Angular Distributions and a Selection Rule in Charge-Pole Reactions*

Daniel Zwanziger
Department of Physics, New York University, New York, New York 10003
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In the scattering of electrically and magnetically charged particles, it is found that, besides the orbital and spin angular momentum of each particle, there is a residual angular momentum in the electromagnetic field of the in or out scattering states given by $M^{\mu\nu} = \pm \sum_{i>j} \mu_{ij} \times \epsilon^{\mu\nu}{}_{\kappa\lambda} p_i^\kappa p_j^\lambda [(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{-1/2}$, where $\mu_{ij} = (4\pi)^{-1}(e_i g_j - g_i e_j)$ and p_i, e_i, g_i are the 4-momentum and electric and magnetic charges of the ith particle. Because of the addition of this $M^{\mu\nu}$ to the generator of Lorentz transformations, the scattering states do not transform like free-particle states, but the modification has a simple group-theoretical description. For each pair of particles i and j, $M^{\mu\nu}$ generates a one-dimensional representation of the little group of the pair of 4-vectors p_i, p_j. This is the subgroup of the Lorentz group which leaves both 4-vectors invariant and is isomorphic to the one-parameter group of rotations about the z axis. The problem of constructing scattering amplitudes satisfying the new kinematics is solved. For two-body decay processes $1 \to 2+3$, there results the selection rule $s_1 + s_2 + s_3 \geq |\mu_{23}| = (4\pi)^{-1}|e_2 g_3 - g_2 e_3|$ relating the spins s_i of the particles to their electric and magnetic charges. Parity- and time-reversal-violating angular distributions are found. For example, in the decay $1 \to 2+3$, if particle 1 has spin one and polarization vector $\vec{\epsilon}$, and particles 2 and 3 are spinless with $\mu_{23} = 1$, the center-of-mass angular distribution is $\vec{\epsilon} \cdot \vec{\epsilon}^* - \vec{\epsilon} \cdot \hat{q}\ \vec{\epsilon}^* \cdot \hat{q} + i\ \vec{\epsilon} \times \vec{\epsilon}^* \cdot \hat{q}$, where $\hat{q}$ is the direction of particle 2. It is found that a consistent Lorentz transformation law requires μ_{ij} to take on integral or half-integral values, but the usual connection between spin and statistics further limits μ_{ij} to integral values only.

I. INTRODUCTION

Whereas the nonrelativistic charge-pole problem has been rather thoroughly explored,[1-8] the relativistic quantum field theory of charges and poles has, in general, received only rather formal treatment.[9-14] In the present note some definite observable consequences[15] concerning angular distributions and a selection rule are extracted from the relativistic theory.

We evaluate the angular momentum contained in the electromagnetic field of the in and out asymptotic states and obtain a definite nonvanishing interaction angular momentum, above and beyond the spin and orbital angular momentum of each particle. This addition to the generator of Lorentz transformations changes the way scattering states transform. The new law has a simple group-theoretical description, however, which is a natural extension of Wigner's result[16] that the spin space of a particle transforms according to a representation of the little group of its momentum 4-vector (the subgroup of the Lorentz group which leaves the 4-vector invariant). The transformation law of scattering states includes, for each pair of particles i and j the (one-dimensional) representation of the little group of the pair of momentum 4-vectors p_i, p_j labeled by $\mu_{ij} = (4\pi)^{-1}(e_i g_j - g_i e_j)$, where e_i and g_i are electric and magnetic charges.[17] This is the subgroup of the Lorentz group which leaves both 4-vectors invariant and is isomorphic to the one-parameter group of rotations about the z axis.

The new kinematics leads to definite experimental predictions, the most striking of which are a selection rule relating spin and electric and magnetic charge, and angular distributions which violate parity and time-reversal invariance.

Let us now outline how the relativistic quantum-mechanical calculation is done. In the asymptotically distant past and future, wave packets concentrated close to p in momentum space move along straight-line trajectories, $\lim_{\tau\to\pm\infty} x^\mu(\tau) = (p^\mu/m)\tau + O(\ln\tau)$. At these times the electromagnetic field deviates negligibly from its classical value in regions spatially far away from the trajectories. As we shall see, the angular momentum (apart from spin and orbital angular momentum) contained in the electromagnetic field of the asymptotic states in fact lies at spatial distances $x \sim c\tau$ from the trajectories so that the angular momentum in an asymptotic state may be evaluated classically. This done in Sec. II. There results a contribution to the angular momentum tensor[18] of in or out states

$$M^{\mu\nu} = \pm \sum_{i>j} \mu_{ij} \frac{\epsilon^{\mu\nu}{}_{\kappa\lambda} p_i^\kappa p_j^\lambda}{[(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}\ ,$$
$$\mu_{ij} = \frac{e_i g_j - g_i e_j}{4\pi}\ . \qquad (1.1)$$

This constitutes a relativistic generalization of the familiar nonrelativistic angular momentum,[1-7] $\vec{J}=\mu\hat{r}$ which, for scattering in or out states of relative momentum $\vec{p}$, becomes $\vec{J}=\mp\mu\hat{p}$, since $\lim_{t\to\pm\infty}\vec{r}(t)=(\vec{p}/m)t$.

To understand the relativistic form consider an asymptotic state of two particles and let particle 1 be at rest, and let particle 2 have 3-momentum $\vec{p}$. Then (1.1) gives zero Lorentz boost ($M^{0i}=0$) and an angular momentum $\vec{J}=\mp\mu_{12}\hat{p}$ ($J_i=\frac{1}{2}\epsilon_{ijk}M^{jk}$). The agreement between the relativistic and nonrelativistic expressions is remarkable, because the angular momentum densities are quite different in the two cases. However, it is necessary to the consistency of the quantization condition

$$\mu_{ij}=(e_ig_j-g_ie_j)/4\pi=0,\pm\tfrac{1}{2},\pm1,\ldots\,. \tag{1.2}$$

We employ two different methods to pass from the expression for the interaction angular momentum to the Lorentz transformation law. The first is purely kinematical, relying solely on group theory, whereas the second is dynamical and relies on the use of Dirac strings. To understand the first method, again consider a pair of spinless particles and observe that if particle 1 is at rest and particle 2 is traveling in the z direction, then rotations about the z axis leave the momenta invariant. These rotations thus constitute the little group of the pair of 4-vectors. However, because of the identification of angular momentum with the generator of rotations, a rotation ϕ about the z axis changes the phase of the state by $\exp(i\mu_{12}\phi)$. We obtain a representation of the little group only if μ_{12} takes on one of the quantized values (1.2). The corresponding representation of the full Poincaré group is found in Sec. III.

