{4.6}$$

To facilitate the solution we substitute $G_l = \alpha_l^*\tilde{G}_l$ and, using the existence of an inverse for $\alpha_l\alpha_l^* = (\alpha_l^*\alpha_l+1)$, obtain

$$R_l^- = (2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1} - \lambda(2\omega_l)\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1} \times F_l\alpha_l^*(\alpha_l^*\alpha_l+1)^{-1} + O(\lambda^2), \tag{4.7}$$

where $\lambda F_l = U_l^- - (2\omega_l)^{-1/2}\alpha_l$.

Similarly, α_l^* and U_l^+ have left inverses and we write $R_l^+ U_l^+ = I$, where

$$R_l^+ = [R_{-l}^-]^*. \tag{4.8}$$

Equation (4.1) takes the form

$$a_l(t) = U_l^-(I + iR_l^-V_l^-t)e^{-i\omega_l t} + e^{i\omega_l t}(I - iV_l^+R_l^+t)U_l^+ + W_l(t). \tag{4.9}$$

The terms enclosed in parentheses can be taken as (first order in λ) approximations to exponentials, particularly for $|\lambda t|$ small as required above. We get

$$a_l(t) = U_l^- e^{-i(\omega_l - R_l^-V_l^-)t} + e^{i(\omega_l - V_l^+R_l^+)t}U_l^+ + W_l(t). \tag{4.10}$$

The operator nature of the frequency renormalization is apparent in Eq. (4.10).

In Sec. V the frequency (or mass) renormalization terms are examined in detail.

V. SIMPLIFIED FORM OF THE RENORMALIZED FREQUENCY OPERATOR

From Eqs. (4.4), (4.5), (4.7), and (4.8) we have to order λ

$$R_l^- V_l^- = \lambda 3(4L)^{-1} \sum_{l_1 l_2 l_3}'' \delta_{l, l_1+l_2+l_3}(\omega_l\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha_l^*(\alpha_l^*\alpha_l+I)^{-1}\alpha_{-l_1}^*\alpha_{l_2}\alpha_{l_3}, \tag{5.1}$$

$$V_l^+ R_l^+ = \lambda 3(4L)^{-1} \sum_{l_1 l_2 l_3}'' \delta_{l, l_1+l_2+l_3}(\omega_l\omega_{l_1}\omega_{l_2}\omega_{l_3})^{-1/2}\alpha_{-l_3}^*\alpha_{-l_2}^*\alpha_{l_1}(\alpha_{-l}^*\alpha_{-l}+I)^{-1}\alpha_{-l}. \tag{5.2}$$

The summations in Eq. (5.1) can be written in a symmetric form as

$$R_l^- V_l^- = \lambda 3(4L)^{-1}\sum_{k,p}''(\omega_l\omega_k\,\omega_{(k+l)/2+p}\,\omega_{(k+l)/2-p})^{-1/2}(\alpha_l^*\alpha_l)^{-1}\alpha_l^*\alpha_k^*\alpha_{(k+l)/2+p}\,\alpha_{(k+l)/2-p}. \tag{5.3}$$

The restriction on the sums indicated by the double prime is given by (3.9) and, in terms of the new variables, becomes

$$|\omega_l + \omega_k - \omega_{(k+l)/2+p} - \omega_{(k+l)/2-p}| \le |\lambda'|. \tag{5.4}$$

The inequality is satisfied for p in sufficiently small neighborhoods of $p = \pm\frac{1}{2}(k-l)$. We expand about these points and approximate Eq. (5.4) by keeping terms to quadratic order.

$$\left|\,|k\omega_k^{-1} - l\omega_l^{-1}|(\pm p - \tfrac{1}{2}|k-l|) + \tfrac{1}{2}m^2(\omega_k^{-3}+\omega_l^{-3})(\pm p - \tfrac{1}{2}|k-l|)^2\right| \le |\lambda'|. \tag{5.5}$$

Here $\pm p$ denotes that if either $+p$ in both places or $-p$ in both places satisfies Eq. (5.5) then Eq. (5.4) is also satisfied. The quadratic approximation leads to small errors of order λ/m^2 in the limits on p given by Eq. (5.4). The coefficient of the linear term vanishes for $k = l$, hence the left-hand side increases slowly for k near l. This implies that many terms contribute to the p summation in Eq. (5.3) for k near l.

