[8]See, e.g., M. Scadron, Phys. Rev. 165, 1640 (1968).

[9]It is worth noting that for physical processes M is a scalar. When vector particles, e.g., photons, are involved a vector index may be contracted with a polarization vector ϵ_μ. Thus M involves invariants such as $\epsilon\cdot p'$ which must also be evaluated numerically. The sum over polarization directions for the vector particle must now be done rather more explicitly than in the usual method. Similar comments apply to higher-spin bosons.

[10]For purposes of counting we take $\sigma_{\mu\nu}=i(\gamma_\mu\gamma_\nu-g_{\mu\nu})$ as contributing two terms to r and multiply out all other brackets appearing in Γ_M. For specific calculations in *either* method, one of course will use as many specific tricks, redefinitions, etc., as possible to minimize the number of traces to be computed. The values of r and n we quote are perhaps slightly ambiguous, as we do not give the exact form of the matrix element used to compute them. They should, however, give an approximately correct idea of the relative number of terms involved in the various processes.

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Effect of Anomalies on Quasi-Renormalizable Theories*

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Apparently nonrenormalizable field theories, such as the new models for weak interactions, can become renormalizable when gauge invariance of the second kind is present. However, the anomaly associated with the axial-vector current may destroy this gauge invariance in perturbation theory, even though it is present in the Lagrangian. When this happens the theory remains nonrenormalizable. Nevertheless it is possible, by enlarging the theory, to remove the anomaly at the expense of introducing additional fermion fields, which correspond to as-yet-unobserved particles.

I. INTRODUCTION

In addition to the usual renormalizable and super-renormalizable field theories, there exist models which apparently yield, in conventional perturbation theory, a finite, well-defined, and unitary S matrix, even though superficial estimates of degree of divergence indicate nonrenormalizability. We call such theories "quasi-renormalizable." A classic example is a massive vector meson coupled to a conserved current.[1] It has been conjectured some time ago that a spontaneously broken gauge theory of the weak interactions also is quasi-renormalizable,[2] and recently arguments have been presented in support of this conclusion.[3]

The essential ingredient, which may convert an apparently nonrenormalizable theory into a quasi-renormalizable one, is gauge invariance of the second kind. In the massive-vector-meson example, the invariance, although "weakly" broken by the meson mass, is sufficiently operative to effect this desirable state of affairs. Similarly in the weak-interaction theories, the gauge principle, though spontaneously broken, allows the argument to proceed to a successful conclusion.

In this paper we demonstrate that the anomalies of the axial-vector current[4] can invalidate the "proof" of quasi-renormalizability. These anomalies, which are in general present when there are fermions in the model, destroy gauge invariance of the second kind. One way of understanding their origin is to observe that any theory, renormalizable or not, must be regulated when perturbative calculations are performed. If the regulator procedure is non-gauge-invariant, there is no guarantee that the gauge principle will be regained when the regulators are removed, for example by sending the mass of the regulator fields to infinity. The anomalies of the axial-vector current are thus a consequence of the absence of a chirally invariant regulator procedure for fermions. However, it must be emphasized that this difficulty is not merely technical; one must not entertain the hope that eventually a proper regulator procedure will be found. It can be shown in certain models that the gauge principle makes theoretically testable predictions, which are violated by explicit calculations.[5] Thus the only way to remove the anomaly is to change the theory. Since quasi-renormalizability is clearly a desirable feature, the anomaly places an important constraint on model building: The theory must be anomaly-free.

In Sec. II we examine a theory of massive vector mesons coupled to an axial-vector current which is constructed from massless fermions. The apparent conservation of this current suggests that the theory is quasi-renormalizable, analogous to the vector interaction case. However, the anomaly of the axial-vector current renders the theory nonrenormalizable. Furthermore, we show that the theory does not possess a zero-boson-mass limit, which in turn means that a theory of *massless* vector mesons axially coupled to fermions is internally inconsistent. A Feynman graph is presented which explicitly exhibits the difficulty. This investigation lays the groundwork for the subsequent discussion, which follows quite closely the pattern exhibited in this simple example.

Section III is devoted to the study of an Abelian version of the spontaneously broken gauge theory of the weak interactions.[6] Such theories can be studied in three stages. In Stage *i*, one has a gauge theory of massless vector mesons coupled to conserved currents. At this stage the theory is apparently renormalizable. In Stage *ii* one allows the symmetry to be spontaneously broken by the vacuum. The vector mesons acquire masses but the theory is still apparently renormalizable. The would-be Goldstone bosons should decouple from physical states due to the Ward identities which incorporate the content of the (broken) gauge invariance. This cancellation is most easily seen by passing to Stage *iii*, in which one exploits the underlying gauge invariance to redefine the fields so as to eliminate explicitly the Goldstone-boson fields. The theory in Stage *iii* is thus manifestly unitary.

After reviewing these standard arguments we show, for the above model, that (1) The theory at Stage *i* is not internally consistent in that it involves massless vector mesons coupled to nonconserved currents. (2) If one ignores this problem and passes to Stage *ii* the theory is either nonrenormalizable or beset by the presence of zero-mass, unphysical singularities. (3) The manifestly unitary theory at Stage *iii*, which one would naively derive, is not renormalizable.

In Sec. IV we show that these problems can be cured by doubling the number of fermions. If the new fermions have opposite axial-vector couplings the anomalies are canceled. In that case the Stage *i* Lagrangian can be written, with a suitable redefinition of fermion fields, with only parity-conserving vector-meson couplings. (We call this Stage 0.) Parity is then broken only by the different couplings of the fermions to the scalar mesons, which leads in Stage *ii* to different masses. We then argue that the Stage *ii* theory is renormalizable and contains no zero-mass singularities. The new set of fermions can have arbitrary mass. However, they are not to be regarded as regulators and their mass cannot be taken to be infinite. We also observe that once one employs this anomaly-removing technique it is natural to construct theories in which parity is spontaneously broken.

Finally in Sec. V we extend the discussion to the non-Abelian case. The anomalies again destroy, in general, the possibility of constructing a renormalizable theory. Once again one must double the number of fundamental fermions in order to remove the anomalies. We discuss the practical feasibility of this procedure, and investigate the possibility of arranging the hadronic and leptonic anomalies to cancel.

II. A MASSIVE-VECTOR-MESON THEORY

Consider a theory described by the Lagrangian

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}\not\partial\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mu^2 A^\mu A_\mu - gJ_5^\mu A_\mu, \\ J_5^\mu &= \bar{\psi}\gamma^\mu\gamma_5\psi, \\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu . \end{aligned} \tag{2.1}$$

A conventional quantization leads to a vector-meson propagator of the form

$$\frac{-i}{k^2-\mu^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{\mu^2}\right),$$

rendering the theory nonrenormalizable.[7] However, the apparent conservation of J_5^μ raises the hope that a Stueckelberg transformation can be performed, which decouples the longitudinal degree of freedom of the vector meson (which is responsible for the $k^\mu k^\nu$ term in the propagator), without loss of unitarity. To examine the feasibility of this, we use the Feynman path-integral approach.[8]

A. Stueckelberg Transformation

The generating functional for meson Green's functions is given by

$$Z(I) = Z_0 \int d\psi d\bar{\psi} dA^\mu \times \exp i\left(S(\psi, \bar{\psi}, A^\mu) + \int d^4x I^\mu(x) A_\mu(x)\right), \tag{2.2}$$

where S is the action

$$S = \int d^4x \mathcal{L}(x) \tag{2.3}$$

and Z_0 is an appropriate normalization constant. (In this paper we shall always use the same symbol Z_0 to describe this normalization, even though it may change from formula to formula.) For simplicity we do not consider the generating functional for fermion Green's functions. Note that the quantity

$$\int dA \exp \frac{-i}{2\alpha} \int d^4x \left(\partial^\mu A_\mu(x) + \frac{1}{\mu}(\Box + \alpha\mu^2) A(x)\right)^2 \tag{2.4}$$

is independent of A_μ. (Here α is a numerical parameter.) This is a consequence of translation invariance of the functional A integral. Consequently, instead of (2.2), we may write

$$Z(I) = Z_0 \int d\psi\, d\bar{\psi}\, dA^\mu dA \exp i\left\{S(\psi, \bar{\psi}, A^\mu) + \int d^4x \left[I^\mu(x) A_\mu(x) - \frac{1}{2\alpha}\left(\partial^\mu A_\mu(x) + \frac{1}{\mu}(\Box + \alpha\mu^2) A\right)^2\right]\right\}. \tag{2.5}$$

The Stueckelberg transformation now corresponds to a change of variables in the functional integral (2.5):

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \frac{1}{\mu}\partial_\mu A(x), \\ \psi'(x) &= \exp\left(-i\frac{g}{\mu}\gamma_5 A(x)\right)\psi(x), \\ \bar{\psi}'(x) &= \bar{\psi}(x)\exp\left(-\frac{ig}{\mu}\gamma_5 A(x)\right), \\ A'(x) &= A(x). \end{aligned} \tag{2.6}$$

Since the Jacobian of this transformation is unity, (2.5) becomes

$$Z(I) = Z_0 \int d\psi\, d\bar{\psi}\, dA^\mu dA \times \exp i\left[S\left(e^{i\gamma_5 Ag/\mu}\psi, \bar{\psi} e^{i\gamma_5 Ag/\mu}, A^\mu - \frac{1}{\mu}\partial^\mu A\right) + \int d^4x \left(I^\mu(x) A_\mu(x) + \frac{1}{\mu}\partial_\mu I^\mu(x) A(x) - \frac{1}{2\alpha}[\partial^\mu A_\mu + \alpha\mu A(x)]^2\right)\right]. \tag{2.7}$$

B. The Anomaly

To complete the analysis, it is necessary to reexpress the action in (2.7) in terms of ψ, $\bar{\psi}$, and A^μ. If one were to take seriously the formula (2.1) for the Lagrangian, then one would conclude that, apart from the mass term, S is invariant under the transformation (2.6). However, this conclusion must be wrong, since it would imply, among other things, that the three-current Green's function

$$T^{\mu\nu\alpha}(p, q) = \int d^4x\, d^4y\, e^{ipx} e^{iqy} \times \langle 0|T^*\{J_5^\mu(x) J_5^\nu(y) J_5^\alpha(0)\}|0\rangle \tag{2.8a}$$

is transverse:

$$p_\mu T^{\mu\nu\alpha}(p, q) = q_\nu T^{\mu\nu\alpha}(p, q) = (p+q)_\alpha T^{\mu\nu\alpha}(p, q) = 0. \tag{2.8b}$$

This is known to be false because of the triangle anomaly.[4] In lowest-order perturbation theory, this Green's function is given by

$$T^{\mu\nu\alpha}(p, q) = \Gamma^{\mu\nu\alpha}(p, q) + \Gamma^{\nu\mu\alpha}(q, p). \tag{2.9}$$

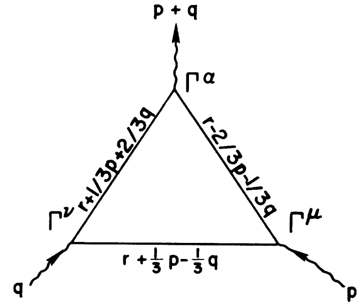

FIG. 1. The anomalous triangle graph. The exhibited routing of internal momenta assures Bose symmetry.