The second, dynamical, method of finding the transformation law for the asymptotic states is to introduce vector potentials for the electromagnetic fields. As usual for magnetic monopoles, these potentials rely on the artifice of the Dirac string. The total angular momentum tensor of the asymptotic state is obtained by adding (1.1) to the angular momentum tensor for each particle which is, apart from spin,

$$M^{\mu\nu}=\lim_{t\to\pm\infty}\sum_i x_i^\mu[p_i^\nu-e_iA^\nu(x_i)-g_iB^\nu(x_i)]-(\mu\leftrightarrow\nu). \tag{1.3}$$

Here $A^\mu(x_i)$ and $B^\mu(x_i)$ are the vector potentials produced by the other charged particles in rectilinear motion in the asymptotic state. This is evaluated in Sec. IV and the result agrees with the group-theoretical expression. The arbitrariness in the choice of the Dirac string is found to coincide with the arbitrary but necessary choice of phase convention in the group-theoretical method. Thus, the rather nonintuitive Dirac string is shown to be identical with the more familiar arbitrariness involved in phase conventions of states.

In Sec. V, the previous results are translated into predictions about scattering and decay processes. When a particle at rest decays into a pair of particles, $1\to2+3$, the interaction angular momentum lies along the momentum $\vec{q}$ of the final particle. This must be balanced by spin angular momentum because orbital angular momentum is perpendicular to $\vec{q}$. The selection rule

$$s_1+s_2+s_3\geq|\mu_{23}| \tag{1.4}$$

results. Scattering and decay amplitudes for arbitrary spin and particle number are constructed which satisfy the restriction imposed by the transformation law of scattering states. They are not invariant functions of momentum and polarization vectors but depend on phase conventions which are equivalent to a dependence on the direction of the Dirac string. This dependence is given explicitly. The resulting cross sections and angular distributions are invariant functions of momentum and polarization vectors which, however, display explicit parity and time-reversal violation. It is observed that the normal correction between spin and statistics restricts μ_{ij} to integral values. Finally, we note that the Lorentz transformation law of scattering amplitudes is not crossing-invariant. For example, if a pair of spinless electrically charged particles annihilate and produce a spinless pole-antipole pair, the amplitude transforms like a scalar. However, in the crossed channel corresponding to charge-pole scattering it does not. Readers primarily interested in applications may turn directly to Sec. V.

II. ELECTROMAGNETIC ANGULAR MOMENTUM IN THE ASYMPTOTIC STATES

From the Maxwell equations generalized to include electric and magnetic currents[18,19] j_e and j_g,

$$\partial\cdot F=j_e, \tag{2.1a}$$

$$\partial\cdot F^d=j_g, \tag{2.1b}$$

we obtain the Yang-Feldman equations

$$F=F^{\text{in,out}}+D^{\text{ret,adv}}[\partial\wedge j_e-(\partial\wedge j_g)^d]. \tag{2.2}$$

Here $F^{\text{in,out}}$ are free fields. We are interested in the retarded and advanced source contributions which are often assumed to vanish as $t\to\pm\infty$, respectively. Consider an asymptotic state made up of particles of electric and magnetic charge e_i and g_i each in a wave-packet state concentrated in momentum space around p_i, with $p_i^2=m_i^2$. For sufficiently large times the wave packets in position space separate and travel along trajectories given by

$$x_i^\mu(\tau_i) = (p_i^\mu/m_i)\tau_i + O(\ln\tau_i) \tag{2.3a}$$

or

$$\vec{x}_i(t) = (\vec{p}_i/E_i)t + O(\ln t), \tag{2.3b}$$

so the distance between wave packets diverges linearly with time[20]

$$\vec{x}_i(t) - \vec{x}_j(t) - (\vec{v}_i - \vec{v}_j)t + O(\ln t)$$

for $\vec{v}_i = \vec{p}_i/E_i$. The asymptotic times are such that the spatial extent of the wave packets is very small compared to the separation between wave packets. Consequently, in the region between wave packets, Eq. (2.2) may be evaluated with the wave packets replaced by point sources moving uniformly according to (2.3). The retarded and advanced potentials are equal for uniform motion and Eq. (2.2) becomes

$$\lim_{t\to\mp\infty} F(x) = F^{\text{in,out}}(x) + \frac{1}{4\pi}\sum_i \frac{e_i(x\wedge u_i) - g_i(x\wedge u_i)^d}{[(x\cdot u_i)^2 - x^2]^{3/2}}, \tag{2.4}$$

where $u_i = p_i/m_i$. The approximation involved in deriving this expression is to neglect the spreading of wave packets in space which grows linearly with time, like $\Delta v t$, where Δv is the uncertainty in velocity. Consequently, Eq. (2.4) is exact for momentum eigenstates.

According to Eq. (2.4) the source contribution to F vanishes like t^{-2} as expected from dimensional considerations. The corresponding energy and momentum densities which are quadratic in F vanish like t^{-4} and extend over a volume of order t^3. So the electromagnetic energy and momentum (apart from the part which is self-energy and self-momentum) contributed by the sources vanish like t^{-1}. Their moment however brings in another power of t, and we thus expect a finite contribution to the angular momentum tensor,

$$M^{\mu\nu} = \lim_{t\to\mp\infty}\int (x^\mu T^{\nu 0} - x^\nu T^{\mu 0})\, d^3x, \tag{2.5}$$

with the energy-momentum tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = \tfrac{1}{2}(F^{\mu\lambda}F_\lambda{}^\nu + F^{d\mu\lambda}F^{d\nu}_\lambda) \tag{2.6a}$$

or in matrix notation

$$T = \tfrac{1}{2}(F\cdot F + F^d\cdot F^d). \tag{2.6b}$$

We are only interested in the contribution to T from the self-fields of the charged particles so we retain only the second term of Eq. (2.4) and find[19]

$$T = \frac{1}{2}\frac{1}{(4\pi)^2}\sum_{i\neq j}\frac{1}{D_i^3 D_j^3}\{(e_ie_j + g_ig_j)[(x\wedge u_i)\cdot(x\wedge u_j) + (x\wedge u_i)^d\cdot(x\wedge u_j)^d] + (e_ig_j - g_ie_j)[(x\wedge u_i)^d\cdot(x\wedge u_j) - (x\wedge u_i)\cdot(x\wedge u_j)^d]\}, \tag{2.7}$$

$$D_i \equiv [(x\cdot u_i)^2 - x^2]^{1/2}. \tag{2.8}$$

Here we have dropped the term with $i=j$ since we are not attempting to caclulate self-interactions.