At this point we consider a further simplifying restriction. In particular, we require that (expectation values of) α_k and α_k^* vary continuously in the parameter k (for all normalizable states). This is a restriction on the representation space. With this, assuming that λ and hence $|p| - \frac{1}{2}|k-l|$ is sufficiently small [according to (5.5)], we approximate Eq. (5.3) by

$$R_l^- V_l^- = 3\lambda(4L)^{-1}\sum_{k,p}{}''(\omega_l\omega_k)^{-1}(\alpha_l^*\alpha_l)^{-1}\alpha_l^*\alpha_k^*\alpha_l\alpha_k . \tag{5.6}$$

We can simplify by using $\alpha_k^*\alpha_l = \alpha_l\alpha_k^* - \delta_{k,l}$, and then the operators in the summation become $(\alpha_l^*\alpha_l)^{-1}(\alpha_l^*\alpha_l\alpha_k^*\alpha_k - \alpha_l^*\alpha_l\delta_{l,k})$; however, we have to be careful in canceling the inverse since $(\alpha_l^*\alpha_l)^{-1}$ only exists as a mapping on the domain which is the subspace of the Hilbert space of states in which there is at least one particle present in the momentum state l. In fact, $(\alpha_l^*\alpha_l)^{-1}\alpha_l^*\alpha_l$ is the unit operator on this subspace, and vanishes on the orthogonal subspace of states in which no particle is present in the state l. Therefore, $(\alpha_l^*\alpha_l)^{-1}\alpha_l^*\alpha_l = I - P_l^0$, where P_l^0 is the orthogonal projection operator on the latter subspace, and Eq. (5.6) becomes

$$R_l^- V_l^- = 3\lambda(4L)^{-1}\sum_{k,p}{}''(\omega_l\omega_k)^{-1}(I - P_l^0)(\alpha_k^*\alpha_k - \delta_{l,k}) . \tag{5.7}$$

We see that the approximated operator $R_l^- V_l^-$ vanishes on all the one-particle states, as was the case in Eq. (5.6). Note that $P_l^0\alpha_l^*\alpha_l = 0$ so that

$$[R_l^- V_l^-]^* = R_l^- V_l^- . \tag{5.8}$$

Then the renormalized frequency operator defined by Eq. (4.10)

$$\Omega_l = \omega_l - R_l^- V_l^- \tag{5.9}$$

is (algebraically) self-adjoint. Thus we replace Eq. (4.10) by

$$a_l(t) = U_l^- e^{-i\Omega_l t} + e^{i\Omega_{-l} t}U_l^+ + W_l(t) \tag{5.10}$$

because $V_l^+R_l^+ = (R_{-l}V_{-l})^*$.

We recall certain properties of the terms appearing in Eq. (5.10). The Fourier transformation of W_l vanishes in the frequency intervals $-\omega_l - |\lambda'| < \omega < -\omega_l + |\lambda'|$ and $\omega_l - |\lambda'| < \omega < \omega_l + |\lambda'|$. The term $U_l^- e^{-i\Omega_l t}$ is the renormalized approximation of $(U_l^- + iV_l^- t)e^{-i\omega_l t}$ whose (singular) Fourier transformation is nonvanishing only at the point ω_l. Similarly the term $e^{i\Omega_{-l} t}U_l^+$ is the renormalized approximation of $(U_l^+ - iV_l^+ t)e^{i\omega_l t}$ whose Fourier transformation vanishes except at the point $-\omega_l$.