$\Gamma^{\mu\nu\alpha}(p,q)$ is represented by the graph of Fig. 1, where the bare vertex Γ^α is $\gamma^\alpha\gamma_5$. This graph, though finite, is superficially linearly divergent; hence it depends on the routing of the internal momenta. The exhibited routing assures that $T^{\mu\nu\alpha}$ is Bose-symmetric:

$$T^{\mu\nu\alpha}(p,q)=T^{\nu\mu\alpha}(q,p)=T^{\alpha\nu\mu}(-p-q,q)\,.$$

The formula for $(p+q)_\alpha T^{\mu\nu\alpha}(p,q)$ can be computed from the Feynman integral representation for $\Gamma^{\mu\nu\alpha}(p,q)$ by purely algebraic manipulations.[9] Rather than (2.8b), we find a nonzero result:

$$i(p+q)_\alpha T^{\mu\nu\alpha}(p,q)=-8c\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta\,. \tag{2.10a}$$

Here $c=-1/48\pi^2$. By Bose symmetry, similar formulas are found for $p_\mu T^{\mu\nu\alpha}(p,q)$ and $q_\nu T^{\mu\nu\alpha}(p,q)$. This can be interpreted as nonconservation of J_5^μ.

$$\partial_\mu J_5^\mu=cg^2F^{\alpha\beta}\tilde F_{\alpha\beta}\,,\qquad \tilde F^{\alpha\beta}=\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}\,. \tag{2.10b}$$

Note that (2.10b) may be rewritten as

$$\partial_\mu \mathcal{J}_5^\mu=0\,,\qquad \mathcal{J}_5^\mu=J_5^\mu-4cg^2\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta\,. \tag{2.11}$$

Thus it is seen that there *does* exist a conserved current associated with gauge transformations, but it does not coincide with the current J_5^μ to which the vector meson couples.

It is now possible to compute the anomalous transformation law for the action, with help of (2.10) and (2.11). The explicit argument is given in the Appendix. The result is

$$\begin{aligned}S\Big(e^{i\gamma_5 A g/\mu}\psi,\bar\psi e^{i\gamma_5 Ag/\mu}\,,A^\mu-\frac{1}{\mu}\partial^\mu A\Big)&=S(\psi,\bar\psi,A^\mu)+\int d^4x\,[\tfrac12\partial^\mu A(x)\partial_\mu A(x)+\mu A^\mu(x)\partial_\mu A(x)]\\&\quad-\frac{g^3c}{\mu}\int d^4x\,F^{\alpha\beta}(x)\tilde F_{\alpha\beta}(x)A(x)\,.\end{aligned} \tag{2.12}$$

The second term on the right-hand side of (2.12) arises from the explicit gauge breaking, viz., the boson mass, while the last term is the effect of the anomaly.

The derivation of (2.12) makes use of the assumption that the anomaly in the *complete* theory is given by the lowest-order result (2.10). That this is true in spinor electrodynamics and in the σ model has been shown by Adler,[4] and explicit second-order calculations have verified the general argument.[4]

C. Absence of Quasi-Renormalizability

We now return to (2.7) and complete the evaluation of the Stueckelberg-transformed $Z(I)$. From (2.7) and (2.12)

$$\begin{aligned}Z(I)=Z_0\int d\psi\, d\bar\psi\, dA^\mu dA\,\exp i\Big\{&S_F(\psi,\bar\psi)+S_I(\psi,\bar\psi,A^\mu)+S_M^\alpha(A^\mu)\\&+\int d^4x\Big[-\tfrac12 A(x)(\Box+\alpha\mu^2)A(x)-\frac{g^3c}{\mu}F^{\alpha\beta}(x)\tilde F_{\alpha\beta}(x)A(x)+I^\mu(x)A_\mu(x)+\frac{1}{\mu}\partial_\mu I^\mu(x)A(x)\Big]\Big\}\,.\end{aligned} \tag{2.13}$$

Here $S_F(\psi,\bar\psi)$ is the free fermion action, $S_I(\psi,\bar\psi,A^\mu)$ is the action associated with the fermion-meson interaction Lagrangian of (2.1), and $S_M^\alpha(A^\mu)$ is the free, modified vector-meson action

$$S_M^\alpha(A^\mu)=\tfrac12\int d^4x\Big\{A^\mu(x)(\Box+\alpha\mu^2)A_\mu(x)+\Big(1-\frac{1}{\alpha}\Big)[\partial_\mu A^\mu(x)]^2\Big\}\,.$$

$S_M^\alpha(A^\mu)$ corresponds to a vector-meson propagator

$$\frac{-i}{k^2-\mu^2}\Big(g^{\mu\nu}-\frac{1-\alpha}{k^2-\mu^2\alpha}k^\mu k^\nu\Big)\,,$$

which has ordinary asymptotic decrease for large k. However, because of the anomaly, the Stueckelberg field A does not decouple, but rather interacts through a nonrenormalizable derivative coupling with A^μ:

$$-4\frac{g^3c}{\mu}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta A\,.$$

Consequently the theory remains nonrenormalizable. (Of course in a theory with a *vector* interaction there is no anomaly; A decouples and the theory is quasi-renormalizable.[1])

The A integral in (2.13) can be performed, with the result

$$Z(I)=Z_0\int d\psi d\bar{\psi} dA^\mu \exp i\Big\{S_F(\psi,\bar{\psi})+S_I(\psi,\bar{\psi},A^\mu)+S_M^\alpha(A^\mu)+\int d^4x I^\mu(x)A_\mu(x)$$
$$-\tfrac{1}{2}\int d^4x d^4y\left[\frac{g^3c}{\mu}F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)-\frac{\partial_\mu I^\mu(x)}{\mu}\right]D(x-y|\mu^2\alpha)\left[\frac{g^3c}{\mu}F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)-\frac{\partial_\mu I^\mu(x)}{\mu}\right]\Big\}. \tag{2.14}$$

Here $D(x|\mu^2)$ is the propagator, normalized such that

$$(\Box+\mu^2)D(x|\mu^2)=-\delta^4(x).$$

Thus even if one is not interested in generating longitudinal vector mesons, i.e., $\partial_\mu I^\mu=0$, there remains an effective action corresponding to a nonlocal interaction arising from the anomaly

$$\frac{-g^6c^2}{2\mu^2}\int d^4x\, d^4y\, F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)D(x-y|\mu^2\alpha)F^{\mu\nu}(y)\tilde{F}_{\mu\nu}(y).$$

D. Zero-Mass Limit

Formulas (2.13) and (2.14) may be examined to study whether or not a zero-mass limit of the theory exists.[7] Such a limit can of course only be taken for transverse components of a vector-meson field; hence we set $\partial_\mu I^\mu=0$. But now it is seen that it is still impossible to pass to zero-mass since the anomaly involves

$$\frac{g^3c}{\mu}F^{\mu\nu}\tilde{F}_{\mu\nu}.$$

Hence a zero-mass theory does not exist. The same result can be established in another way. Consider the vacuum-vacuum amplitude for a zero-mass version of this story. The analog of (2.2) is

$$Z=Z_0\int d\psi d\bar{\psi} dA^\mu \exp iS(\psi,\bar{\psi},A^\mu). \tag{2.15}$$

S is constructed as in (2.1) and (2.3), except that the boson mass is now zero. Next consider an arbitrary functional $\bar{S}$, depending on ψ, $\bar{\psi}$, and A, which satisfies

$$\int d\theta \exp i\bar{S}(e^{-ig\gamma_5\theta}\psi,\bar{\psi}e^{-ig\gamma_5\theta},A^\mu+\partial^\mu\theta)=1. \tag{2.16}$$

Consequently (2.15) is equivalent to

$$\begin{aligned}Z&=Z_0\int d\psi d\bar{\psi} dA^\mu d\theta \exp i\,[S(\psi,\bar{\psi},A^\mu)+\bar{S}(e^{-ig\gamma_5\theta}\psi,\bar{\psi}e^{-ig\gamma_5\theta},A^\mu+\partial^\mu\theta)]\\
&=Z_0\int d\psi d\bar{\psi} dA^\mu d\theta \exp i[\bar{S}(\psi,\bar{\psi},A^\mu)+S(e^{ig\gamma_5\theta}\psi,\bar{\psi}e^{ig\gamma_5\theta},A^\mu-\partial^\mu\theta)]\\
&=Z_0\int d\psi d\bar{\psi} dA^\mu d\theta \exp i\left[S(\psi,\bar{\psi},A^\mu)+\bar{S}(\psi,\bar{\psi},A^\mu)-g^3c\int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)\theta(x)\right]\\
&=Z_0\int d\psi d\bar{\psi} dA^\mu \exp i[S(\psi,\bar{\psi},A^\mu)+\bar{S}(\psi,\bar{\psi},A^\mu)]\delta(F^{\alpha\beta}\tilde{F}_{\alpha\beta}).\end{aligned} \tag{2.17}$$

Thus the zero-mass theory, by virtue of the δ function in the last term of (2.17), requires that the anomaly be absent; this is not the case.