Upon examining this expression for $T^{\mu\nu}$, we see that each pair of particles contributes a term proportional to $(e_ie_j + g_ig_j)$ which is Coulombic, and a term proportional to $(e_ig_j - g_ie_j)$ which is the charge-pole interaction that interests us. The asymptotic Coulombic interaction will be discussed in detail elsewhere. At present we note only that the Coulombic contribution to $M^{\mu\nu}$, Eq. (2.5), must vanish because for each pair of particles i and j the total contribution to $M^{\mu\nu}$ is symmetric in i and j, whereas the only antisymmetric covariant tensor that can be formed out of u_i and u_j is antisymmetric in i and j [i.e., $u_i\wedge u_j$ or $(u_i\wedge u_j)^d$], and in the Coulombic term this is multiplied by the symmetric coefficient $e_ie_j + g_ig_j$. (Actually, the Coulombic contribution to the integral (2.5) is ambiguous and depends on the order of integration.) The charge-pole contribution to T may be easily rewritten as

$$T_{eg} = \frac{1}{(4\pi)^2}\sum_{i>j}(e_ig_j - g_ie_j)[xx\cdot(u_i\wedge u_j)^d + x\cdot(u_i\wedge u_j)^d x]\frac{1}{D_i^3 D_j^3}, \tag{2.9a}$$

$$T_{eg}^{\mu\nu} = \frac{-1}{(4\pi)^2}\sum_{i>j}(e_ig_j - g_ie_j)[x^\mu(\epsilon^\nu{}_{\rho\sigma\tau}x^\rho u_i^\sigma u_j^\tau) + (\epsilon^\mu{}_{\rho\sigma\tau}x^\rho u_i^\sigma u_j^\tau)x^\nu]\frac{1}{D_i^3 D_j^3}. \tag{2.9b}$$

We thus get for the contribution to the angular momentum tensor from the self-fields of the charged particles

$$M^{\mu\nu} = \lim_{t\to\mp\infty}\frac{-t}{(4\pi)^2}\sum_{i>j}(e_ig_j - g_ie_j)\int[x^\mu(\epsilon^\nu{}_{\rho\sigma\tau}x^\rho u_i^\sigma u_j^\tau) - x^\nu(\epsilon^\mu{}_{\rho\sigma\tau}x^\rho u_i^\sigma u_j^\tau)]D_i^{-3}D_j^{-3}\,d^3x. \tag{2.10}$$

The evaluation of this integral is simplified by noting that the contributions to $T_{eg}^{\mu\nu}$, from each pair i,j has vanishing 4-divergence, as is easily verified in the form (2.9). So $x^\mu T^{\nu\kappa} - x^\nu T^{\mu\kappa}$ has vanishing divergence on the index κ, and the value of the integral (2.10) is independent of the surface of integration chosen for each pair i,j. Let us evaluate the contribution to $M^{\mu\nu}_{ij}$ from a single pair i,j, in the frame where $u^\mu_i=(1,0,0,0)$ and $u^\mu_j=((1+\vec{u}^2)^{1/2},\vec{u})$. In this frame one finds directly that the Lorentz boosts vanish

$$M^{0m}_{ij}=0 \tag{2.11}$$

(the upper indices are Lorentz indices $m,n=1,2,3$, the lower indices label a particular particle pair i,j). For the angular momentum $\vec{J}$, $J_l=\frac{1}{2}\epsilon_{lmn}M^{mn}$, one finds

$$\vec{J}_{ij}=\lim_{t\to\mp\infty}\frac{+t}{(4\pi)^2}(e_ig_j-g_ie_j)\int\frac{\vec{x}\times(\vec{x}\times\vec{u})\,d^3x}{|\vec{x}|^3\{[t(\vec{u}^2+1)^{1/2}-\vec{x}\cdot\vec{u}]^2-t^2+\vec{x}^2\}^{3/2}}\;.$$

The t dependence may be eliminated by making the change of variables

$$\vec{x}=|t|\,\vec{x}'\,, \tag{2.12}$$

which gives, after dropping the prime

$$\vec{J}_{ij}=\frac{\mp1}{(4\pi)^2}(e_ig_j-g_ie_j)\int\frac{\vec{x}\times(\vec{x}\times\vec{u})\,d^3x}{|\vec{x}|^3\{[\mp(\vec{u}^2+1)^{1/2}-\vec{x}\cdot\vec{u}]^2-1+\vec{x}^2\}^{3/2}}\;. \tag{2.13}$$

This form for $\vec{J}_{ij}$ shows the important result that J has finite asymptotic values independent of t, and the change of variable (2.12) shows that the contribution to $\vec{J}$ comes from a region with volume $\sim|t|^3$, as argued above. By symmetry one has that $\vec{J}_{ij}$ is parallel to $\vec{u}$ so

$$\vec{J}_{ij}=\frac{\pm\vec{u}}{(4\pi)^2}(e_ig_j-g_ie_j)\int\frac{[\vec{x}^2-(\vec{x}\cdot\hat{u})^2]\,d^3x}{|\vec{x}|^3[\vec{u}^2\pm2(\vec{u}^2+1)^{1/2}\vec{u}\cdot\vec{x}+(\vec{x}\cdot\vec{u})^2+\vec{x}^2]^{3/2}}\;. \tag{2.14}$$

This integral is easily evaluated in cylindrical coordinates with the result

$$\vec{J}_{ij}=\frac{\pm\hat{u}}{4\pi}(e_ig_j-g_ie_j). \tag{2.15}$$

It is remarkable that this expression is independent of the magnitude of $\vec{u}$ and agrees with the nonrelativistic value for which the second denominator in Eqs. (2.13) and (2.14) would be $|\vec{x}-\vec{u}|^{3/2}$. This is necessary for quantization of $\mu_{ij}=(e_ig_j-g_ie_j)/(4\pi)$. The expressions (2.11) and (2.15) may be written covariantly,[4]

$$M^{\mu\nu}_{ij}=\pm\frac{1}{4\pi}(e_ig_j-g_ie_j)\epsilon^{\mu\nu}{}_{\kappa\lambda}\frac{u_i^\kappa u_j^\lambda}{[(u_i\cdot u_j)^2-1]^{3/2}}\,,$$

which is the contribution of each pair of charged particles to $M^{\mu\nu}$. We thus find for an asymptotic state containing electrically and magnetically charged particles, a net contribution to the angular momentum tensor from the self-fields of the particles given by

$$M^{\mu\nu}=\sum_{i>j}\pm\frac{e_ig_j-g_ie_j}{4\pi}\,\frac{\epsilon^{\mu\nu}{}_{\kappa\lambda}u_i^\kappa u_j^\lambda}{[(u_i\cdot u_j)^2-1]^{3/2}}\;. \tag{2.16}$$

The upper sign holds for in states, the lower for out states.