We close this section with a change of notation to facilitate the study of the limit $L\to\infty$. Let σ_λ be the characteristic function of the set of points $\{(l,k,p)\}$ satisfying the inequality (5.4) or the (approximate) inequality (5.5). Then we have $\sigma_\lambda(l,k,p) = 1$, if the inequality is satisfied and $\sigma_\lambda(l,k,p) = 0$, otherwise. The double prime on the summation in Eq. (5.7) may be dropped by using σ_λ for the renormalized frequency, giving

$$\Omega_l = \omega_l - 3\lambda(4\omega_l L)^{-1}(I - P_l^0) \times \sum_{k,p}\omega_k^{-1}\sigma_\lambda(l,k,p)(\alpha_k^*\alpha_k - \delta_{k,l}) . \tag{5.11}$$

We define the phase-space function ρ_λ by

$$\sum_p \sigma_\lambda(l,k,p) = L\rho_\lambda(l,k) . \tag{5.12}$$

Then $L\rho_\lambda(l,k)$ gives the number of independent two-particle states (of mass m) with energy E lying within the range $\omega_l + \omega_k - |\lambda'| < E < \omega_l + \omega_k + |\lambda'|$ and having total momentum $l+k$. In the limit of large L, the phase-space function becomes independent of L, and Eq. (5.11) can then be written as

$$\Omega_l = \omega_l - 3\lambda(4\omega_l)^{-1}(I - P_l^0) \times \sum_k \omega_k^{-1}\rho_\lambda(l,k)(\alpha_k^*\alpha_k - \delta_{k,l}) . \tag{5.13}$$

VI. LIMIT OF A LARGE SYSTEM

If the terms with small denominators were not treated as quasisecular, Eq. (5.6) would include contributions from only the exactly secular terms. The function σ_λ would be replaced by σ_0 (i.e., σ_λ with $\lambda = 0$) and the summations would include only two terms in p for each value in k. Then the right-hand side of Eq. (5.6) would vanish in the limit $L\to\infty$, and the operator frequency renormalization would be lost. Thus, the quasisecular terms play a crucial role in the existence of operator renormalizations.

We start the study of the limit $L\to\infty$ with a short review.[14] The large L limit involves changes in the commutation relations as well as the replacement of summations by integrations. The Fourier decomposition Eq. (2.2) can be written as

$$\phi(t,x) = (2\pi)^{-1/2}\sum_l (\Delta l)\left(\frac{L}{2\pi}\right)^{1/2} a_l(t)e^{ilx} , \tag{6.1}$$

where $\Delta l = 2\pi L^{-1}$. In the limit $L\to\infty$ we write

$$\phi(t,x) = (2\pi)^{-1/2}\int dl\, a(l,t)e^{ilx} , \tag{6.2}$$

where

$$\left(\frac{L}{2\pi}\right)^{1/2} a_l(t) \to a(l,t) . \tag{6.3}$$

Here both L and n go to infinity in such a way that $2\pi n L^{-1}$ tends to l.

The commutation relation Eq. (2.11a) is written as

$$\left[\left(\frac{L}{2\pi}\right)^{1/2}\alpha_k, \left(\frac{L}{2\pi}\right)^{1/2}\alpha_l^*\right] = \frac{L}{2\pi}\delta_{k,l} . \tag{6.4}$$

If $L\to\infty$ is taken as above the commutation relation becomes

$$[\alpha(k), \alpha(l)^*] = \delta(k-l), \tag{6.5}$$

where again we take

$$\left(\frac{L}{2\pi}\right)^{1/2} \alpha_l \rightarrow \alpha(l), \tag{6.6}$$

where L and n go to infinity so that $2\pi n L^{-1}$ tends to l. We have used

$$1 = \sum_k \delta_{k,l} = \sum_k (\Delta k) \frac{L}{2\pi} \delta_{k,l} \xrightarrow[L \to \infty]{} \int dk\, \delta(k-l), \tag{6.7}$$

where $\Delta k = 2\pi L^{-1}$.

