Method for Expressing Dirac Spinor Amplitudes in Terms of Invariants and Application to the Calculation of Cross Sections*

Harold W. Fearing and Richard R. Silbar
University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544
(Received 28 May 1971)

We describe here a method for expressing spinor amplitudes $M = \bar{u}(p', s')\Gamma u(p, s)$ in terms of invariants. The method also provides a simple formula for the square of M summed on spins. When Γ contains more than five terms or has more than two γ-matrices per term this formula involves the evaluation of fewer and shorter traces than the usual trace technique. Thus this should be a useful computational device for calculating cross sections. The methods presented here are independent of the spinor representation, but can be easily specialized to the important helicity case. A number of other possible applications of the amplitude formula are also briefly discussed.

I. INTRODUCTION

In many practical calculations one must deal with amplitudes of the form

$$M = M(p', s', p, s) = \bar{u}(p', s')\Gamma_M u(p, s), \tag{1}$$

where $u(p, s)$ is a free Dirac spinor and Γ_M is a sum of products of Dirac γ-matrices γ_μ. Usually one goes on to calculate a cross section by squaring and summing over spins, e.g.,[1]

$$|M|^2_{\text{unpol}} = \sum_{s,s'} |M(p', s', p, s)|^2$$
$$= \mathrm{Tr}\{(\gamma\cdot p' + m')\Gamma_M(\gamma\cdot p + m)\bar{\Gamma}_M\}. \tag{2}$$

If Γ_M is at all complicated, the evaluation of this trace, although perfectly straightforward, is often an extremely lengthy and tedious process. Alternatively, one can avoid the trace evaluation by choosing a particular representation for the γ_μ and $u(p, s)$ and evaluating $M(p', s', p, s)$ by simply multiplying out the matrices; the resulting complex numbers are then squared and the appropriate spin sum is taken directly. This procedure has the advantage of transferring much of the labor to a machine, but the disadvantage of losing sight of the mathematical structure of the quantity being computed.

On the other hand, there are times when one wants an analytical expression for the amplitude itself. In many calculations, such as those involving dispersion relations or the Regge or absorption models, a necessary first step is to relate the helicity amplitudes $\langle\lambda'\cdots|T|\lambda\cdots\rangle$ to those obtained from certain Feynman graphs (e.g., Born terms) or to the invariant functions for the process considered [e.g., the $A(s, t)$ and $B(s, t)$ of π-N elastic scattering]. In such cases it is customary to evaluate the $M(p', s', p, s)$ involved using helicity spinors in a particular frame, again a fairly complicated calculation that must be done separately for each set of helicities and for each tensor index present in Γ_M.[2]

The amplitude of Eq. (1) is, however, a Lorentz covariant. As such, it should be expressible in terms of invariants and of covariants formed from the various spin and momentum four-vectors of the problem, including p'_μ, s'_μ, p_μ, and s_μ. We show here how this can be done in an almost trivial fashion which allows one to treat all tensor indices in Γ_M and all choices of s' and s at once. The resulting expressions are independent of any particular representation of γ_μ or $u(p, s)$ but can be easily specialized to the often-used helicity representation.

For most calculations one really wants $|M|^2_{\text{unpol}}$, which is normally obtained via Eq. (2). Our method, however, gives $|M|^2_{\text{unpol}}$ as a square of an analytic result for the scalar amplitude M. In principle one could multiply out the square but the result would be more complicated than that obtained from Eq. (2) because of the presence of invariants containing spin vectors. However, for numerical purposes one just evaluates M and squares the resulting number. We will show that in complicated cases the effort required to obtain M is much less than to get $|M|^2$ via Eq. (2) and that the result involves fewer terms. This may make it possible to do calculations by hand (or with the aid of a small, limited computer) that were previously inaccessible. Even at laboratories with large computers and access to algebraic trace-taking programs[3] the simplifications may make more complicated calculations practical.

In view of the utility of these results and the fact that they do not seem to be generally well known, we present here a unified discussion of these methods and of the simplifications they allow.[4,5] In Sec. II of the paper we describe the

trick for expressing a spinor amplitude in terms of covariants and make a number of comments regarding the method. We then show in Sec. III how to evaluate the squared, spin-averaged amplitude and discuss when this method is simpler than the usual trace method indicated in Eq. (2). Finally, Sec. IV is devoted to a discussion of a number of other applications and the Appendix contains formulas derived by these methods for a number of common Dirac covariants.