III. GROUP-THEORETICAL DERIVATION OF THE TRANSFORMATION LAW

Let us consider an in or out state containing a pair of charged particles with 4-momenta p_1 and p_2 and charges (e_1,g_1) and (e_2,g_2), and let the particles be spinless for the moment. Further, suppose particle 1 is at rest and particle 2 is moving along the $+z$ axis. We call this the standard configuration for the pair

$$p_1=p_1^s\equiv m_1(1,0,0,0), \tag{3.1a}$$

$$p_2=p_2^s\equiv((m_2{}^2+p^2)^{1/2},0,0,p)\;. \tag{3.1b}$$

Both of these 4-momenta are invariant under rotations around the z axis. In fact this is the largest subgroup of the Lorentz group which leaves both 4-vectors invariant; for p_1^s is invariant only under rotations, and of these, only the rotations about the z axis leave p_2^s invariant. Thus, rotations about the z axis constitute the little group of p_1^s and p_2^s.

Suppose we apply the operator of rotation through an angle ϕ about the z axis to an in or out state with the pair of particles in the standard configuration,

$$[R_z(\phi)]\,|p_1^s,p_2^s\rangle\,. \tag{3.2}$$

The momenta are left invariant, and since by assumption there are no other labels for the state, the state itself is unchanged, which means the vector $|p_1^s,p_2^s\rangle$ will change at most by a phase factor. One usually assumes that the scattering state transforms like the product of free-particle states, so that this phase factor is unity. However, according to Sec. II, Eq. (2.15), there is an interaction angular momentum in this in or out state, with z component

$$m_{12} = \pm \mu_{12}, \tag{3.3}$$

$$\mu_{12} \equiv (e_1 g_2 - g_1 e_2)/4\pi. \tag{3.4}$$

Thus, under the transformation (3.2) the in or out state must pick up the phase $e^{\pm i \mu_{12}\phi}$,

$$[R_z(\phi)]|p_1^s, p_2^s\rangle = e^{\pm i\mu_{12}\phi}|p_1^s\rangle. \tag{3.5}$$

This transformation law must be a representation of the group of rotations around the z axis, [more precisely of the subgroup of the covering group, SL(2, C) of the Lorentz group corresponding to rotations about the z axis], which means that μ_{12} must be integer or half-integer:

$$\mu_{12} \equiv (e_1 g_2 - g_1 e_2)/4\pi = 0, \pm\tfrac{1}{2}, \pm 1, \ldots .$$

The charge quantization condition is thus necessary for a consistent transformation law of the scattering states.

Besides yielding the quantization condition, the relation (3.5) also determines the transformation law of an arbitrary configuration under an arbitrary Lorentz transformation. This depends on phase conventions for the states $|p_1 p_2\rangle$ which we determine as follows. The argument occurs in deriving the transformation law of single-particle states.[16] Let $\Lambda_{p_1 p_2}$ be a Lorentz transformation which brings the standard configuration p_1^s, p_2^s into the configuration p_1, p_2:

$$p_1 = \Lambda_{p_1 p_2} p_1^s, \quad p_2 = \Lambda_{p_1 p_2} p_2^s. \tag{3.6}$$

Obviously, $\Lambda_{p_1 p_2}$ is indeterminate to within a right-factor of rotation about the z axis, and the phase convention consists in some arbitrary but definite choice of $\Lambda_{p_1 p_2}$. (We will specify a particular choice later.)

The phase of the state $|p_1, p_2\rangle$ is specified by

$$|p_1, p_2\rangle = U(\Lambda_{p_1 p_2})|p_1^s, p_2^s\rangle. \tag{3.7}$$

Now consider an arbitrary Lorentz transformation Λ which takes p_1, p_2 into p_1', p_2',

$$p_1' = \Lambda p_1, \quad p_2' = \Lambda p_2, \tag{3.8}$$

and apply it to $|p_1, p_2\rangle$

$$\begin{aligned} U(\Lambda)|p_1, p_2\rangle &= U(\Lambda)U(\Lambda_{p_1 p_2})|p_1^s, p_2^s\rangle \\ &= U(\Lambda\Lambda_{p_1 p_2})|p_1^s, p_2^s\rangle. \end{aligned}$$

Now $\Lambda\Lambda_{p_1 p_2}$ is a Lorentz transformation which takes p_1^s, p_2^s into p_1', p_2' so it differs from $\Lambda_{p_1' p_2'}$ at most by a rotation about the z axis, which may depend on $p_1 p_2$ and Λ,

$$\Lambda\Lambda_{p_1 p_2} = \Lambda_{p_1' p_2'} R_z(\phi(p_1, p_2, \Lambda)). \tag{3.9}$$

Hence we have

$$U(\Lambda)|p_1, p_2\rangle = U(\Lambda_{p_1' p_2'})U(R_z(\phi(p_1 p_2, \Lambda)))|p_1^s, p_2^s\rangle,$$

or by Eq. (3.5)

$$U(\Lambda)|p_1, p_2\rangle = U(\Lambda_{p_1' p_2'})|p_1^s, p_2^s\rangle \exp[i\mu_{12}\phi(p_1, p_2, \Lambda)].$$

This yields the desired transformation law for in or out states:

$$U(\Lambda)|p_1, p_2\rangle = |p_1', p_2'\rangle \exp[\pm i\mu_{12}\phi(p_1, p_2, \Lambda)], \tag{3.10}$$

with the phase angle ϕ determined by Eq. (3.9),

$$R_z(\phi(p_1, p_2, \Lambda)) = \Lambda_{p_1' p_2'}{}^{-1}\Lambda\Lambda_{p_1 p_2}. \tag{3.11}$$