We are ready to treat the $L \rightarrow \infty$ behavior of Ω_l. Using (6.6) in Eq. (6.13) we find

$$\Omega_l = \omega_l - 3\lambda(4\omega_l)^{-1} \times \int dk\, \omega_k^{-1} \rho_\lambda(k, l) [\alpha^*(k)\alpha(k) - \delta(k-l)] . \tag{6.8}$$

The operator renormalization clearly remains in the limit $L \rightarrow \infty$. We take $I - P_l^0$ equal to the identity in the limit $L \rightarrow \infty$.

VII. PHASE-SPACE FUNCTION

The phase-space function $\rho_\lambda(l, k)$ determines the renormalized frequency operator through Eq. (6.8), so it is desirable to have a simple picture of its behavior. Exact analysis of the inequality (5.5) which determines ρ_λ is very difficult, however, for $|k-l|$ sufficiently small the analysis is made easy because the linear term in p vanishes approximately on the left-hand side of (5.5). To see this, note that the coefficient of the first term in p is

$$k\omega_k^{-1} - l\omega_l^{-1} = (k-l) \frac{d}{dq}(q\omega_q^{-1}), \tag{7.1}$$

where q is some point in the closed interval with end points l and k, by the mean-value theorem of calculus. The derivative is $(d/dq)(q\omega_q^{-1}) = m^2 \omega_q^{-3}$ and if $|k-l|$ is sufficiently small, it can be approximated by $\frac{1}{2}(\omega_l^{-3} + \omega_k^{-3})$ which is the mean of the values at the two end points. This approximation makes the linear term in p vanish. The relative error in the phase-space function induced by the approximation is equal to the relative error in the approximation itself, i.e.,

$$e_r = \frac{|\omega_q^{-3} - \frac{1}{2}(\omega_l^{-3} + \omega_k^{-3})|}{\frac{1}{2}(\omega_l^{-3} + \omega_k^{-3})} .$$

Since q lies between l and k, e_r is less than $|\omega_k^{-3} - \omega_l^{-3}| / |\omega_k^{-3} + \omega_l^{-3}|$, so if we require this expression to be less than δ (say $\delta = 0.1$) then also $e_r \leq \delta$. The requirement leads to the condition

$$|k-l| \leq [(1+\gamma)l^2 + \gamma m^2] - |l|, \tag{7.2}$$

where

$$\gamma = \frac{(1+\delta)^{2/3}}{(1-\delta)^{2/3} - 1} \approx \tfrac{4}{3}\delta .$$

In the extreme nonrelativistic limit, $|l| \ll m\sqrt{\gamma}$, the condition (7.2) becomes $|k-l| \ll m\sqrt{\gamma}$, while in the opposite limit $|l| \gg m\sqrt{\gamma}$ it becomes $|k-l| \leq \frac{1}{2}\gamma |l|$. The condition $|k-l| \leq \frac{1}{2}\gamma\omega_l$ provides a simpler form of bound for all l, but it is too restrictive in the nonrelativistic region where the bound is proportional to $\sqrt{\gamma}$ rather than γ.

After approximation the inequality (5.5) becomes

$$|p^2 - \tfrac{1}{4}|k-l|^2| \leq A^{-1}|\lambda'|, \tag{7.3}$$

where $A = \frac{1}{2}m^2(\omega_k^{-3} + \omega_l^{-3})$. For small values of $|k-l|$ the parabola $p^2 - \frac{1}{4}|k-l|^2$ cuts only the upper boundary line $p = A^{-1}|\lambda'|$ and

$$\rho_\lambda(l, k) = \pi^{-1}[A^{-1}|\lambda'| + \tfrac{1}{4}|k-l|^2]^{1/2} . \tag{7.4}$$