II. COVARIANT EVALUATION OF SPINOR AMPLITUDES

The trick for evaluating the amplitude M of Eq. (1) in terms of covariants consists simply of multiplying and dividing by some particular spinor amplitude R^* and using the projection operator for $u(p, s)\bar{u}(p, s)$ to get a trace.[1] We write

$$\begin{aligned} M &= R^*M/R^* \\ &= \bar{u}(p, s)\bar{\Gamma}_R u(p', s')\bar{u}(p', s')\Gamma_M u(p, s)/R^* \\ &= \tfrac{1}{4}\,\mathrm{Tr}\{(\gamma\cdot p + m)(1+\gamma_5\gamma\cdot s)\bar{\Gamma}_R \\ &\qquad \times(\gamma\cdot p' + m')(1+\gamma_5\gamma\cdot s')\Gamma_M\}/R^*, \end{aligned} \tag{3}$$

where $R = \bar{u}(p', s')\Gamma_R u(p, s)$. Thus, knowing R, which we work out in a few special cases below, we can express any arbitrary M in terms of covariants, for any set of spin quantum numbers and any set of tensor indices which may be present in Γ_M, by evaluating a single trace.

The magnitude of R can be obtained at once, i.e.,

$$\begin{aligned} |R|^2 &= \bar{u}(p, s)\bar{\Gamma}_R u(p', s')\bar{u}(p', s')\Gamma_R u(p, s) \\ &= \tfrac{1}{4}\,\mathrm{Tr}\{(\gamma\cdot p + m)(1+\gamma_5\gamma\cdot s)\bar{\Gamma}_R \\ &\qquad \times(\gamma\cdot p' + m')(1+\gamma_5\gamma\cdot s')\Gamma_R\}. \end{aligned} \tag{4}$$

Thus, for a given choice for Γ_R, R is known up to a (possibly complex) phase $\eta = \eta(p', s', p, s)$. Alternatively, if the phase is needed one can choose a particular representation for the γ_μ and $u(p, s)$ and evaluate R directly, thus obtaining both $|R|$ and η in that representation.

The simplest choice for Γ_R, since it reduces the number of γ matrices in the trace for M to a minimum, is $\Gamma_R = 1$. In this case R becomes the usual Dirac scalar covariant,

$$\begin{aligned} S &= \bar{u}(p', s')u(p, s) \\ &= \eta[(p'\cdot p + m'm)(1 - s'\cdot s) + p'\cdot s p\cdot s']^{1/2}. \end{aligned} \tag{5}$$

In this case the phase η must satisfy the conditions $\eta(p, s, p, s) = 1$ and $\eta^*(p', s', p, s) = \eta(p, s, p', s')$ which follow from the constraints $S(p, s, p, s) = 2m$ and $S^*(p', s', p, s) = S(p, s, p', s')$. For most purposes it is sufficient to simply choose $\eta = 1$. (See, however, the discussion below.)

We have so far expressed all results in covariant form using the spin four-vector s^μ. Formally this four-vector is defined by a Lorentz transformation applied to $(0, \hat{n})$ where $\hat{n}$ is a unit vector defining the direction of the spin in some rest frame. Alternatively using the conditions $s^2 = -1$ and $s\cdot p = 0$ one can solve for $s^\mu = (s_0, \vec{s})$ in a general frame where the particle has momentum $p^\mu = (E, \vec{p})$ to obtain[6]

$$\begin{aligned} s_0 &= \hat{s}\cdot\vec{\beta}[1 - (\hat{s}\cdot\vec{\beta})^2]^{-1/2}, \\ \vec{s} &= \hat{s}[1 - (\hat{s}\cdot\vec{\beta})^2]^{-1/2}, \end{aligned} \tag{6}$$

where $\vec{\beta} = \vec{p}/E$ and $\hat{s}$ is a unit vector in the direction of $\vec{s}$ in this general frame. This unit vector can be expressed in terms of the rest frame $\hat{n}$ by considering the Lorentz transformation connecting the frames. This gives, with $\gamma = (1-\beta^2)^{-1/2}$,

$$\hat{s} = [\hat{n} + (\gamma - 1)(\hat{\beta}\cdot\hat{n})\hat{\beta}][1 + \gamma^2(\beta\cdot\hat{n})^2]^{-1/2}. \tag{7}$$

For convenience in eventually summing on spins we rewrite our results in terms of a parameter λ with values $\pm\frac{1}{2}$ corresponding to the two possible spin states which are defined in the rest frame by $(0, \pm\hat{n})$. Since s^μ is linear in $\hat{n}$, in a general frame these two states correspond to $\pm s^\mu$. Thus, we simply must make the replacement

$$s^\mu \to 2\lambda s^\mu \tag{8}$$

in all of our results. Since s^μ is a four-vector the quantities $s\cdot p'$, etc., appearing in the final expressions will be invariants (depending however on the choice of the rest frame vector $\hat{n}$) and can be evaluated in any frame.