The absence of the zero-mass limit is of course a consequence of the fact that the equations of motion are inconsistent:

$$\begin{aligned}&\partial_\mu F^{\mu\nu}=gJ_5^\nu,\\ &\partial_\mu\partial_\nu F^{\mu\nu}=0=g\partial_\nu J_5^\nu=cg^3F^{\alpha\beta}\tilde{F}_{\alpha\beta}\neq 0.\end{aligned} \tag{2.18}$$

E. Discussion

It is not hard to present a Feynman graph which exhibits the difficulties which we have encountered. Consider the contribution to the vacuum polarization given in Fig. 2. The integral is

$$\Pi^{\alpha\beta}(k)=-\tfrac{1}{2}g^6\int\frac{d^4q}{(2\pi)^4}\,T^{\mu'\nu'\alpha}(-k+q,-q)D_{\mu'\mu}(k-q)\times D_{\nu'\nu}(q)T^{\mu\nu\beta}(k-q,q). \tag{2.19}$$

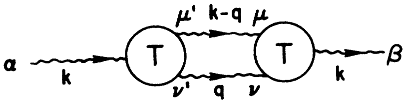

FIG. 2. A contribution to the vacuum polarization tensor. The bubble labeled T is the AAA vertex.

If the theory is quantized by ordinary methods, then the boson propagators $D^{\mu\nu}$ have a bad asymptotic behavior. The hope for quasi-renormalizability rests on the ability to remove the $k_\mu k_\nu$ terms in the propagators with the help of the Ward identities. If $T^{\mu\nu\alpha}$ satisfied the naive Ward identities (2.8b), then indeed one could drop the longitudinal parts of the propagator, and one would be left with the normal part $\Pi_N^{\alpha\beta}(k)$,

$$\Pi_N^{\alpha\beta} = -\tfrac{1}{2}g^6 \int \frac{d^4q}{(2\pi)^4}\, T^{\mu\nu\alpha}(-k+q, q) \times D(k-q)D(q)T_{\mu\nu}{}^{\beta}(k-q, q)\,, \qquad D(k) = \frac{i}{k^2-\mu^2}\,. \tag{2.20}$$

However, because of the anomaly there are also anomalous contributions to $\Pi^{\alpha\beta}(k)$. These are

$$\Pi_A^{\alpha\beta}(k) = \left(\frac{8cg^3}{\mu}\right)^2 \int \frac{d^4q}{(2\pi)^4}\,\epsilon^{\alpha\mu\omega\gamma}k_\omega q_\gamma \epsilon^{\beta\nu\omega'\gamma'}k_{\omega'}q_{\gamma'} \times g_{\mu\nu}D(k-q)D(q) \neq 0\,. \tag{2.21}$$

The decomposition $\Pi^{\alpha\beta} = \Pi_N^{\alpha\beta} + \Pi_A^{\alpha\beta}$ corresponds to the Stueckelberg-transformed generating functional (2.13), with $\alpha = 1$. $\Pi_N^{\alpha\beta}$ is the vector-meson part, where the meson propagator is $-g_{\mu\nu}D(k)$, while $\Pi_A^{\alpha\beta}$ is given by an anomalous interaction of the Stueckelberg A field (mass μ) with the vector-meson field. This is represented by Fig. 3.

$\Pi^{\alpha\beta}(k)$ also demonstrates the problems with a zero-mass theory. Suppose we compute (2.19) for massless vector mesons. The logitudinal part should be zero. However, we find

$$\begin{aligned} k_\alpha k_\beta \Pi^{\alpha\beta}(k) &= \tfrac{1}{2}(8cg^3)^2 \int \frac{d^4q}{(2\pi)^4}\,\epsilon^{\mu'\nu'\gamma'\delta'}k_{\gamma'}q_{\delta'}\epsilon^{\mu\nu\gamma\delta}k_\gamma q_\delta \times D_{\mu\mu'}(k-q)D_{\nu'\nu}(q) \\ &= \tfrac{1}{2}(8cg^3)^2 \int \frac{d^4q}{(2\pi)^4}\,\epsilon^{\mu\nu\gamma\delta}k_\gamma q_\delta \epsilon_{\mu\nu\gamma'\delta'}k^{\gamma'}q^{\delta'} \times D(k-q)D(q)\,. \end{aligned} \tag{2.22}$$

This quantity, though quadratically divergent, has an unambiguous absorptive part proportional to $k^4\theta(k^2)$. Hence the vector meson acquires a mass.

III. A SPONTANEOUSLY BROKEN GAUGE MODEL

An Abelian version of the spontaneously broken gauge theory, which has been proposed as a model for weak interactions, is given by the following

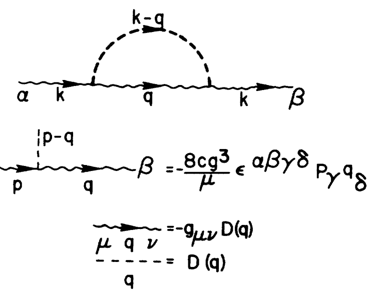

FIG. 3. The contribution of the Stueckelberg field (dotted line) to the vacuum polarization of the vector meson (wavy line).

Lagrangian, called Stage i:[6]

$$\begin{aligned} \mathcal{L}_1 = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + 2igA_\mu)\phi^*(\partial^\mu - 2igA^\mu)\phi \\ &+ \mu^2\phi^*\phi - h(\phi^*\phi)^2 + \bar\psi[i\partial\!\!\!/ + A\!\!\!/(f - g\gamma_5)]\psi \\ &- \tfrac{1}{2}G\bar\psi\psi(\phi+\phi^*) - \tfrac{1}{2}G\bar\psi\gamma_5\psi(\phi-\phi^*)\,, \\ F^{\mu\nu} = &\,\partial^\mu A^\nu - \partial^\nu A^\mu\,. \end{aligned} \tag{3.1}$$

We shall first review the "proof of quasi-renormalizability" and then show how the triangle anomaly affects the result.

A. Argument for Quasi-Renormalizability

The Lagrangian (3.1) formally possesses a gauge symmetry of the second kind: $\{q\} \to \{q^\theta\}$, where $\{q\}$ stand for all the fields ψ, $\bar\psi$, A^μ, ϕ, ϕ^*, and $\{q^\theta\}$ is given by

$$\{q^\theta\} = \begin{cases} \exp[i\theta(f - g\gamma_5)]\psi \\ \bar\psi\exp[-i\theta(f + g\gamma_5)] \\ A^\mu + \partial^\mu\theta \\ \exp[2ig\theta]\phi \\ \exp[-2ig\theta]\phi^*\,. \end{cases} \tag{3.2}$$

The apparently conserved current is

$$J^\mu = -\bar\psi\gamma^\mu(f - g\gamma^5)\psi - 2ig\phi^*(\partial^\mu - 2igA^\mu)\phi + 2ig\phi(\partial^\mu + 2igA^\mu)\phi^*\,. \tag{3.3}$$

Because of the gauge symmetry, the vacuum-to-vacuum transition amplitude Z,

$$Z = Z_0 \int dq \exp iS(q)\,, \qquad S(q) = \int d^4x\,\mathcal{L}_1(x)\,,$$

is undefined, which reflects the absence of a canonical formalism. (The canonical momentum conjugate to A^0 vanishes, and A^0 cannot be expressed in terms of canonically independent quantities.)

To remedy this one uses the Faddeev-Popov device.[10] Consider an arbitrary functional of q, $\bar S(q)$, which satisfies

$$\int d\theta \exp i\bar{S}(q^\theta) = 1 . \tag{3.4}$$

Z may now be written as

$$Z = Z_0 \int d\theta dq \exp i[S(q) + \bar{S}(q^\theta)] \tag{3.5a}$$

$$= Z_0 \int d\theta dq \exp i[S(q^{-\theta}) + \bar{S}(q)] \tag{3.5b}$$

$$= Z_0 \int d\theta dq \exp i[S(q) + \bar{S}(q)] \tag{3.5c}$$

$$= Z_0 \int dq \exp i[S(q) + \bar{S}(q)] . \tag{3.5d}$$

Equation (3.5b) follows from (3.5a) by a change of variables $q = q^\theta$ which has unit Jacobian; $q^{-\theta}$ is the inverse gauge transformation. Equation (3.5c) is a statement of the gauge invariance of S, while in (3.5d) we have absorbed the (infinite) constant $\int d\theta$ into Z_0. Thus the Faddeev-Popov prescription is to employ the action $S + \bar{S}$, rather than just S, where $\bar{S}$ is arbitrary except that it satisfies (3.4). One can show that vacuum matrix elements of quantities $F(q)$, which are gauge-invariant, $F(q) = F(q^\theta)$, are independent of the choice for $\bar{S}$.[11]

The theory may now be quantized, using the gauge-dependent Lagrangian $\mathcal{L} = \mathcal{L}_1 + \bar{\mathcal{L}}$, where

$$\bar{S} = \int d^4x\, \bar{\mathcal{L}}(x) . \tag{3.6}$$

The spontaneous symmetry breaking is introduced by assigning a real vacuum expectation value to ϕ and ϕ^*:

$$\phi = \frac{1}{\sqrt{2}}(v + \varphi + i\chi) , \qquad \langle\varphi\rangle = \langle\chi\rangle = 0 . \tag{3.7}$$

We then arrive at Stage *ii*, which is characterized by the Lagrangian

$$\begin{aligned}\mathcal{L}_2 = \bar{\mathcal{L}} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(2gv)^2 A_\mu A^\mu + \tfrac{1}{2}(\partial_\mu\varphi - 2gA_\mu\chi)^2 + \tfrac{1}{2}(\partial_\mu\chi + 2gA_\mu)^2 - \tfrac{1}{2}(3hv^2 - \mu^2)\varphi^2 + 2gvA^\mu(\partial_\mu\chi + 2gA_\mu\varphi) \\ - hv\varphi(\varphi^2 + \chi^2) - \tfrac{1}{4}h(\varphi^2 + \chi^2)^2 + \bar{\psi}\left[i\partial\!\!\!/ - \frac{Gv}{\sqrt{2}} + A\!\!\!/(f - g\gamma_5) - \frac{G}{\sqrt{2}}(\varphi + i\gamma_5\chi)\right]\psi - (hv^2 - \mu^2)(v\varphi - \tfrac{1}{2}\chi^2) ,\end{aligned} \tag{3.8}$$

where to lowest order $hv^2 = \mu^2$. The vector meson and the fermion both acquire masses, $2gv$ and $(G/\sqrt{2})v$, respectively. Nevertheless, with an appropriate choice for $\bar{\mathcal{L}}$ a renormalizable perturbation theory emerges. Thus, if

$$\bar{\mathcal{L}} = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 , \tag{3.9}$$

the A^μ propagator is

$$D_{\mu\nu}(k) = \int d^4x e^{ikx}\langle 0|TA_\mu(x)A_\nu(0)|0\rangle = -\frac{i(g_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2 - (2gv)^2} - \frac{i\alpha k^\mu k^\nu}{k^4} , \tag{3.10}$$

whereas the χ propagator is[12]

$$D(k) = \int d^4x\, e^{ikx}\langle 0|T\chi(x)\chi(0)|0\rangle = \frac{i}{k^2} - \frac{i\alpha(2gv)^2}{k^4} . \tag{3.11}$$

The gauge invariance should ensure that the theory be independent of α, and α may be conveniently chosen to be zero,[13] eliminating the k^{-4} terms. This theory is manifestly renormalizable,[12] but it appears to be beset with poles at $k^2 = 0$. However, they should in fact be absent due to the underlying gauge invariance, which leads to Ward identities that ensure the cancellation of the Goldstone boson (χ) poles and the negative-metric ghost poles in the vector-meson propagator. Of course one must establish that the procedure of renormalization does not invalidate the formal Ward identities. A complete analysis of an Abelian gauge model without fermions has been performed by Lee,[12] who showed that one can make the necessary subtractions in a way consistent with the Ward identities and that these indeed lead to the decoupling of the massless particles.