This transformation law clearly satisfies the group property as well as its obvious generalization to an arbitrary number of particles,

$$U(\Lambda)|\cdots p_i \cdots p_j \cdots\rangle = |\cdots \Lambda p_i \cdots \Lambda p_j \cdots\rangle \exp\left[\pm i \sum_{i>j} \mu_{ij}\phi(p_i, p_j, \Lambda)\right]. \tag{3.12}$$

The Lorentz transformation law of the in or out states is completely specified by Eq. (3.12). In order to make use of this formula, however, it is necessary to obtain a more explicit formula for ϕ than (3.11). The Lorentz transformation corresponding to rotation by ϕ about the z axis is given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so from Eq. (3.11) we have in terms of matrix elements

$$\cos\phi = [\Lambda_{p_1' p_2'}{}^{-1}\Lambda\Lambda_{p_1 p_2}]^2{}_2,$$

$$\sin\phi = [\Lambda_{p_1' p_2'}{}^{-1}\Lambda\Lambda_{p_1 p_2}]^2{}_1,$$

where we have suppressed the argument of $\phi(p_1, p_2, \Lambda)$. More explicitly, using $\Lambda^{-1} = g\Lambda^t g$, we have

$$\cos\phi = g_{\mu\sigma}(\Lambda_{p_1' p_2'})^\sigma{}_\tau g^{\tau 2}\Lambda^\mu{}_\nu(\Lambda_{p_1 p_2})^\nu{}_2, \tag{3.13a}$$

$$\sin\phi = g_{\mu\sigma}(\Lambda_{p_1' p_2'})^\sigma{}_\tau g^{\tau 2}\Lambda^\mu{}_\nu(\Lambda_{p_1 p_2})^\nu{}_1. \tag{3.13b}$$

Thus to obtain ϕ we need $(\Lambda_{p_1 p_2})^\nu{}_2$ and $(\Lambda_{p_1 p_2})^\nu{}_1$.

Let us now determine a convention for $\Lambda_{p_1 p_2}$. From Eq. (3.6) we have

$$p_1^\mu = (\Lambda_{p_1 p_2})^\mu{}_0 m_1$$

and

$$p_2^\mu = (\Lambda_{p_1 p_2})^\mu{}_0 (m_2{}^2 + p^2)^{1/2} + (\Lambda_{p_1 p_2})^\mu{}_3 p,$$

or

$$(\Lambda_{p_1p_2})^\mu{}_0 = p_1^\mu/m_1, \tag{3.14a}$$

$$(\Lambda_{p_1p_2})^\mu{}_3 = [(p_1\cdot p_2)^2 - p_1^2p_2^2]^{-1/2}\left[m_1p_2^\mu - \frac{p_1\cdot p_2 p_1^\mu}{m_1}\right]. \tag{3.14b}$$

The columns of $\Lambda^\mu{}_\nu$ are orthonormal 4-vectors, so $\Lambda^\mu{}_1$ and $\Lambda^\mu{}_2$ are orthogonal to p_1 and p_2. The arbitrariness in $\Lambda_{p_1p_2}$ which is a factor of right multiplication by a rotation about the z axis results in an arbitrary recombination of $\Lambda^\mu{}_1$ and $\Lambda^\mu{}_2$. We fix this arbitrariness in $\Lambda_{p_1p_2}$ by requiring that $\Lambda^\mu{}_2$ be orthogonal to a fixed 4-vector n. In this case $\Lambda^\mu{}_2$ is orthogonal to p_1, p_2 and n and may be written[19]

$$\Lambda^\mu{}_2 = \frac{\epsilon^\mu(p_1,p_2,n)}{|\epsilon(p_1,p_2,n)|}, \tag{3.15a}$$

and $\Lambda^\mu{}_1$ is now determined up to sign, which is fixed by requiring $|\Lambda_{p_1p_2}|=1$,

$$\Lambda^\mu{}_1 = \frac{-\epsilon^\mu{}_{\nu\kappa\lambda}p_1^\nu p_2^\kappa \epsilon^\lambda(p_1,p_2,n)}{[(p_1\cdot p_2)^2 - p_1^2p_2^2]^{1/2}|\epsilon(p_1,p_2,n)|}. \tag{3.15b}$$

Equation (3.13a) then gives for $\cos\phi$,

$$\cos\phi = -(\Lambda_{p_1'p_2'})^\sigma{}_2(\Lambda^{-1})^\rho{}_\sigma g_{\rho\nu}(\Lambda_{p_1p_2})^\nu{}_2$$

$$= -\frac{\epsilon^\sigma(p_1',p_2',n)}{|\epsilon(p_1',p_2',n)|}\Lambda^{-1\rho}{}_\sigma\frac{\epsilon_\rho(p_1,p_2,n)}{|\epsilon(p_1,p_2,n)|},$$

or, using $\Lambda^{-1}\epsilon(p_1',p_2',n) = \epsilon(\Lambda^{-1}p_1', \Lambda^{-1}p_2', \Lambda^{-1}n) = \epsilon(p_1,p_2,\Lambda^{-1}n)$, we get

$$\cos\phi(p_1,p_2,\Lambda) = -\frac{\epsilon(p_1,p_2,\Lambda^{-1}n)}{|\epsilon(p_1,p_2,\Lambda^{-1}n)|}\cdot\frac{\epsilon(p_1,p_2,n)}{|\epsilon(p_1,p_2,n)|}. \tag{3.16a}$$

Similarly, from Eqs. (3.13b) and (3.15b) we get

$$\sin\phi(p_1,p_2,\Lambda) = \frac{\epsilon(p_1,p_2,\Lambda^{-1}n)}{|\epsilon(p_1,p_2,\Lambda^{-1}n)|}\cdot\frac{\epsilon(p_1,p_2,\epsilon(p_1p_2n))}{[(p_1\cdot p_2)^2-p_1^2p_2^2]^{1/2}|\epsilon(p_1p_2n)|}, \tag{3.16b}$$

$$\sin\phi(p_1,p_2,\Lambda) = \frac{\epsilon(p_1,p_2,n,\Lambda^{-1}n)[(p_1\cdot p_2)^2-p_1^2p_2^2]^{1/2}}{|\epsilon(p_1,p_2,\Lambda^{-1}n)||\epsilon(p_1,p_2,n)|}. \tag{3.16c}$$

These formulas are easily understood by observing that

$$\frac{\epsilon^\mu(p_1,p_2,n)}{|\epsilon(p_1,p_2,n)|}$$

and

$$\frac{\epsilon^\mu(p_1,p_2,\epsilon(p_1,p_2,n))}{[(p_1\cdot p_2)^2-p_1^2p_2^2]^{1/2}|\epsilon(p_1,p_2,n)|} \tag{3.17}$$

are a pair of unit orthonormal vectors in the two-dimensional space orthogonal to p_1 and p_2. Any change in n, in particular from n to $\Lambda^{-1}n$, leaves the vectors (3.17) orthonormal in this two-dimensional space, so it produces at most a rotation of the vectors (3.17) through some angle ϕ, with $\cos\phi$ and $\sin\phi$ given by Eqs. (3.16).