The parameter λ denotes the component of spin along the axis $\hat{n}$ in the rest frame. Thus for example if $\hat{n} = \hat{z}$, $\lambda = m_z$, the z component of spin in the rest frame. If $\hat{n} = \hat{p}$ where $\hat{p}$ is the direction of the particle momentum after the boost, then λ is the helicity. In this case s^μ takes, in this boosted frame, the simple form

$$s^\mu = (p/m, E\hat{p}/m). \tag{9}$$

Using the substitution given in Eq. (8) we can rewrite Eq. (5) for S in a general frame as

$$\begin{aligned} S &= \eta(K + 4\lambda\lambda' L)^{1/2}, \\ K &= p\cdot p' + mm', \\ L &= p'\cdot s p\cdot s' - s'\cdot s(p\cdot p' + mm'), \end{aligned} \tag{10}$$

where, of course, each s^μ is now and henceforth understood to be expressed in terms of p^μ and its quantization axis by means of Eqs. (6) through (8). Thus with the choice $R = S$, Eq. (3) for a general amplitude M becomes

$$
\begin{aligned}
M &= \eta(A + 2\lambda' B' + 2\lambda B + 4\lambda\lambda' C)/(K + 4\lambda\lambda' L)^{1/2}, \\
A &= \tfrac{1}{4}\,\mathrm{Tr}\{(\gamma\cdot p + m)(\gamma\cdot p' + m')\Gamma_M\}, \\
B &= \tfrac{1}{4}\,\mathrm{Tr}\{(\gamma\cdot p + m)\gamma_5\gamma\cdot s(\gamma\cdot p' + m')\Gamma_M\}, \\
C &= -\tfrac{1}{4}\,\mathrm{Tr}\{(\gamma\cdot p + m)\gamma\cdot s\gamma\cdot s'(\gamma\cdot p' + m')\Gamma_M\},
\end{aligned}
\tag{11}
$$

with B' the same as B but with s' in place of s.

As an illustration of this method we give in an appendix the results for the usual five Dirac covariants S, V_μ, $T_{\mu\nu}$, A_μ, and P corresponding to the choices $\Gamma_M = 1$, γ_μ, $\sigma_{\mu\nu}/\sqrt{2}$, $i\gamma_\mu\gamma_5$, and γ_5. An arbitrary Γ_M can of course be decomposed in terms of this complete set of matrices. However, it is usually simpler to apply Eq. (11) to Γ_M directly.

To evaluate M or S in the important case of the helicity representation we simply substitute the specific choice for s^μ given in Eq. (9) into the general expressions. In particular, using Eq. (5) S becomes

$$
\begin{aligned}
S = \eta[2(E' + m')(E + m)]^{-1/2} & \\
\times[(E' + m')(E + m) - 4\lambda\lambda' p' p] & \\
\times(1 + 4\lambda\lambda'\cos\theta)^{1/2}, &
\end{aligned}
\tag{12}
$$

where $\cos\theta = \hat{p}\cdot\hat{p}'$.

When discussing helicity amplitudes it is conventional to use Jacob and Wick[7] phase conventions. This amounts to a specific choice for η which can be determined by adopting for the spinors the specific representation (for a spinor with momentum p in direction Θ)

$$
u(p, \lambda) = (E + m)^{-1/2} e^{-i\sigma_2\Theta/2} \begin{pmatrix} (E + m)\chi_\lambda \\ 2\lambda p\chi_\lambda \end{pmatrix}, \tag{13}
$$

where

$$
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \chi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

and evaluating S directly. This gives

$$
S = [(E' + m')(E + m)]^{-1/2}[(E' + m')(E + m) - 4\lambda\lambda' pp'][\,|\lambda' + \lambda|\cos^2\tfrac{1}{2}(\Theta' - \Theta) + (\lambda' - \lambda)\sin^2\tfrac{1}{2}(\Theta' - \Theta)\,]. \tag{14}
$$

Clearly this result is the same as Eq. (12) up to a phase factor which is ± 1 according to the values of λ, λ', $\hat{p}$, $\hat{p}'$.

We conclude this section with a number of comments.

(1) A similar procedure holds for amplitudes involving charge conjugate spinors $v(p, s)$. All that changes is the projection operator, i.e., $(\gamma\cdot p + m)$ becomes $(\gamma\cdot p - m)$, which amounts to a replacement of m by $-m$ for each antiparticle in all of our general formulas.