Another way of seeing that the Goldstone boson ghost cancellation occurs is to pass to the Stage *iii* Lagrangian. This is achieved by a change of variables in (3.5), for any $\bar{S}(q)$. Contrary to assertions frequently made in the literature, this is *not* a choice of gauge but merely a redefinition of fields in (3.5) for *any gauge* (a point transformation). The desired change is from ψ, $\bar{\psi}$, A^μ, ϕ, ϕ^* to a new set ψ', $\bar{\psi}'$, $A^{\mu\prime}$, ρ, θ given by

$$\begin{aligned}\psi &= \exp[-i\theta(f - g\gamma_5)]\psi' , \\ \bar{\psi} &= \bar{\psi}'\exp[i\theta(f + g\gamma_5)] , \\ A^\mu &= A'^\mu - \partial^\mu\theta , \\ \phi &= \frac{1}{\sqrt{2}}\rho\exp[-2ig\theta] , \\ \phi^* &= \frac{1}{\sqrt{2}}\rho\exp[2ig\theta] .\end{aligned} \tag{3.12}$$

The Jacobian is $\det\rho$. Consequently we have from (3.5)

$$
\begin{aligned}
Z &= Z_0 \int dq \exp i[S(q) + \bar{S}(q)] &(3.13\text{a})\\
&= Z_0 \int \det\rho\, d\rho\, d\theta\, d\psi\, d\bar{\psi}\, dA^\mu \exp i \Big\{ S\Big(e^{-i\theta(f-g\gamma_5)}\psi, \bar{\psi}e^{i\theta(f+g\gamma_5)}, A^\mu - \partial^\mu\theta, \tfrac{1}{\sqrt{2}}\rho e^{-2ig\theta}, \tfrac{1}{\sqrt{2}}\rho e^{2ig\theta}\Big)\\
&\qquad + \bar{S}\Big(e^{-i\theta(f-g\gamma_5)}\psi, \bar{\psi}e^{i\theta(f+g\gamma_5)}, A^\mu - \partial^\mu\theta, \tfrac{1}{\sqrt{2}}\rho e^{-2ig\theta}, \tfrac{1}{\sqrt{2}}\rho e^{2ig\theta}\Big)\Big\} &(3.13\text{b})\\
&= Z_0 \int \det\rho\, d\rho\, d\theta\, d\psi\, d\bar{\psi}\, dA^\mu \exp i \Big\{ S\Big(\psi, \bar{\psi}, A^\mu, \tfrac{1}{\sqrt{2}}\rho, \tfrac{1}{\sqrt{2}}\rho\Big)\\
&\qquad + \bar{S}\Big(e^{-i\theta(f-g\gamma_5)}\psi, \bar{\psi}e^{i\theta(f+g\gamma_5)}, A^\mu - \partial^\mu\theta, \tfrac{1}{\sqrt{2}}\rho e^{-2ig\theta}, \tfrac{1}{\sqrt{2}}\rho e^{2ig\theta}\Big)\Big\} &(3.13\text{c})\\
&= Z_0 \int \det\rho\, d\rho\, d\psi\, d\bar{\psi}\, dA^\mu \exp iS\Big(\psi, \bar{\psi}, A^\mu, \tfrac{1}{\sqrt{2}}\rho, \tfrac{1}{\sqrt{2}}\rho\Big). &(3.13\text{d})
\end{aligned}
$$

Equation (3.13c) follows from (3.13b) by the *formal* observation that the change of variables (3.12) leaves S invariant; (3.13d) is a consequence of the normalization condition (3.4) satisfied by $\bar{S}$. Thus the Stage *iii* Lagrangian is

$$
\begin{aligned}
\mathcal{L}_3 = &-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \tfrac{1}{2}\mu^2\rho^2 - \tfrac{1}{4}h\rho^4\\
&+ 2g^2\rho^2 A^\mu A_\mu + \bar{\psi}\Big[i\not{\partial} + \not{A}(f - g\gamma_5) - \tfrac{1}{\sqrt{2}}G\rho\Big]\psi. \qquad (3.14)
\end{aligned}
$$

This no longer posesses any gauge freedom.

Spontaneous symmetry breaking is now introduced by assigning a vacuum expectation to ρ: $\langle 0|\rho|0\rangle = \lambda$. The perturbation series is then developed by shifting ρ to $\hat{\rho} + \lambda$. This gives a mass $\mu = 2g\lambda$ to the vector meson and a mass $M = G\lambda/\sqrt{2}$ to the fermion. The Feynman rules are the usual, nonrenormalizable ones, except that the Jacobian, $\det\rho$, must be taken into account. Since $\det\rho = \exp[\mathrm{Tr}\ln(\lambda + \hat{\rho})] = \exp i\int d^4x\{-i\delta^4(0)\ln(\lambda + \hat{\rho})\}$, this merely adds a contact term $-i\delta^4(0)\ln(1 + \hat{\rho}/\lambda)$ to the interaction Lagrangian, which serves to cancel some divergences.

The importance of exhibiting the Stage *iii* Lagrangian, (3.14), is that it shows that all unphysical singularities are absent. This strongly suggests quasi-renormalizability, since the *same* theory is seen to be renormalizable in Stage *ii* and free of unwanted zero-mass singularities in Stage *iii*.

B. Failure of Quasi-Renormalizability

The argument sketched above fails because the triangle anomaly is present as a consequence of the axial-vector coupling between A^μ and the fermions. The relevant graph is as in Fig. 1, except that the bare vertex is now $\Gamma^\alpha = -\gamma^\alpha(f - g\gamma_5)$. Arguments completely analogous to those in Sec. II show that

$$
\begin{aligned}
&\partial_\mu J^\mu = cg(g^2 + 3f^2)F^{\alpha\beta}\tilde{F}_{\alpha\beta},\\
&\partial_\mu \mathcal{J}^\mu = 0, \qquad (3.15)\\
&\mathcal{J}^\mu = J^\mu - 4cg(g^2 + 3f^2)\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta.
\end{aligned}
$$

(We again make use of the assumption that Adler's argument[4] establishing the absence of anomalies in higher-order perturbation theory is applicable in the present context; see also the last paragraph in Sec. II B.) The conserved current $\mathcal{J}^\mu$ does not coincide with the current J^μ to which A^μ couples. Our first conclusion therefore is that Stage *i* Lagrangian (3.1) is inconsistent, since it involves a massless vector meson coupled to a nonconserved current:

$$
\begin{aligned}
&\partial_\mu F^{\mu\nu} = J^\nu,\\
&\partial_\mu\partial_\nu F^{\mu\nu} = 0 = \partial_\nu J^\nu \qquad (3.16)\\
&\qquad = cg(g^2 + 3f^2)F^{\alpha\beta}\tilde{F}_{\alpha\beta} \neq 0.
\end{aligned}
$$

The same inconsistency emerges if we consider Z,

$$
\begin{aligned}
Z &= Z_0 \int dq \exp iS(q)\\
&= Z_0 \int dq\, d\theta \exp i[S(q) + \bar{S}(q^\theta)]\\
&= Z_0 \int dq\, d\theta \exp i[S(q^{-\theta}) + \bar{S}(q)]. \qquad (3.17)
\end{aligned}
$$

Here $\bar{S}(q)$ is normalized as in (3.4). Because of (3.15) and (3.16), we have by the same arguments that lead to (2.12)

$$
S(q^{-\theta}) = S(q) - cg(g^2 + 3f^2)\int d^4x\, F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)\theta(x). \qquad (3.18)
$$

[The second term in (2.12) is absent here, since it arises from the vector-meson mass, which is not contained in (3.1).] From (3.17) we see that

$$Z = Z_0 \int dq d\theta \exp i\Big[S(q) + \bar{S}(q) - cg(g^2+3f^2)\int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)\theta(x)\Big]$$

$$= Z_0 \int dq\, \delta[F^{\alpha\beta}\tilde{F}_{\alpha\beta}] \exp i[S(q) + \bar{S}(q)]. \quad (3.19)$$

Thus the theory described by (3.1) is consistent only if $F^{\alpha\beta}\tilde{F}_{\alpha\beta}$ vanishes.

Let us ignore this difficulty and pass to the Stage *ii* Lagrangian, by using (3.5d)

$$Z = Z_0 \int dq \exp i[S(q) + \bar{S}(q)]. \quad (3.20)$$

Thus we are considering a *different* theory from that described by $\mathfrak{L}_1$: A theory in which gauge invariance is explicitly broken by $\bar{S}(q)$. This may not be unreasonable, since in the end (Stage *iii*) we are not interested in a gauge-invariant theory anyway. So if it were possible to produce a renormalizable, physically acceptable theory, we would be willing to abandon Stage *i*. However, it will be seen that this does not lead to a satisfactory theory.