Our formulas for $\phi(p_1,p_2,\Lambda)$ correspond to a particular choice of $\Lambda_{p_1p_2}$. The most general choice of $\Lambda_{p_1p_2}$ differs from the one we have given by right multiplication of a rotation about the z axis through some angle χ depending on p_1 and p_2,

$$\Lambda'_{p_1p_2} = \Lambda_{p_1p_2}R_z(\chi(p_1,p_2)). \tag{3.18}$$

From Eq. (3.11), we see immediately that the most general form of ϕ is given by

$$\phi(p_1,p_2,\Lambda)\to\phi(p_1,p_2,\Lambda)+\chi(p_1,p_2)-\chi(\Lambda p_1,\Lambda p_2). \tag{3.19}$$

Let us now find the form of the generators of Lorentz transformations. This will allow us in Sec. IV to make contact with dynamical theory, for the generators correspond to physical observables. The transformation law (3.10) for in or out momentum eigenstates

$$U(\Lambda)|p_1,p_2\rangle = |\Lambda p_1,\Lambda p_2\rangle\exp[\pm i\mu_{12}\phi(p_1,p_2,\Lambda)], \tag{3.20}$$

leads to the transformation law for wave functions

$$U(\Lambda)\int f(p_1,p_2)|p_1,p_2\rangle\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} = \int f(p_1,p_2)|\Lambda p_1,\Lambda p_2\rangle\exp[\pm i\mu_{12}\phi(p_1,p_2,\Lambda)]\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}$$

$$= \int f(\Lambda^{-1}p_1,\Lambda^{-1}p_2)\exp[\pm i\mu_{12}\phi(\Lambda_1^{-1}p_1,\Lambda_1^{-1}p_2,\Lambda)]|p_1,p_2\rangle\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}$$

or

$$f(p_1,p_2)\to f_\Lambda(p_1,p_2) = f(\Lambda^{-1}p_1,\Lambda^{-1}p_2)\exp[\pm i\mu_{12}\phi(\Lambda^{-1}p_1,\Lambda^{-1}p_2,\Lambda)]. \tag{3.21}$$

Now let Λ be an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \Omega^\mu{}_\nu,$$

where $\Omega^{\mu\nu}$ is an infinitesimal antisymmetric matrix. The form of the Lorentz generators $M_{\mu\nu}$ is determined by

$$f_\Lambda(p_1,p_2) - f(p_1,p_2) = \tfrac{1}{2}i\Omega^{\mu\nu}M_{\nu\mu}f(p_1,p_2), \tag{3.22}$$

to first order in Ω. From Eq. (3.16c), we get

$$\phi(\Lambda^{-1}p_1, \Lambda^{-1}p_2, \Lambda) = \frac{\epsilon_{\kappa\lambda\sigma\mu}p_1^\kappa p_2^\lambda n^\sigma \Omega^\mu{}_\nu n^\nu}{-\epsilon^2(p_1, p_2, n)} [(p_1 \cdot p_2)^2 - p_1^2 p_2^2]^{1/2}, \tag{3.23}$$

so we may write

$$\phi(\Lambda^{-1}p_1, \Lambda^{-1}p_2, \Lambda) = \tfrac{1}{2}\Omega^{\mu\nu}\phi_{\mu\nu}(p_1, p_2),$$

with

$$\phi_{\mu\nu}(p_1, p_2) = [n_\mu \epsilon_\nu(p_1, p_2, n) - n_\nu \epsilon_\mu(p_1, p_2, n)] \frac{[(p_1 \cdot p_2)^2 - p_1^2 p_2^2]^{1/2}}{\epsilon^2(p_1, p_2, n)} . \tag{3.24}$$

The most general infinitesimal form of ϕ is found from Eq. (3.19),

$$\phi_{\mu\nu}(p_1, p_2) \to \phi_{\mu\nu}(p_1, p_2) - \left[p_{1\mu}\frac{\partial}{\partial p_1^\nu} - p_{1\nu}\frac{\partial}{\partial p_1^\mu} + p_{2\mu}\frac{\partial}{\partial p_2^\nu} - p_{2\nu}\frac{\partial}{\partial p_2^\mu}\right]\chi(p_1, p_2), \tag{3.25}$$

where $\chi(p_1, p_2)$ is assumed to be differentiable.[21] It is natural to restrict $\chi(p_1, p_2)$ to be antisymmetric in p_1 and p_2, since the original expression (3.24) is.

Combining Eqs. (3.21) and (3.24), we obtain

$$M_{\mu\nu} = M^f_{\mu\nu} \pm \mu_{12}\phi_{\mu\nu}(p_1, p_2), \tag{3.26}$$

with

$$M^f_{\mu\nu} \equiv i\left(p_{1\mu}\frac{\partial}{\partial p_1^\nu} - p_{1\nu}\frac{\partial}{\partial p_1^\mu} + p_{2\mu}\frac{\partial}{\partial p_2^\nu} - p_{2\nu}\frac{\partial}{\partial p_2^\mu}\right) . \tag{3.27}$$

In order to make the physical meaning of this formula more transparent, we write it out explicitly in matrix notation, so as not to be burdened by indices,

$$M = i\left(p_1 \wedge \frac{\partial}{\partial p_1} + p_2 \wedge \frac{\partial}{\partial p_2}\right) \pm \mu_{12}\left[n \wedge \epsilon(p_1, p_2, n)\frac{[(p_1 \cdot p_2)^2 - p_1^2 p_2^2]^{1/2}}{\epsilon^2(p_1, p_2, n)} - p_1 \wedge \frac{\partial \chi}{\partial p_1}(p_1, p_2) - p_2 \wedge \frac{\partial \chi}{\partial p_2}(p_1, p_2)\right] . \tag{3.28}$$