(2) We have so far taken the simplest and most natural choice $\Gamma_R = 1$. Under some circumstances however, other choices, e.g., $\Gamma_R = \gamma_5$, may be advantageous in that they may simplify the trace in Eq. (3) or change the singularity structure in accordance with comment (4) below.

(3) The trick used in Eq. (3) only works if $R \neq 0$. The factor $(1 + 4\lambda\lambda'\cos\theta)^{1/2}$ in Eq. (12) indicates, however, that $R = S$ vanishes whenever $\theta = 0$ and $\lambda = -\lambda'$, or $\theta = \pi$ and $\lambda = \lambda'$. This same factor also occurs in all of the other simple choices for Γ_R, i.e., γ_0, γ_5, or $\gamma_0\gamma_5$, as can be seen from the formulas in the Appendix. These are isolated zeros, however, and M can be evaluated at such points by considering the limit as $\theta \to 0$ or π.

(4) From Eq. (12) and the Appendix we see that $R = S, P$ also vanish in the massless case ($m = m' = 0$) when $\lambda' = \lambda$ regardless of the value of θ. For $R = V_0 = \gamma_0$ or $R = A_0 = i\gamma_0\gamma_5$, the zero occurs when $\lambda = -\lambda'$. Thus in a calculation such as Ref. 5, where the original amplitude vanishes for $\lambda = -\lambda'$, the choice $R = V_0$ is quite convenient.

(5) We have considered specifically only situations in which two fermions (one fermion line) are present. However, these methods generalize in a straightforward manner to cases with more than one fermion line. For example, for an amplitude of the form $M = \bar{u}_3\Gamma u_1\bar{u}_4\Gamma u_2$ one simply chooses a multiplying factor $R = R_1R_2$ with $R_1 = \bar{u}_3\Gamma_1 u_1$ and $R_2 = \bar{u}_4\Gamma_2 u_2$. The two factors R_1 and R_2 are independent and can be separately chosen in the most convenient fashion.

(6) In some cases one might want to choose different multiplying factors, say $R_{\rm I}$ and $R_{\rm II}$ for different pieces of the amplitude. It is then necessary to know the relative phase of $R_{\rm I}$ and $R_{\rm II}$. This situation arises most naturally in processes with identical particles. For example, consider an antisymmetrized amplitude of the form (as in proton-proton elastic scattering)

$$
M = \bar{u}_3\Gamma u_1\bar{u}_4\Gamma u_2 - \bar{u}_4\Gamma u_1\bar{u}_3\Gamma u_2 .
$$

Convenient multiplying factors would then be $R_{\rm I} = \bar{u}_3 u_1\bar{u}_4 u_2$ for the first term and $R_{\rm II} = \bar{u}_4 u_1\bar{u}_3 u_2$ for the second. The magnitudes $|R_{\rm I}|$ and $|R_{\rm II}|$ can be found as before and their relative phase $\eta_{\rm rel}$ determined from $R_{\rm I}R_{\rm II}^* = \eta_{\rm rel}|R_{\rm I}||R_{\rm II}|$ by evaluating the trace $R_{\rm I}R_{\rm II}^*$.[5] As this trace can be rather long it is perhaps easier to simply calculate the first term using the multiplying factor $R_{\rm I}$, determine the phase of $R_{\rm I}$ as a function of the momenta and spins by adopting a specific representation for the spinors, and then antisymmetrize the result. A

third alternative, which avoids phase complications completely, is to use the factor R_I for both terms. This has the severe disadvantage, however, of converting the second term to a trace which is as long as or longer than $R_I R_{II}^*$, though it still may be shorter than some of the cross term traces arising in the usual method of calculating $|M|^2$.

(7) The method can be generalized to spinor amplitudes involving higher-spin particles. The analog of Eq. (3) requires only a knowledge of the projection operator corresponding to $u\bar{u}$ for all values of the spin projections.[8] (In this regard one should be aware of the dimensions arising from $\bar{u}u = 2m$ rather than 1.)

III. APPLICATIONS TO SUMMING SQUARED AMPLITUDES ON SPINS

To calculate the cross section for a process having amplitude M it is necessary to sum $|M|^2$ over the fermion spins. Clearly one can use the explicit expressions for M given in Eq. (11) to do the sum on λ, λ' numerically. It is very simple, however, to do this sum analytically. Thus from Eq. (11) we have

$$|M|^2_{\text{unpol}} = \sum_{\lambda,\lambda'=\pm 1/2} |M|^2 = 4(K^2 - L^2)^{-1}\{K(|A|^2 + |B|^2 + |B'|^2 + |C|^2) - 2L\,\text{Re}(A^*C + B^*B')\}. \tag{15}$$