With the choice $\bar{S}(q) = -(1/2\alpha)\int d^4x[\partial_\mu A^\mu(x)]^2$ the theory described by $\mathfrak{L}_2$ is renormalizable, since the vector-meson propagator $D_{\mu\nu}(k)$ is given by (3.10). However, the Ward identities which normally would effect the cancellation of the zero-mass singularities in the S matrix are anomalous due to the triangle graph. In the Abelian case, without fermions, considered by Lee it was possible to prove the validity of the Ward identities by using Pauli-Villars regulators for the vector-meson and scalar fields that preserved the gauge invariance. In our case it is impossible to introduce massive regulators at Stage *i* for the fermion fields due to their axial-vector couplings, and in fact the Ward identities are anomalous.

It is perhaps instructive to exhibit a physical amplitude which contains an unphysical singularity due to the anomaly in the Ward identity. Consider the fourth-order (in g and f) contribution to fermion-antifermion annihilation to a vector-meson pair. The four graphs that have zero-mass poles are shown in Fig. 4. It is easy to see that these graphs are independent of the gauge parameter α, which we therefore set equal to zero, in which case the graphs given by Fig. 4(c) and Fig. 4(d) vanish. The residue of the pole at $k^2 = (p+q)^2 = 0$ is proportional to

$$\Delta^{\mu\nu} = k_\alpha T^{\alpha\mu\nu}(k,p,q) - 2igvT^{\mu\nu}(k,p,q), \quad (3.21)$$

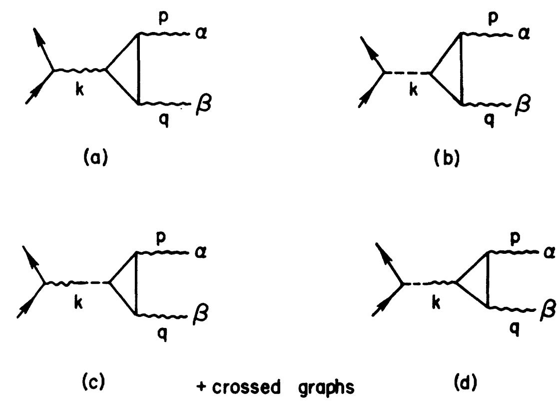

FIG. 4. The graphs that contain zero-mass poles for fermion-antifermion annihilation. The wavy (dashed) lines represent the vector-meson (χ) propagators; solid lines are fermions.

where $T^{\alpha\mu\nu}$ ($T^{\mu\nu}$) is the irreducible 3-point function (2-point function) of A^α, A^μ, and A^ν. Recalling that the mass of the fermion is given by $(1/\sqrt{2})Gv$, we see that the vanishing of $\Delta^{\mu\nu}$ is equivalent to the validity of partial conservation of axial-vector current (PCAC):

$$\partial^\mu \bar{\psi}\gamma_\mu(f - g\gamma_5)\psi = 2\frac{G}{\sqrt{2}} v\bar{\psi}\gamma_5\psi.$$

This is false. To lowest order $T^{\alpha\mu\nu}$ and $T^{\mu\nu}$ are uniquely calculable and $\Delta^{\mu\nu}$ is in fact given by the term previously calculated, which is independent of the fermion mass (or of G):

$$\Delta^{\mu\nu} = -8c\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta g(g^2+3f^2). \quad (3.22)$$

Thus the Goldstone boson (χ)–ghost (A^μ) cancellation does not occur.

One might attempt to remedy this by adding counterterms to the Lagrangian which cancel the anomaly. In fact to this order one could add the term

$$\mathfrak{L}_c = -\frac{g^2+3f^2}{2v} cF_{\alpha\beta}\tilde{F}^{\alpha\beta}\chi \quad (3.23)$$

to $\mathfrak{L}_2$, thus canceling the anomaly. Furthermore, one can guarantee the cancellation of all anomalies by adding to the action $S(q)$ terms that render it invariant, at Stage *i*, under gauge transformations. From (3.2) and (3.18) it is clear that

$$\tilde{S}(q) = S(q) + \tfrac{1}{4}ic(g^2+3f^2) \times \int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x) \ln\frac{\phi(x)}{\phi^*(x)} \quad (3.24)$$

is gauge-invariant, i.e., $\tilde{S}(q^{-\theta}) = \tilde{S}(q)$. Because of the logarithm of fields ϕ and ϕ^*, this action cannot be used for practical calculations at Stage *i*, but at Stage *ii* it corresponds to the Lagrangian

$$\tilde{\mathcal{L}}_2 = \mathcal{L}_2 + \tfrac{1}{4} i c (g^2 + 3f^2) F_{\alpha\beta} \tilde{F}^{\alpha\beta} \ln\{[1 + (\varphi + i\chi)/v][1 + (\varphi - i\chi)/v]^{-1}\}$$
$$= \mathcal{L}_2 - \frac{c}{2v}(g^2 + 3f^2) F_{\alpha\beta} \tilde{F}^{\alpha\beta} \chi + O(g^4), \tag{3.25}$$

which coincides with the counterterm (3.23) to lowest order in g [recall that $2gv$ = mass of the vector meson, and thus $v \approx O(1/g)$]. This Lagrangian is clearly nonrenormalizable.

Thus at Stage *ii* we are fixed with two unacceptable options: (a) If we adopt $\mathcal{L}_2$ as our Lagrangian then the Ward identities are invalid and the theory contains zero-mass unphysical singularities. (b) If we instead use the manifestly gauge-invariant nonpolynomial Lagrangian $\tilde{\mathcal{L}}_2$ we destroy the renormalizability of the theory. Thus the theory is not quasi-renormalizable.

Once again we can exhibit the difficulties by passing to Stage *iii*. If we start with (3.20) and effect the change of variables (3.12) we find

$$Z = Z_0 \int \det\rho\, d\rho\, d\theta\, d\psi\, d\bar{\psi}\, dA^\mu \exp i\Big[S\Big(e^{-i\theta(f - g\gamma_5)}\psi, \bar{\psi} e^{i\theta(f+g\gamma_5)}, A^\mu - \partial^\mu\theta, \frac{\rho}{\sqrt{2}} e^{-2ig\theta}, \frac{\rho}{\sqrt{2}} e^{2ig\theta}\Big)$$
$$- \frac{1}{2\alpha}\int d^4x [\partial_\mu A^\mu(x) - \Box\theta(x)]^2\Big]$$
$$= Z_0 \int \det\rho\, d\rho\, d\theta\, d\psi\, d\bar{\psi}\, dA^\mu \exp i\Big[S\Big(\psi, \bar{\psi}, A^\mu, \frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}\Big)$$
$$- \int d^4x\Big(g(g^2 + 3f^2) c \tilde{F}^{\alpha\beta}(x) F_{\alpha\beta}(x)\theta(x) + \frac{1}{2\alpha}[\partial_\mu A^\mu(x) - \Box\theta(x)]^2\Big)\Big]. \tag{3.26}$$

We have used the anomalous transformation law for the action (3.18). It is now impossible to exploit the normalization condition on $\bar{S}$,

$$\int d\theta \exp\Big(\frac{-i}{2\alpha}\int d^4x[\partial_\mu A^\mu(x) - \Box\theta(x)]^2\Big) = \text{constant},$$

to perform the θ integral, since an additonal θ dependence has been introduced by the anomaly.

To evaluate the θ integral we shift the θ integration in (3.26)

$$\theta(x) = \theta'(x) - \int d^4y\, D(x-y|0)\partial_\mu A^\mu(y). \tag{3.27}$$

Then the θ part of the integral in (3.26) is

$$\int d\theta \exp - i\int d^4x\Big(\frac{1}{2\alpha}[\Box\theta(x)]^2 + cg(g^2 + 3f^2) F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)\theta(x)\Big)$$
$$\times \exp icg(g^2 + 3f^2)\int d^4x\, d^4y\, F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(y) D(x-y|0)\partial_\mu A^\mu(x)$$
$$= \exp icg(g^2 + 3f^2)\int d^4x\, d^4y\, F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)[D(x-y|0)\partial_\mu A^\mu(y) + \tfrac{1}{2}\alpha c g(g^2 + 3f^2) G(x-y) F^{\mu\nu}(y)\tilde{F}_{\mu\nu}(y)]. \tag{3.28}$$

An over-all constant has been dropped. G here is

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^4 + i\epsilon}. \tag{3.29}$$

(This is infrared-divergent.) Thus (3.26) becomes

$$Z = Z_0 \int \det\rho\, d\rho\, d\psi\, d\bar{\psi}\, dA^\mu$$
$$\times \exp i\int d^4x\Big(\mathcal{L}_3(x) + cg(g^2 + 3f^2) F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)\int d^4y[D(x-y|0)\partial_\mu A^\mu(y) + \tfrac{1}{2}\alpha c g(g^2 + 3f^2) G(x-y)\tilde{F}^{\mu\nu}(y) F_{\mu\nu}(y)]\Big). \tag{3.30}$$

It is seen that the Stage *iii* Lagrangian, which arises from the renormalizable but non-gauge-invariant Stage *ii* Lagrangian, contains in addition to $\mathcal{L}_3$, interactions with the anomaly $F\tilde{F}$. These lead, as in Stage *ii*, to $1/k^2$ singularities which arise from $D(x-y|0)$. Moreover, there remain an α dependence and $1/k^4$ singularities from $G(x-y)$.

If we were instead to work with the gauge-invariant Lagrangian $\tilde{\mathcal{L}}_2$ we would have had no problem integrating out the θ field. In fact, the Stage *iii* theory corresponding to $\tilde{\mathcal{L}}_2$ is given by $\mathcal{L}_3$, which is manifestly physical. However, $\tilde{\mathcal{L}}_2$ is not renormalizable, and it is easy to convince oneself that the same is true of $\mathcal{L}_3$.

IV. A QUASI-RENORMALIZABLE MODEL WITH AXIAL-VECTOR COUPLINGS

A. The Model

We have shown that a gauge theory of vector mesons coupled axially to fermions is not quasi-renormalizable. In order to be able to construct a Stage *ii* Lagrangian in which the Goldstone bosons decouple it is necessary to preserve gauge invariance at Stage *i*. We showed that this could be achieved by adding terms to the Lagrangian, which restored the gauge invariance; however, these new interactions were nonrenormalizable. Is there a way out?