The generalization to an arbitrary number of particles is clearly

$$M = \sum_i p_i \wedge i\frac{\partial}{\partial p_i} \pm \sum_{i>j}\mu_{ij}\left[n \wedge \epsilon(p_i, p_j, n)\frac{[(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}{\epsilon^2(p_i, p_j, n)} - \left(p_i \wedge \frac{\partial}{\partial p_i} + p_j \wedge \frac{\partial}{\partial p_j}\right)\chi(p_i, p_j)\right] . \tag{3.29}$$

The first term is the Lorentz generator for free particles. The second term, proportional to $\pm\mu_{12}$, is an additional contribution from the charge-pole interaction to the Lorentz generators for in or out states. In Sec. IV we will derive this form of $M_{\mu\nu}$ from a specific dynamical theory involving electromagnetic potentials depending on a Dirac string. The arbitrary fixed vector n which appears in $M_{\mu\nu}$ as a result of choosing phases of states will turn out to be the direction of the Dirac string. The arbitrariness in $M_{\mu\nu}$, due to the appearance of the arbitrary $\chi(p_1, p_2)$, which expresses the freedom in the choice of the phases, will turn out to correspond to the gauge freedom of the electromagnetic potential.

IV. DERIVATION OF THE TRANSFORMATION LAW USING THE DIRAC STRING

In a classical theory the orbital angular momentum of a set of particles is given by

$$L_{\mu\nu} = \sum_i m_i(x_{i\mu}u_{i\nu} - x_{i\nu}u_{i\mu}) \tag{4.1a}$$

or[19]

$$L = \sum_i m_i x_i \wedge u_i , \tag{4.1b}$$

where u_i is the 4-velocity $u_i^2 = 1$. If the particles are in an asymptotic state and carry charges e_i, g_i then to this should be added the angular momentum carried in the electromagnetic field, Eq. (2.16), and we get for the total angular momentum in an in (out) state,

$$M = \sum_i m_i x_i \wedge u_i \pm \sum_{i>j}\mu_{ij}\frac{(u_i \wedge u_j)^d}{[(u_i \cdot u_j)^2 - 1]^{1/2}} . \tag{4.2}$$

To pass to the quantum theory of electric and magnetic charges, we use the formalism developed in the last reference of Ref. 14. Basically the transition is made by the substitution

$$m_i u_i^\mu \to p_i^\mu - e_i A^\mu(x_i) - g_i B^\mu(x_i), \tag{4.3}$$

where $A^\mu(x)$ and $B^\mu(x)$ are 4-potentials described below. The meaning of asymptotic states is that at asymptotic times the dynamical variables have their free-particle Heisenberg representation. Thus at asymptotic times, when acting on the in or out asymptotic states $|p_1, p_2, \ldots\rangle$, p_i^μ is the operator of multiplication by

$$p_i^\mu(E_i, \vec{p}_i) = ((m_i^2 + \vec{p}_i^{\,2})^{1/2}, \vec{p}_i) \tag{4.4a}$$

and x_i^μ is represented by

$$x_i^\mu = \left(t, -i\nabla_{p_i} + \frac{\vec{p}_i}{E_i}\,t\right) = p_i^\mu t/E_i + (0, -i\nabla_{p_i}). \tag{4.4b}$$

(There are actually Coulombic terms of order $\ln t$, which do not concern us here.) The potentials are of order $1/t$ at asymptotic times, as we shall verify below, so Eq. (4.3) may be written

$$M = \sum_i x_i \wedge [p_i - e_i A(x_i) - g_i B(x_i)] \pm \sum_{i>j} \mu_{ij} \frac{(p_i \wedge p_j)^d}{[(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}\,. \tag{4.5}$$

On making the substitutions (4.4a) and (4.4b), we find, on dropping terms that vanish like $1/t$, for the angular momentum tensor

$$M = \sum_i \left[p_i \wedge i\frac{\partial}{\partial p_i} - \lim_{t \to \mp\infty}\left(\frac{t}{E_i} p_i \wedge [e_i A_i(p_i t/E_i) + g_i B(p_i t/E_i)]\right)\right] \pm \sum_{i>j} \mu_{ij} \frac{(p_i \wedge p_j)^d}{[(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}\,. \tag{4.6}$$

To obtain the asymptotic form of M, all we need are the potentials $A(x_i)$ and $B(x_i)$ corresponding to the fields produced by the other charged particles given in Eq. (2.4),

$$F(x_i) = \frac{1}{4\pi} \sum_{j \neq i} \frac{e_i (x_i \wedge u_j) - g_j (x_i \wedge u_j)^d}{[(x_i \cdot u_j)^2 - x_i^2]^{3/2}}\,, \tag{4.7}$$

where $u_j = p_j/m_j$, which is accurate to the order considered. The potentials $A(x_i)$ and $B(x_i)$ are defined by[14]

$$F(x_i) = \partial_i \wedge A(x_i) - (n \cdot \partial)^{-1}(n \wedge j_g)^d, \tag{4.8a}$$

$$F^d(x_i) = \partial_i \wedge B_i + (n \cdot \partial)^{-1}(n \wedge j_e)^d\,. \tag{4.8b}$$

Here j_e and j_g are the electric and magnetic currents and $(n\cdot\partial)^{-1}$ is an integral operator with kernel $(n\cdot\partial)^{-1}(x-y)$ given by

$$(n \cdot \partial)^{-1}(x) = \frac{1}{2}\int_{-\infty}^{\infty} \delta^4(x - ns)\epsilon(s)\,ds, \tag{4.9}$$

where n is a unit spacelike 4-vector, $n^2 = -1$. The support of $(n\cdot\partial)^{-1}(x)$ is on the Dirac string $x = ns$, $-\infty < s < \infty$. According to Eq. (4.7), F is a sum of separate contributions from electric and magnetic sources,

$$F = F_e + F_g\,. \tag{4.10}$$

Correspondingly, let us put

$$A(x_i) = A_e(x_i) + A_g(x_i), \tag{4.11}$$

with $A_e(x)$ the usual

$$A_e(x_i) = \sum_j \frac{e_j u_j}{[(x_i \cdot u_j)^2 - x_i^2]^{1/2}}\,. \tag{4.12}$$