In principle Eq. (15) gives an analytic expression for $|M|^2_{\text{unpol}}$, written in factored form. To evaluate $|M|^2_{\text{unpol}}$, however, one would never carry out the implied multiplications algebraically. Instead one should evaluate $A, B, \ldots$ numerically from the results of Eq. (11) and substitute the numbers into Eq. (15). Since the total number of terms contained in $A, B, \ldots$ is in general less than the number in $|M|^2$ evaluated from Eq. (2), this technique can lead to a saving in computer time.[9]

Despite its appearance, this last equation must be independent of quantization axes, a fact which could be invoked as a check on numerical work. It also is fairly simple because the denominator of $|M|^2$, i.e., $|S|^2$, did not contain terms linear in λ or λ', as can be seen from Eq. (10). One can in fact show that $|R|^2$ can contain such linear terms only if Γ_R is a sum of two or more terms, e.g., $\Gamma_R = 1 - \gamma_5$, a choice which might be of use in some weak interaction processes. This means, however, that for any of the other simple choices of R, viz., V_μ, $T_{\mu\nu}$, A_μ, P, we get after summing on spins an expression of the same form as Eq. (15), but with different values for A, B, B', C, K, and L.

It is clear that this same procedure works when only one of the spins is summed over. Thus for example summing on the final polarization λ'

$$\sum_{\lambda'=\pm 1/2} |M|^2 = \tfrac{1}{2}|M|^2_{\text{unpol}} + 4\lambda(K^2 - L^2)^{-1} \times\{K\,\text{Re}(A^*B + B'^*C) - L\,\text{Re}(A^*B' + B^*C)\}. \tag{16}$$

In this case one of course chooses a quantization axis $\hat{n}$ appropriate to the desired result. This formula also holds when only the incident polarization λ is summed but with $B \rightleftarrows B'$ and $\lambda \rightleftarrows \lambda'$.

The evaluation of $|M|^2$ using the results of Eq. (11) in Eqs. (15) or (16) may, when Γ_M is sufficiently complicated, have distinct advantages over the usual trace techniques. The Dirac algebra and the trace which must be computed are shortened considerably with this method at the expense, however, of some preliminary algebra necessary to express the extra invariants containing s and s', e.g., $s'\cdot p$, in terms of standard ones. This is not necessarily much of a disadvantage since $|M|^2$ is evaluated numerically anyway and thus, once obtained, the required transformations become just a few extra steps in a computer program.

To get a more precise feeling for how complex Γ_M must be before this method becomes simpler than the usual one, suppose Γ_M is the sum of r terms, each of which contains a maximum of n γ-matrices (not counting possible γ_5's). Consider first the case in which spins are not summed. Then with the usual method $|M|^2$ is obtained by calculating $4r^2$ traces, where by "trace" we mean a trace containing two energy projection operators, one term consisting of $\leq n$ γ-matrices from each of the two Γ_M, and one of the four possible terms coming from the spin projection operators, i.e., 1, $\gamma_5\gamma\cdot s$, $\gamma_5\gamma\cdot s'$, or $\gamma_5\gamma\cdot s\gamma_5\gamma\cdot s'$. However, the off-diagonal pairs of terms are complex conjugates of one another and so the actual number of different traces to be calculated in the usual method is $4[r + \frac{1}{2}r(r-1)] = 2r(r+1)$ with a maximum of $2n+4$ γ-matrices per trace. Using the method described here one must calculate A, B, C of Eq. (11), that is, one must calculate $3r$ traces with a maximum of $n+4$ γ-matrices per trace. Thus, when spins are not summed, this method *always* gives fewer traces and *always* gives a smaller maximum number of γ-matrices per trace than the usual one. When all spins are summed however, the usual trace method requires only $\frac{1}{2}r(r+1)$ traces with a maximum of $2n+2$ γ-matrices per trace while the method described here of course requires the same number of traces and γ-matrices per trace as when spins are not summed. Thus, formally, Eq. (15) is preferable when

$3r < \frac{1}{2}r(r+1)$, i.e., $r > 5$ or when $n+4 < 2n+2$, i.e., $n > 2$.

In actual practice before using this method, one probably wants r and n somewhat larger than their minimum values in order to offset the extra work involved in calculating the new invariants $s' \cdot p$, etc. One other consideration may also affect the usefulness of this method. The usual trace technique gives $|M|^2_{\text{unpol}}$ directly in terms of invariants such as $p \cdot p'$ which are easily expressed in terms of the total energy s and momentum transfer t. With this method, however, one must also deal with quantities such as $s \cdot p'$ which can easily be expressed in a given frame and a given choice of quantization axis $\hat{n}$ in terms of energies, angles, and momenta but less easily as functions of s and t. This makes no difference for numerical purposes, but may be a disadvantage if one wants a final analytic result in terms of s and t.