The only other modification of the theory which seems possible is *the addition of other fermions*, with opposite axial-vector couplings. Since the triangle anomaly is odd in the axial-vector coupling one might hope that the anomalies would disappear and that the resulting theory would be quasi-renormalizable.[14]

Consider the following Stage *i* Lagrangian:

$$\begin{aligned}\mathcal{L}_1^R = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + 2igA_\mu)\phi^*(\partial^\mu - 2igA^\mu)\phi \\ &+ \mu^2(\phi^*\phi) - h(\phi^*\phi)^2 + \bar{\psi}[i\partial\!\!\!/ + A\!\!\!/(f - g\gamma_5)]\psi \\ &+ \bar{\psi}'[i\partial\!\!\!/ + A\!\!\!/(f+g\gamma_5)]\psi' - \tfrac{1}{2}G\bar{\psi}[\phi(1+\gamma_5) + \phi^*(1-\gamma_5)]\psi \\ &- \tfrac{1}{2}G'\bar{\psi}'[\phi(1-\gamma_5) + \phi^*(1+\gamma_5)]\psi'. \end{aligned} \tag{4.1}$$

This Lagrangian differs from (3.1) by the addition of a fermion ψ' with opposite axial-vector coupling and equal vector coupling to A^μ. It is formally invariant under gauge transformations of the second kind, where ψ, $\bar{\psi}$, A^μ, ϕ, ϕ^* transform as in (3.2) and where

$$\begin{aligned}\psi' &\to \exp[i\theta(f+g\gamma_5)]\psi'. \\ \bar{\psi}' &\to \bar{\psi}'\exp[-i\theta(f-g\gamma_5)].\end{aligned} \tag{4.2}$$

It is clear that when one adds the contributions of the two triangle graphs, one for each fermion, the anomaly is removed. In fact, in lowest order $[g(g^2+3f^2)]$ the three-point function of A^μ is identically zero, since the two fermions have identical masses and the triangle graph is odd in g. It is then possible to show that there are no additional anomalies,[15] and that therefore the above Lagrangian is in fact gauge-invariant and that this Stage-*i* theory is consistent and renormalizable.

B. Proof of Quasi-Renormalizability

We shall now outline a proof of the quasi-renormalizability of the Stage *ii* theory which emerges by adding to $\mathcal{L}_1^R$ the term $\bar{\mathcal{L}}$ given by (3.9) and by introducing spontaneous symmetry breaking. This theory is described by the manifestly renormalizable Lagrangian

$$\mathcal{L}_2^R = \mathcal{L}_2 + \bar{\psi}'\left[i\partial\!\!\!/ - \frac{G'v}{\sqrt{2}} + A\!\!\!/(f+g\gamma_5) - \frac{G'}{\sqrt{2}}(\varphi - i\gamma_5\chi)\right]\psi', \tag{4.3}$$

where $\mathcal{L}_2$ is given by (3.8). The new fermion has acquired a mass equal to $G'v/\sqrt{2}$, which in general differs from the mass $Gv/\sqrt{2}$ of the original fermion. The sum of the two triangle graphs no longer vanishes, however the anomaly, which is independent of the fermion mass, cancels in the two graphs.

As shown by Lee,[12] the proof of quasi-renormalizability consists of three steps: (1) The theory is rendered finite by a regularization procedure which preserves the underlying gauge invariance and the formal Ward identities. (2) One then specifies a finite number of subtractions for the primitively divergent, one-particle-irreducible Green's functions, consistent with the Ward identities relating the various vertices. This ensures that when the cutoffs are removed the renormalized Green's functions obey these identities. (3) One then proves that the Ward identities lead to the cancellation of the Goldstone boson and the negative-metric vector-meson poles.

It is at Step 1 that our procedure must differ from that of Lee; the remaining steps are identical. It is still possible to introduce (as long as the gauge group is Abelian) Pauli-Villars regulators for the vector and scalar mesons. We do so in exactly the same manner as Lee.[12] With a sufficiently large number of boson regulators all primitively divergent graphs can be made finite except for those involving internal fermion loops. These we will deal with separately, since we cannot introduce fermion regulators without destroying the gauge invariance of the theory. (This is most simply seen at Stage *i*, where all fermions must be massless.) Following Adler,[15] we can make the theory finite by performing intermediate *explicit subtractions* on all divergent one-fermion-loop Green's functions with no radiative corrections. Thus a typical second-order contribution to the vector-meson self-energy, Fig. 5(a), will be made finite by an explicit subtraction of the *AAAA* one-loop vertex [Fig. 5(b)] and the use of the vector-meson regulators; whereas the sixth-order contribution depicted in Fig. 5(c) will be finite once the one-loop *AAAA* [Fig. 5(b)] and *AA* [Fig. 5(d)] vertices have been explicitly subtracted. Step 1 will be completed if we can make the explicit subtractions in a way consistent with the Ward identities relating Green's functions of A^μ to Green's functions of χ. The (quadratically divergent) two-point functions and the (logarithmically divergent) four-point functions are anomaly-free and can easily be subtracted in a way consistent with the Ward identities, even if we did not have the additional fermions. These are only required to deal with the *AAA* triangle, which is finite and anomalous. With the additional fermions present the anomaly cancels, and the Ward identity which is necessary for (3.21) to vanish is satisfied. Since it has been shown that all other triangle graphs are nonanomalous,[4] all boson one-loop graphs can be rendered finite consistent with the Ward identities. This regularization procedure will guarantee that in the cutoff finite theory the Ward identities are indeed true. The remaining steps are then identical to those of Lee,[12] except that there are additional primitively divergent Green's functions. This theory is therefore quasi-renormalizable.

C. Discussion

We have shown that quasi-renormalizable gauge theories with axial-vector couplings can be constructed if one adds a new fermion whose couplings to the vector meson differ only by interchanging left and right: $\gamma_5 \to -\gamma_5$. Since the coupling strength of the new fermion to the scalar mesons G' is arbitrary, one can arrange its mass to be arbitrarily large at Stage *ii*. However, the additional fermion, which removes the anomalies, cannot be regarded as a regulator since its mass cannot be taken to be infinite. This is because

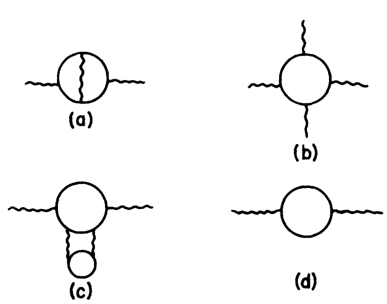

FIG. 5. Some typical contributions to vertex functions of vector mesons.

$$M_{\psi'} = M_\psi G'/G\,, \tag{4.4}$$

and $M_{\psi'} \to \infty$ implies that $G' \to \infty$. Whenever one lets a *dimensionless* coupling constant become infinite one is inviting disaster. It is easy to exhibit renormalized amplitudes which, although finite and unitary for finite G', diverge as $G' \to \infty$.

Consider the triangle-graph contribution of the new fermion to $T^{\mu\nu}(k, p, q)$ [the χAA vertex function, see Fig. 4 and Eq. (3.21)]. This graph is given by (apart from numerical constants)

$$T_\Delta^{\mu\nu}(k,p,q) = \frac{G}{M_\psi}(g^2+3f^2)M_{\psi'}{}^2\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta I(p,q,M_{\psi'})\,, \tag{4.5}$$

where

$$I(p,q,M_{\psi'}) = \int_0^1 dx \int_0^{1-x} dy[y(1-y)p^2 + x(1-x)q^2 - 2xypq - M_{\psi'}{}^2]^{-1}\,. \tag{4.6}$$

For large, Euclidean momenta (4.6) vanishes:

$$I(p,q,M_{\psi'}) \underset{p,q\approx\Lambda\to\infty}{\sim} \Lambda^{-2}\ln\Lambda\,, \tag{4.7}$$

and therefore $T_\Delta^{\mu\nu}$ approaches a constant (up to logarithmic terms). This behavior is that given by Weinberg's theorem,[16] and ensures that when $T^{\mu\nu}$ is inserted into larger diagrams they will have their normal degree of divergence.

On the other hand, if we first let $M_{\psi'} \to \infty$, we see that $I \to \frac{1}{2}M_{\psi'}{}^{-2}$, so that

$$T_\Delta^{\mu\nu}(k,p,q) \xrightarrow[M_{\psi'}\to\infty]{} \frac{G}{2M_\psi}(g^2+3f^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta\,, \tag{4.8}$$

which has quadratic growth for large momenta. In fact in this limit we just recover the contribution to $T_\Delta^{\mu\nu}$ of the nonrenormalizable counterterm (3.23) considered previously. The effect of this will be that finite renormalized S-matrix elements, which contain $T_\Delta^{\mu\nu}$ as a subgraph, will diverge as $M_{\psi'} \to \infty$ (e.g., fermion-fermion scattering to sixth order, Fig 6).

Therefore we may not let $M_{\psi'} \to \infty$. In fact if we wish to keep G' small, say of order g, then we must keep $M_{\psi'}$ of the order of the vector-meson mass M_v, since $M_{\psi'} = \sqrt{2}\,G'M_v/g$.

The underlying reason that the anomalies are canceled for $\mathcal{L}_2^R$ is that the fermion–vector-meson couplings are parity-invariant, if the parity transformation for fermion fields is defined to be

$$P\psi(\vec{x}, t)P^{-1} = \gamma^0\psi'(-\vec{x}, t)\,. \tag{4.9}$$

In fact, we can define linear combinations of ψ and ψ', which are parity eigenstates under (4.9):

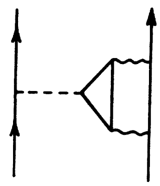

FIG. 6. A contribution to fermion-fermion scattering which contains a triangle subgraph in fourth order; conventions are as in Fig. 4.

$$\xi_\pm = \frac{1}{\sqrt{2}}(\psi \pm \psi'), \tag{4.10}$$

and the Lagrangian can be rewritten, at Stage i, as

$$\begin{aligned}\mathcal{L}_1^R = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}(\phi, \phi^*) \\ &+ \bar{\xi}_+(i\partial\!\!\!/ + fA\!\!\!/)\xi_+ + \bar{\xi}_-(i\partial\!\!\!/ + fA\!\!\!/)\xi_- \\ &- g[\bar{\xi}_+ A\!\!\!/\gamma_5\xi_- + \bar{\xi}_- A\!\!\!/\gamma_5\xi_+] \\ &+ G_+[(\bar{\xi}_+\xi_+ + \bar{\xi}_-\xi_-)\varphi + i(\bar{\xi}_-\gamma_5\xi_+ + \bar{\xi}_+\gamma_5\xi_-)\chi] \\ &+ G_-[(\bar{\xi}_+\xi_- + \bar{\xi}_-\xi_+)\varphi + i(\bar{\xi}_+\gamma_5\xi_+ + \bar{\xi}_-\gamma_5\xi_-)\chi],\end{aligned} \tag{4.11}$$

where $G_\pm = 1/2\sqrt{2}\,(G \pm G')$ and $\mathcal{L}(\phi, \phi^*)$ contains only scalar field variables. This form shows that all couplings are parity conserving except for the coupling proportional to G_- of the scalar mesons to the fermions. We call this form of the theory Stage 0.