This expression is valid as long as $|k-l| \leq 2A^{-1/2}|\lambda'|^{1/2}$, so that in the nonrelativistic region, where $A = m^{-1}$, we require $|k-l| \leq 2m^{1/2}|\lambda'|^{1/2}$. On the other hand, in the highly relativistic region $|k-l| \leq 2|\lambda'|^{1/2} m^{-1} l^{3/2}$ (provided that $|\lambda'|^{1/2} \ll m^{1/2}$), so that the limit on the validity of approximation given by (7.2) is reached before the limit on the validity of the expression (7.4). This is somewhat unfortunate because (7.4) indicates that $\rho_\lambda(l, k)$ is monotonically increasing as $|k-l|$ increases from 0 to the limit, and at the limit $\rho_\lambda(l, k) = \sqrt{2}\,\rho_\lambda(l, l) = \sqrt{2}\,\pi^{-1} A^{-1/2}|\lambda'|^{1/2}$. As a result our approximation is too crude to give a complete picture of the function $\rho_\lambda(l, k)$ in the relativistic region.

For $|k-l|$ larger than the limit of validity of (7.4) the parabola cuts the lower boundary $p = -A^{-1}|\lambda'|$ as well and

$$\rho_\lambda(l, k) = \pi^{-1}(A^{-1}|\lambda'| + \tfrac{1}{4}|k-l|^2)^{1/2} - \pi^{-1}(\tfrac{1}{4}|k-l|^2 - A^{-1}|\lambda'|), \tag{7.5}$$

if $|k-l| \geq 2A^{-1/2}|\lambda'|^{1/2}$. As $|k-l|$ increases from this value, the function drops down, at first with infinite slope and then decreases more and more slowly toward zero. For $|k-l|$ very large, the asymptotic behavior is $2\pi^{-1}A^{-1}|\lambda'||k-l|^{-1}$, which is a slowly decreasing tail of magnitude proportional to $|\lambda'|$, and hence small compared to the maximum value of the function which is proportional to $|\lambda'|^{1/2}$. The approximation gives a fair picture of the phase-space function in the nonrelativistic region if $|\lambda'| m^{-1} \ll \gamma$ because at the limit of the approximation $|k-l| = m\sqrt{\gamma}$ the value of the function is

$$\rho_\lambda(l,k) = 2\pi^{-1}|\lambda'|\gamma^{-1/2} = 2|\lambda'|^{1/2}m^{-1/2}\gamma^{-1/2}\rho_\lambda(l,l),$$

so that it has decreased by a reasonable factor of $(|\lambda'|/m\gamma)^{1/2}$ from its value at $k=l$. However, even if we choose $\gamma = 0.1$ for a 10% accuracy in ρ_λ and if $|\lambda'| = 0.01m$ which is a rather small coupling constant then $(|\lambda'|/m\gamma)^{1/2} = 0.31$ is still not a very small factor. This indicates the restrictive nature of the approximation. We can, however, draw some conclusions about ρ_λ: (1) the maximum of ρ_λ is proportional to $|\lambda|^{1/2}$; (2) the width of ρ_λ is proportional to $|\lambda|^{1/2}$; (3) the function first increases as $|k-l|$ increases, and then decreases, but rather slowly with a long tail.

VIII. CONCLUDING REMARKS

It is shown in the preceding work that the inclusion of quasisecular terms (small energy denominators) is important in renormalization. These terms contribute a nonvanishing frequency (mass) and amplitude (wave-function) renormalization in the limit of an infinite system. The renormalized frequency operator depends on a certain phase-space function giving the density of two-particle states in energy and momentum. The form of the phase-space function shows that field modes of nearly the same momentum have a large coupling to each other. This situation is somewhat analogous to the situation in the BCS theory of superconductivity where electrons of opposite momentum are strongly coupled.[12] However, the BCS model neglects interaction between electron pairs of nonzero total momentum. Recently the model was extended to allow for the interaction of pairs of total momentum slightly differing from zero.[15] It was found that the phase space available for scattering of pairs is sharply peaked at zero total momentum, and that repeated interactions (leading to binding of the pairs) emphasize more and more the peak of the phase-space function. It is expected that the phase-space function of Sec. VII will play a similar role when the two-particle states of the ϕ^4 model are calculated. In that case the peaking of the phase-space function for modes of nearly the same momentum could be related to binding effects between two co-moving particles.

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