For some typical processes the values of r and n which apply are as follows[10]:

(1) For the Born contributions to Compton scattering (nucleon pole graphs) $r = 2$ and $n = 3$. If we include both electric and magnetic coupling at the $NN\gamma$ vertices, these numbers become $r = 18$ and $n = 5$.

(2) For πN scattering with the usual invariant amplitudes, A and B, $r = 2$, and $n = 1$.

(3) For NN scattering using the usual Goldberger-Grisaru-MacDowell-Wong (GGMW)[11] invariant amplitudes, $r = 6$, and $n = 2$.

(4) For neutron β decay, keeping only the four first-class terms, $r = 5$ and $n = 2$.

Thus, even for some very simple processes, the method of calculating $|M|^2$ described is perhaps on the borderline of usefulness. Two much more complicated calculations with which the authors have been separately involved are

(5) soft-pion production in $NN \to NN\pi$,[12] where $r = 24$ and $n = 4$, and

(6) W-boson production in $\mu p \to \mu n W^+$,[13] where $r = 56$ and $n = 4$.

These last calculations both used standard methods but are obviously sufficiently complicated to be prime candidates for the techniques described here. Actually in Ref. 12 the amplitudes $M_{\lambda'\lambda}$ were evaluated numerically from expressions worked out algebraically in the helicity representation in the center-of-mass frame, and then $|M|^2_{\text{unpol}}$ was computed. In Ref. 13 an algebraic expression for $|M|^2_{\text{unpol}}$ was found from Eq. (2) using the program SCHOONSCHIP to take the traces.

As a test of our method we repeated parts of this latter calculation. Originally it was necessary to divide $|M|^2$ into eight parts to avoid swamping the program, and the final run required a total of some 30 minutes of CDC 6600 computer time and resulted in about 5000 terms. Using this method, Eq. (11), with essentially the same SCHOONSCHIP program we obtained a factor of 20 to 25 saving in time required to obtain a result which could be evaluated numerically. Furthermore the result contained a factor of about 4 fewer terms, which in general were each simpler than before. Thus we estimate a factor of 3 or so saving in the time required for numerical evaluation of $|M|^2$, even allowing for the extra work involved in calculating $s' \cdot p$, etc., and in summing over the W-boson polarization. Thus for calculations as complicated as this, our method seems to give significant savings.

IV. OTHER APPLICATIONS

We conclude by briefly mentioning a number of the, to us, more obvious applications of formulas for spinor amplitudes in terms of Lorentz invariants.

(1) One use of Eq. (11) is to provide analytic expressions for amplitudes in order to compare the relative importance of various contributions, e.g., contributions of different graphs. These can also be converted into scalar expressions in terms of the kinematic variables (such as s and t) and then studied, for example, at large s or at small t to look for regularities, good approximations, etc.

(2) In somewhat the same vein, one can easily compare the values of M for different choices of spin states, i.e., different helicity amplitudes. A recent example of this is a discussion of s-channel helicity conservation by Gilman *et al.*[14]

(3) As mentioned in the Introduction, it is often necessary to find the contributions to the helicity amplitudes from certain Feynman graphs. Such is the case with the absorption model,[15] where the partial wave amplitudes in the helicity representation, as obtained from a Born graph, are modified to take account of the absorption. The method of Sec. II might simplify the algebra considerably.[2]

(4) As also mentioned, it is often necessary to relate the helicity amplitudes for a process to the invariant amplitudes (invariant functions) appearing in $\bar{u}(p')\Gamma_\mu u(p)$. The former have definite and convenient angular momentum properties while the latter possess definite analytic properties in the energy and momentum transfer variables. (In addition to the analyticity which can be proved from field theory, this might include model assumptions such as Regge behavior at high energies.) Usually these relations have been found by adopting a specific representation and multiplying out the matrices; see, for example, the classic

discussions of πN interactions by Frazer and Fulco[16] and Chew-Goldberger-Low-Nambu(CGLN)[17] or of NN scattering by GGMW.[11] Again, the method of Sec. II may be useful for obtaining these relations in a quick and representation-free way.

The above list of applications for this method of evaluating a spinor amplitude in terms of Lorentz invariants is obviously not exhaustive but should give at least an indication of the general usefulness of such techniques in many areas of particle physics.

ACKNOWLEDGMENTS

We wish to acknowledge useful conversations with many colleagues including A. M. Bolsterli, L. Heller, M. D. Scadron, M. E. Schillaci, and J. Smith.