One can go even farther and construct a theory which is parity-invariant at Stage 0, and in which parity is broken spontaneously at Stage ii. This we do by introducing an additional set of pseudoscalar mesons φ' and χ',

$$\phi' = \frac{1}{\sqrt{2}}(\varphi' + i\chi')$$

[coupled by $\mathcal{L}(\phi', \phi'^*)$ to $\mathcal{L}_1^R$] and by replacing the parity-violating coupling of the fermions with

$$G_-[(\bar{\xi}_+\xi_- + \bar{\xi}_-\xi_+)\varphi' + i(\bar{\xi}_+\gamma_5\xi_+ + \bar{\xi}_-\gamma_5\xi_-)\chi'].$$

At Stage ii both φ and φ' have nonvanishing vacuum expectation values, and this then breaks parity conservation (as long as $G_- \neq 0$). The resulting theory is the same as before, except that one now is left with an additional pseudoscalar meson φ'. The only advantage of doing this is the esthetic appeal of having a theory which in Stage 0 is both gauge- and parity-invariant, and in which all symmetries are broken spontaneously.

V. NON-ABELIAN MODELS

A. The Anomaly

In this section we shall extend our analysis to a discussion of models which might describe the weak interactions[2] and therefore must involve non-Abelian vector-meson gauge fields with axial-vector couplings to fermions. In the absence of any known gauge-invariant regularization technique for non-Abelian theories, the possibility exists that new anomalies associated with the vector mesons might appear. In any case the fermion anomalies are, in general, present and must be canceled by the previously discussed mechanism.

Consider the gauge theory described by the Lagrangian (Stage i)

$$\mathcal{L}_1 = -\tfrac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu} + \bar{\psi}[i\partial\!\!\!/ - g\gamma^\mu A_\mu(1-\gamma_5)]\psi, \tag{5.1}$$

where

$$\begin{aligned}G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu, \\ A_\mu &= \tfrac{1}{2}\lambda^a A^a_\mu, \\ [\lambda^a, \lambda^b] &= 2if_{abc}\lambda^c,\end{aligned} \tag{5.2}$$

and the λ^a's form a representation of an (arbitrary) Lie group. (We have suppressed the scalar mesons.) This Lagrangian is formally invariant under the infinitesimal gauge transformation

$$\begin{aligned}A^a_\mu &\to A^a_\mu - \partial_\mu\theta^a + gf_{abc}\theta^b A^c_\mu, \\ \psi &\to [1 - ig\theta^a\lambda^a(1-\gamma_5)]\psi.\end{aligned} \tag{5.3}$$

However, because of the triangle anomaly, this formal invariance is broken. The anomalous triangle graph can be calculated as in Sec. II. In the non-Abelian case, however, there are additional anomalies. Specifically, the square graph (the $AAAA$ vertex) is anomalous since the derivation of the naive Ward identity involves a translation of integration variables in linearly divergent triangle diagrams. These linearly divergent integrals are made finite and unique by symmetric integration and the requirement of Bose symmetry. These calculations can be summarized by the statement that the naively conserved currents

$$J^a_\mu = \bar{\psi}\gamma_\mu(1-\gamma_5)\tfrac{1}{2}\lambda^a\psi$$

satisfy

$$\partial^\mu J^a_\mu = W^a = cg^3\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\{\lambda^a[2\partial_\mu A_\nu\partial_\alpha A_\beta - i\partial_\mu(A_\nu A_\alpha A_\beta)]\}. \tag{5.4}$$

This form for the non-Abelian anomaly differs from that usually given in the literature; since we treat the vector-axial-vector ($V-A$) current as a unit, whereas the usual derivation treats vector and axial-vector currents asymmetrically.[17] Since W^α is a total divergence,

$$W^a = \partial^\mu \mathcal{W}^a_\mu = \partial^\mu cg^3\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\{\lambda^a[2A^\nu\partial^\alpha A^\beta - iA^\nu A^\alpha A^\beta]\}, \tag{5.5}$$

the current $\mathcal{J}^\mu_a = J^\mu_a - \mathcal{W}^\mu_a$ is conserved; and as in Sec. II we can then show that the infinitesimal variation of the action under (5.3) is $\delta S = \theta^a(x) W^a(x)$.

Clearly unless W^a vanishes (for all a) the theory will not be quasi-renormalizable. The trace in (5.4) can be evaluated directly, yielding

$$W^a = -(\tfrac{1}{4} g^3 c) d_{abc} \epsilon^{\mu\nu\alpha\beta} \partial_\mu [A_\nu^b (4\partial_\alpha A_\beta^c + f_{cde} A_\alpha^d A_\beta^e)],$$
$$d_{abc} = \tfrac{1}{4} \mathrm{Tr}[\lambda^a \{\lambda^b \lambda^c\}]. \quad (5.6)$$

Thus unless d_{abc} vanishes the anomaly will be present.[18]

If we apply this analysis to Weinberg's models for the weak and electromagnetic interactions of leptons,[2,19] we see that both are plagued by anomalies. In the first model,[2] built on a $SU(2)_L \otimes U(1)$ gauge group, the anomalies do not appear for the SU(2) gauge group [since $d_{abc}=0$ for all representations of SU(2)]; however, they are present for the U(1) gauge group (the hypercharge current). In this model the electron and the muon are independent multiplets. However, one cannot use the muon to cancel the electron anomaly, since both the electron and the muon neutrino are left-handed. In the second model[19] all the leptons are put into a triplet, and the gauge group is $SU(3)_L \otimes SU(3)_R$. The anomalies are certainly present, since $d_{abc} \neq 0$.

In order to remove the anomalies we must have for every right-handed (left-handed) fermion a corresponding left-handed (right-handed) partner with identical couplings to the vector gauge fields. Since in the real world we are interested in the weak interactions of both leptons and hadrons there are two possibilities for anomaly cancellation: (1) Both the leptonic and the hadronic weak currents are nonanomalous, and we double the number of fundamental leptons and hadrons. (2) Only the full weak current is nonanomalous, due to the cancellation of leptonic and hadronic anomalies.

B. Separate Leptonic and Hadronic Cancellations

The most straightforward way to cancel the anomalies in the leptonic weak currents is to double the number of leptons, introducing another electron, e'; a muon, μ'; and their massless right-handed neutrinos, $\nu_{e'}$ and $\nu_{\mu'}$. One then couples these to the gauge fields, interchanging right and left, with the same couplings as the ordinary leptons. The arguments presented in Sec. IV indicate that the resulting theory will be quasi-renormalizable, barring new anomalies associated with the non-Abelian gauge fields.[20] As in Sec. IV, one could also construct models in which parity was spontaneously broken at Stage *ii*. The only upper bound on the masses of the additional fermions is given by the vector-boson mass, following from the requirement of weak coupling. However, this is hardly stringent, since the vector mesons must have typical masses greater than 40 BeV. There is no experimental evidence against the existence of such leptons with masses greater than the mass of the K meson.

A similar mechanism would then be required to deal with the anomalies in the hadronic weak currents. Thus in a quark model one would need a new set of quarks, with chirally opposite weak couplings. Furthermore, if vector gluon fields are the mediators of the strong interactions, then both sets of quarks must couple with identical strength to the gluons; otherwise one would have anomalies in Green's functions involving the gauge fields and the gluons. One then has a new, conserved, baryon number associated with the new quarks. However, since these quarks could have an arbitrary mass and different scalar gluon couplings, one is not forced to expect their bound-state spectrum to be similar to that of the usual hadrons.

The disadvantages of the above anomaly-canceling mechanism are the following: (1) One must postulate at least two new conserved quantum numbers (lepton and baryon number) and the existence of new leptons for which there is no experimental evidence. (2) Having canceled the anomalies in the weak hadronic current, it is difficult to evade the consequences of the Sutherland-Veltman theorem, and PCAC therefore cannot be applied to $\pi^0 \to 2\gamma$.[4] In fact, experimental confirmation of the relations between various reactions derived from anomalous Ward identities[4,21] would be evidence against the above anomaly-canceling mechanism.