APPENDIX

For reference we list here the amplitudes corresponding to the Dirac covariants V, T, A, P. [Formulas for S are given in Eqs. (5) and (12).] We give the formulas first in the general spin representation with the choice of multiplying factor $R=S$ (here $\epsilon_{\alpha\beta\gamma\delta}$ satisfies $\epsilon_{0123}=-\epsilon^{0123}=+1$):

$$V_\mu=\bar{u}(p',s')\gamma_\mu u(p,s)$$

$$=(1/S^*)[(m'p_\mu+mp'_\mu)(1-s\cdot s')+mp'\cdot ss'_\mu+m'p\cdot s's_\mu-i\epsilon_{\mu\alpha\beta\gamma}p^\alpha p'^\beta(s+s')^\gamma], \tag{A1}$$

$$T_{\mu\nu}=\bar{u}(p's')\sigma_{\mu\nu}u(p,s)/\sqrt{2}=i\bar{u}(p',s')(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)u(p,s)/2\sqrt{2}$$

$$=(i/S^*\sqrt{2})[(1-s\cdot s')(p_\nu p'_\mu-p_\mu p'_\nu)-(mm'+p\cdot p')(s_\nu s'_\mu-s_\mu s'_\nu)$$

$$+s\cdot p(s'_\mu p_\nu-p_\mu s'_\nu)+s'\cdot p(p'_\mu s_\nu-s_\mu p'_\nu)-i\epsilon_{\mu\nu\alpha\beta}(m'p+mp')^\alpha(s+s')^\beta], \tag{A2}$$

$$A_\mu=\bar{u}(p',s')i\gamma_\mu\gamma_5 u(p,s)$$

$$=(i/S^*)[(m'm+p'\cdot p)(s_\mu+s'_\mu)-p\cdot s'p'_\mu-p'\cdot sp_\mu+i\epsilon_{\mu\alpha\beta\gamma}s^\alpha s'^\beta(m'p+mp')^\gamma], \tag{A3}$$

$$P=\bar{u}(p',s')\gamma_5 u(p,s)$$

$$=(1/S^*)(mp'\cdot s-m'p\cdot s'+i\epsilon_{\alpha\beta\gamma\delta}p^\alpha s^\beta p'^\gamma s'^\delta). \tag{A4}$$

In the helicity representation, with $\hat{p}'\cdot\hat{p}=\cos\theta$ and

$$s^\mu=(2\lambda/m)(p,E\hat{p}),\quad s'^\mu=(2\lambda'/m')(p',E'\hat{p}'), \tag{A5}$$

we have S given by Eq. (12) of the text and, with $X^\mu=(X_0,\vec{\mathbf{X}})$ and $Y^\mu=(Y_0,\vec{\mathbf{Y}})$ arbitrary four-vectors,

$$V_\mu X^\mu=(1/S^*)[(m'E+mE')(1+4\lambda\lambda'\cos\theta)X_0-(2\lambda'mp'+2\lambda m'p)(2\lambda\hat{p}\cdot\vec{\mathbf{X}}+2\lambda'\hat{p}'\cdot\vec{\mathbf{X}}+4i\lambda\lambda'\vec{\mathbf{X}}\cdot\hat{p}\times\hat{p}')], \tag{A6}$$

$$A_\mu X^\mu=(i/S^*)[(2\lambda'mp'+2\lambda m'p)(1+4\lambda\lambda'\cos\theta)X_0-(m'E+mE')(2\lambda\hat{p}\cdot\vec{\mathbf{X}}+2\lambda'\hat{p}'\cdot\vec{\mathbf{X}}+4i\lambda\lambda'\vec{\mathbf{X}}\cdot\hat{p}\times\hat{p}')], \tag{A7}$$

$$X^\mu T_{\mu\nu}Y^\nu=(i/S^*\sqrt{2})(2\lambda\hat{p}+2\lambda'\hat{p}'+4i\lambda\lambda'\hat{p}\times\hat{p}')\cdot[(2\lambda'Ep'-2\lambda E'p)(X_0\vec{\mathbf{Y}}-Y_0\vec{\mathbf{X}})-i(mm'+EE'-4\lambda\lambda'pp')\vec{\mathbf{X}}\times\vec{\mathbf{Y}}], \tag{A8}$$

$$P=(1/S^*)(2\lambda E'p-2\lambda'Ep')(1+4\lambda\lambda'\cos\theta). \tag{A9}$$

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]We use the notation and conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964) especially Appendix A. We, however, normalize spinors so that $\bar{u}(p,s)u(p,s)=2m$. As a result the projection operator $\sum_s u(p,s)\times\bar{u}(p,s)=\gamma\cdot p+m$.