C. A Lepton-Hadron Anomaly Cancellation

It would be much more economical to cancel the leptonic anomalies with hadronic anomalies, obviating the introduction of new leptons and hadrons. To do this one must construct models in which the leptons and the fundamental fermion hadrons belong to identical multiplets of a given group, and couple in chirally opposite ways to the gauge vector fields. In fact such a cancellation naturally takes place if one attempts to incorporate nucleons into Weinberg's $SU(2)_L \otimes U(1)$ theory of electrons.[2] This is because the antinucleon doublet

$$\bar{N} = \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$$

has the same charge structure as the left-handed neutrino electronic doublet

$$l = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}.$$

Since $G_A/G_V \approx +1$ it is natural to assume that the unrenormalized nucleon weak current is $-(V-A)$. However, this means that the antinucleon doublet couples to $+(V+A)$, and this is exactly what is required to cancel the electronic and nucleonic anomalies.[22] We can then couple the nucleons directly to the scalar mesons, or indirectly through a $(\sigma, \vec{\pi})$ multiplet of scalar mesons as suggested by Weinberg.[23]

The above model does not incorporate muons and strange hadrons; however, it is easy to construct models that do. In fact models involving leptons and quarks in a symmetric fashion have been introduced before.[24] These models are based on an SU(4) quartet of quark fields, and have the important advantage of eliminating first-order weak neutral strangeness-changing currents. The quark quartet is composed of the usual $\mathcal{P}$, $\mathfrak{N}$, and λ SU(3) quarks with the addition of a $\mathcal{P}'$ quark, which differs from the triplet by one unit of a new quantum number (charm). In analogy with the above scheme for electrons and neutrons we assign to the antiquark the same charges as the four leptons. (One could always assign the quarks the same charges as the leptons; however, it would then be hard to understand why $G_A/G_V \approx +1$. The leptons and the quarks are then represented by the vector spinors[25]

$$l = \begin{pmatrix} \nu_e \\ \nu_\mu \\ e^- \\ \mu^- \end{pmatrix}, \quad q = \begin{pmatrix} \mathfrak{N} \\ \lambda \\ \mathcal{P}' \\ \mathcal{P} \end{pmatrix}, \tag{5.7}$$

and the quarks are assigned charges 0, 0, +1, and +1 for $\mathfrak{N}$, λ, $\mathcal{P}'$ and $\mathcal{P}$, respectively. The antiquark and lepton quartets are then symmetric. The leptonic charged weak current belongs to the multiplet of currents

$$\vec{J}^{\,l}_\mu = \bar{l}\vec{C}_L \gamma_\mu (1+\gamma_5) l, \tag{5.8}$$

where

$$C_L^+ = (C_L^-)^\dagger = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad C_L^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.9}$$

(all entries in 4×4 matrices are 2×2 matrices), whereas the hadron charged weak currents belong to the multiplet

$$\vec{C}_H = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \vec{C}_L \begin{pmatrix} u^\dagger & 0 \\ 0 & u^\dagger \end{pmatrix}, \tag{5.10}$$

$$u = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \tag{5.11}$$

(θ is the Cabibbo angle).

It is now easy to generalize Weinberg's model to incorporate these leptons and hadrons. We exhibit the couplings of the leptons and the quarks to the $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)$ gauge fields:

$$\begin{aligned} \mathcal{L}_{\text{int}} = {} & \bar{l}[g\vec{C}_L \cdot \vec{A\!\!\!/} + g' B\!\!\!/]\tfrac{1}{2}(1+\gamma_5) l \\ & - \bar{q}[g\vec{C}_L \cdot \vec{A\!\!\!/} + g' B\!\!\!/]\tfrac{1}{2}(1+\gamma_5) q \\ & + \tfrac{1}{2} g' [\bar{l} B\!\!\!/(1 - C_L^3)(1-\gamma_5) l - \bar{q} B\!\!\!/(1 - C_H^3)(1-\gamma_5) q]. \end{aligned} \tag{5.12}$$

From the discussion above we know that there will be no anomalies associated with the SU(2) gauge fields alone (since $d_{abc} = 0$), and the anomalies in triangles with one or three B^μ fields clearly cancel. Furthermore, there are no neutral strangeness-changing weak currents, and the hadron weak current is still anomalous by itself, so π^0 can decay into two photons. The disadvantages of this scheme are (1) It forces us to a somewhat strange charge assignment for the quarks. In particular the charge operator has a singlet piece and the baryon octet cannot be built up from a single quartet of quarks.[26] As in the previous scheme, we must introduce a new quantum number (charm), for which there is no evidence. (2) The above theory places severe constraints on the strong interactions. Thus, for example, one cannot introduce neutral vector or axial-vector gluons, which mediate the strong interactions, since these would give rise to new anomalies. Scalar or pseudoscalar neutral gluons above are unacceptable, since the strong interactions would then be invariant under four charge-conjugation operations that act separately on each quark.[27] This would lead to an unacceptable degeneracy of charmed and uncharmed states. Therefore, in the context of the above anomaly-canceling scheme, one cannot mediate the strong interactions with neutral singlet gluons. One could, of course, introduce multiplets of scalar gluons (say, a generalized σ model); however, we feel a bit uneasy about a scheme which is so unstable with respect to the strong interactions.

In conclusion we see that although the previously suggested models of the weak interactions are not quasi-renormalizable there are a variety of ways to cure the difficulties. It is clear the anomaly cancellation will place additional constraints on any model which attempts to describe the real world.

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APPENDIX: ANOMALOUS TRANSFORMATION LAW FOR THE ACTION

We derive Eq. (2.12) using (2.10) and (2.11). The total action S is separated into three pieces: The free fermion term S_F, the free meson term S_M, and the interaction term S_I. Consider first the infinitesimal change

$$\delta\psi = ig\frac{\gamma_5}{\mu}A\psi, \quad \delta\bar{\psi} = i\frac{g}{\mu}\bar{\psi}\gamma_5 A$$

applied to $S_F + S_I$. Since $S_F + S_I$ corresponds to the interaction of a fermion with an *external* vector-meson field, and since there *does* exist a conserved current $\mathcal{J}_5^\mu$ associated with gauge transformations, we conclude that

$$\begin{aligned} S_F(\psi+\delta\psi, \bar{\psi}+\delta\bar{\psi}) + S_I(\psi+\delta\psi, \bar{\psi}+\delta\bar{\psi}, A^\mu) \\ = S_F(\psi,\bar{\psi}) + S_I(\psi,\bar{\psi},A^\mu) - \frac{g}{\mu}\int d^4x\, \mathcal{J}_5^\mu(x)\partial_\mu A(x). \end{aligned} \tag{A1}$$

(Equations of motion must not be used to drop the last term, since we need the variation of S for arbitrary fields, not just fields satisfying the Euler-Lagrange equations.) Equivalently we have to first order in A

$$\begin{aligned} S_F(\psi+\delta\psi, \bar{\psi}+\delta\bar{\psi}) + S_I(\psi+\delta\psi, \bar{\psi}+\delta\bar{\psi}, A^\mu+\delta A^\mu) \\ = S_F(\psi,\bar{\psi}) + S_I(\psi,\bar{\psi},A^\mu+\delta A^\mu) - \frac{g}{\mu}\int d^4x\, \mathcal{J}_5^\mu(x)\partial_\mu A(x), \end{aligned} \tag{A2}$$

with $\delta A^\mu = -(1/\mu)\partial^\mu A$. Furthermore we also have

$$\begin{aligned} S_F(\psi,\bar{\psi}) + S_I(\psi,\bar{\psi},A^\mu+\delta A^\mu) \\ = S_F(\psi,\bar{\psi}) + S_I(\psi,\bar{\psi},A^\mu) + \frac{g}{\mu}\int d^4x\, J_5^\mu(x)\partial_\mu A(x). \end{aligned} \tag{A3}$$

This follows from the definition that J_5^μ is the object to which A^μ couples. Combining (A2) and (A3) yields

$$\begin{aligned} \delta(S_F+S_I) &= \frac{g}{\mu}\int d^4x[J_5^\mu(x) - \mathcal{J}_5^\mu(x)]\partial_\mu A(x) \\ &= \frac{g}{\mu}\int d^4x\, 4cg^2\epsilon^{\mu\nu\alpha\beta}A_\nu(x)\partial_\alpha A_\beta(x)\partial_\mu A(x) \\ &= -\frac{g^3c}{\mu}\int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)A(x). \end{aligned} \tag{A4}$$

Here we have used (2.11). Since $F^{\mu\nu}$ is gauge-invariant, the second and higher variations of $S_F + S_I$ vanish. Thus we conclude that

$$\begin{aligned} S_F(e^{i\gamma_5 Ag/\mu}\psi, \bar{\psi}e^{i\gamma_5 Ag/\mu}) \\ + S_I\left(e^{i\gamma_5 Ag/\mu}\psi, \bar{\psi}e^{i\gamma_5 Ag/\mu}, A^\mu - \frac{1}{\mu}\partial^\mu A\right) \\ = S_F(\psi,\bar{\psi}) + S_I(\psi,\bar{\psi},A^\mu) \\ - \frac{g^3c}{\mu}\int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)A(x). \end{aligned} \tag{A5}$$

Finally we calculate the change of S_M, making use of the explicit formula for the free meson Lagrangian; no known anomalies are associated with it.

$$\begin{aligned} S_M\left(A^\mu - \frac{1}{\mu}\partial^\mu A\right) \\ = S_M(A^\mu) + \int d^4x\left[\tfrac{1}{2}\partial^\mu A(x)\partial_\mu A(x) + \mu A^\mu(x)\partial_\mu A(x)\right]. \end{aligned}$$

In summary we have

$$\begin{aligned} S\left(e^{i\gamma_5 Ag/\mu}\psi, \bar{\psi}e^{i\gamma_5 Ag/\mu}, A^\mu - \frac{1}{\mu}\partial^\mu A\right) \\ = S(\psi,\bar{\psi},A^\mu) \\ + \int d^4x\left[\tfrac{1}{2}\partial^\mu A(x)\partial_\mu A(x) + \mu A^\mu(x)\partial_\mu A(x)\right] \\ - \frac{g^3c}{\mu}\int d^4x F^{\alpha\beta}(x)\tilde{F}_{\alpha\beta}(x)A(x). \end{aligned} \tag{A6}$$

This establishes the result (2.12).

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$$\exp\left\{\frac{-i}{2\alpha}\int d^4x\,[\partial_\mu A^\mu(x)]^2\right\}$$

in the limit of $\alpha \to 0$. The answer may be shown to be $\delta[\partial^\mu A_\mu]$. Recall that

$$\lim_{\alpha\to 0}\frac{1}{\sqrt{4\pi\alpha}}\,e^{-z^2/4\alpha}=\delta(z).$$

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Asymptotic Expansions and Nonlinear Field Theories

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The Efimov-Fradkin method in nonlinear field theories, in the context of a simple rational function Lagrangian, is analyzed in the light of the Carleman uniqueness theorem for asymptotic expansion. It is proven that the Carleman theorem allows the elimination of the ambiguities related to the formal summation of the perturbation expansion in the minor coupling constant. This result however implies that the two-point Green's function to second order in the major coupling constant does not have the correct analyticity properties as required by unitarity. These results are easily generalized to other classes of nonlinear theories.

I. INTRODUCTION

Nonlinear field theories, which arise in connection with physical requirements, such as for example chiral symmetry,[1] are defined by interaction Lagrangians of the type

$$L_I(\phi)=\lambda : F(g\phi(x)):, \tag{1.1}$$

where F is some nonpolynomial function of the field $\phi(x)$, and it is customary to define λ as the