[2]See, e.g., S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), pp. 462ff.

[3]A. C. Hearn, Reduce 2 User's Manual, Stanford Artificial Intelligence Project Memo AIM-133 (unpublished); M. Veltman, SCHOONSCHIP, A CDC 6600 Program for the Symbolic Evaluation of Algebraic Expressions, CERN report, 1967 (unpublished).

[4]These methods are sufficiently simple that certain special cases may well have been discussed before with regard to calculations of specific processes. The only such instance of which we are aware, however, is a calculation of muon trident production which we discovered somewhat accidentally when this work was nearly complete. See Ref. 5.

[5]J. D. Bjorken and M. C. Chen, Phys. Rev. 154, 1335 (1967); G. R. Henry, *ibid.* 154, 1534 (1967).

[6]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, Ref. 1, p. 141.

[7]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959). Note that in general, e.g., for production processes, one requires a somewhat more general expression for $u(p,\lambda)$ involving the azimuthal angle ϕ.

[8]See, e.g., M. Scadron, Phys. Rev. 165, 1640 (1968).

[9]It is worth noting that for physical processes M is a scalar. When vector particles, e.g., photons, are involved a vector index may be contracted with a polarization vector ϵ_μ. Thus M involves invariants such as $\epsilon \cdot p'$ which must also be evaluated numerically. The sum over polarization directions for the vector particle must now be done rather more explicitly than in the usual method. Similar comments apply to higher-spin bosons.

[10]For purposes of counting we take $\sigma_{\mu\nu} = i(\gamma_\mu\gamma_\nu - g_{\mu\nu})$ as contributing two terms to r and multiply out all other brackets appearing in Γ_M. For specific calculations in *either* method, one of course will use as many specific tricks, redefinitions, etc., as possible to minimize the number of traces to be computed. The values of r and n we quote are perhaps slightly ambiguous, as we do not give the exact form of the matrix element used to compute them. They should, however, give an approximately correct idea of the relative number of terms involved in the various processes.

[11]M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

[12]C. T. Grant, M. E. Schillaci, and R. R. Silbar, Phys. Rev. 184, 1737 (1969).

[13]H. W. Fearing, M. Pratap, and J. Smith, Phys. Rev. D 5, 158 (1972).

[14]F. J. Gilman, J. Pumplin, A. Schwimmer, and L. Stodolsky, Phys. Letters 31B, 387 (1970).

[15]K. Gottfried and J. D. Jackson, Nuovo Cimento 34, 735 (1964).

[16]W. R. Frazer and J. R. Fulco, Phys. Rev. 117, 1603 (1960).

[17]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

PHYSICAL REVIEW D VOLUME 6, NUMBER 2 15 JULY 1972

Effect of Anomalies on Quasi-Renormalizable Theories*

David J. Gross†
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

and

R. Jackiw
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 1 March 1972)

Apparently nonrenormalizable field theories, such as the new models for weak interactions, can become renormalizable when gauge invariance of the second kind is present. However, the anomaly associated with the axial-vector current may destroy this gauge invariance in perturbation theory, even though it is present in the Lagrangian. When this happens the theory remains nonrenormalizable. Nevertheless it is possible, by enlarging the theory, to remove the anomaly at the expense of introducing additional fermion fields, which correspond to as-yet-unobserved particles.

I. INTRODUCTION

In addition to the usual renormalizable and super-renormalizable field theories, there exist models which apparently yield, in conventional perturbation theory, a finite, well-defined, and unitary S matrix, even though superficial estimates of degree of divergence indicate nonrenormalizability. We call such theories "quasi-renormalizable." A classic example is a massive vector meson coupled to a conserved current.[1] It has been conjectured some time ago that a spontaneously broken gauge theory of the weak interactions also is quasi-renormalizable,[2] and recently arguments have been presented in support of this conclusion.[3]

The essential ingredient, which may convert an apparently nonrenormalizable theory into a quasi-renormalizable one, is gauge invariance of the second kind. In the massive-vector-meson example, the invariance, although "weakly" broken by the meson mass, is sufficiently operative to effect this desirable state of affairs. Similarly in the weak-interaction theories, the gauge principle, though spontaneously broken, allows the argument to proceed to a successful conclusion.

In this paper we demonstrate that the anomalies of the axial-vector current[4] can invalidate the "proof" of quasi-renormalizability. These anomalies, which are in general present when there are fermions in the model, destroy gauge invariance of the second kind. One way of understanding their origin is to observe that any theory, renormalizable or not, must be regulated when pertur-