By contracting Eq. (4.8a) with n, we get

$$n \cdot F_g(x_i) = n \cdot \partial_i A_g(x_i) - \partial_i n \cdot A_g(x_i). \tag{4.13}$$

By imposing the axial-gauge condition

$$n \cdot A_g(x_i) = 0, \tag{4.14}$$

we find

$$A_g(x_i) = (n \cdot \partial_i)^{-1} n \cdot F_g(x_i) \tag{4.15}$$

or, from Eq. (4.7),[19]

$$A_g^\mu(x_i) = \frac{1}{4\pi}\sum_{j\neq i} g_j \int_{-\infty}^{\infty} ds\, \frac{\epsilon(s)}{2}\, \frac{\epsilon^\mu(n, x_i - ns, u_j)}{\{[(x_i - ns)\cdot u_j]^2 - (x_i - ns)^2\}^{3/2}} \,. \tag{4.16}$$

Evaluation of this integral is easily done by aligning the third axis along n, $n^\mu = (0, 0, 0, 1)$, which yields

$$A_g^\mu(x_i) = \frac{1}{4\pi}\sum_{j\neq i} g_j \frac{\epsilon^\mu(n, x_i, u_i)[n\cdot x_i - n\cdot u_j u_j\cdot x_i]}{[(x_i\cdot u_j)^2 - x_i^2]^{1/2}\epsilon^2(n, x_i, u_j)} \,. \tag{4.17}$$

One easily verifies that this expression for $A_g^\mu(x_i)$ satisfies

$$(n\cdot\partial)A_g^\mu(x_i) = \frac{1}{4\pi}\sum_j g_j \frac{\epsilon^\mu(n, x_i, u_j)}{[(x_i\cdot u_j)^2 - x_i^2]^{3/2}} = n\cdot F_g(x_i)\,.$$

Hence we have for the total vector potential A^μ,

$$A^\mu(x_i) = \frac{1}{4\pi}\sum_{j\neq i}\frac{1}{[(x_i\cdot u_j)^2 - x_i^2]^{1/2}}\left\{e_j u_j^\mu + g_j\frac{\epsilon^\mu(n, x_i, u_j)[n\cdot x_i - n\cdot u_j u_j\cdot x_i]}{\epsilon^2(n, x_i, u_j)}\right\} + \partial_i^\mu \sum_{j\neq i} g_j \Lambda(x_i, u_j), \tag{4.18a}$$

and similarly for B^μ

$$B^\mu(x_i) = \frac{1}{4\pi}\sum_{j\neq i}\frac{1}{[(x_i\cdot u_j)^2 - x_i^2]^{1/2}}\left\{g_j u_j^\mu - e_j\frac{\epsilon^\mu(n, x_i, u_j)[n\cdot x_i - n\cdot u_j u_j\cdot x_i]}{\epsilon^2(n, x_i, u_j)}\right\} + \partial_i^\mu \sum_{j\neq i} e_j \Lambda(x_i, u_j). \tag{4.18b}$$

Here we have added possible gauge terms which are of the general form of the string-dependent contribution and which correspond to different directions n or more generally different paths of the string, in Eqs. (4.8). Finally, we require the potentials only for $x_i^\mu = \lim_{t\to\pm\infty} p_i^\mu t/E_i$,

$$\lim_{t\to\pm\infty} A^\mu(x_i) = \frac{E_i}{|t|}\frac{1}{4\pi}\sum_{j\neq i}\frac{1}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}\left\{e_j p_j^\mu + g_j\frac{\epsilon^\mu(n, p_i, p_j)[n\cdot p_i p_j^2 - n\cdot p_j p_j\cdot p_i]}{\epsilon^2(n, p_i, p_j)}\right\}$$
$$+\frac{E_i}{|t|}\frac{\partial}{\partial p_{i\mu}}\frac{1}{4\pi}\sum_{j\neq i} g_j \Lambda(p_i, p_j), \tag{4.19a}$$

$$\lim_{t\to\pm\infty} B^\mu(x_i) = \frac{E_i}{|t|}\frac{1}{4\pi}\sum_{j\neq i}\frac{1}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}\left\{g_j p_j^\mu - e_j\frac{\epsilon^\mu(n, p_i, p_j)[n\cdot p_i p_j^2 - n\cdot p_j p_j\cdot p_i]}{\epsilon^2(n, p_i, p_j)}\right\}$$
$$-\frac{E_i}{|t|}\frac{\partial}{\partial p_{i\mu}}\frac{1}{4\pi}\sum_{j\neq i} e_j \Lambda(p_i, p_j), \tag{4.19b}$$

which is of order $1/t$ as advertised. The gauge term was assumed to have the same asymptotic behavior as the principal term.

These expressions may be used to find the form of the angular momentum operator acting on in or out asymptotic states. From Eq. (4.6), we find, with $\mu_{ij} = (e_i g_j - g_i e_j)/4\pi$,

$$M = \sum_i p_i\wedge i\frac{\partial}{\partial p_i} + \sum_{i,j}\mu_{ij}\frac{p_i\wedge\epsilon(n, p_i, p_j)[n\cdot p_i p_j^2 - n\cdot p_j p_j\cdot p_i]}{(p_i\cdot p_j)^{1/2}\epsilon^2(n, p_i, p_j)}$$
$$\pm\sum_{i,j}\mu_{ij}p_i\wedge\frac{\partial}{\partial p_i}\Lambda(p_i, p_j) \pm \sum_{i>j}\mu_{ij}\frac{(p_i\wedge p_j)^d}{[(p_i\cdot p_i)^2 - p_i^2 p_j^2]^{1/2}} \,. \tag{4.20}$$

The first term is the angular momentum tensor for free particles. The second arises from the $-\sum x_i\wedge[e_i A(x_i) + g_i B(x_i)]$ contribution and depends explicitly on the directions n of the Dirac string, Eq. (4.9). The third gives the change in M due to a gauge transformation. The last term is the angular momentum in the asymptotic electromagnetic fields, as calculated in Sec. II.

We must now compare this expression for M derived using string dependent potentials with the expressions for M derived group theoretically in Sec. III, Eq. (3.29). The two expressions are found to agree upon making use of the identity[19]

$$n\wedge\epsilon(p_i, p_j, n) = \frac{1}{[p_i^2 p_j^2 - (p_i\cdot p_j)^2]}\{p_i\wedge\epsilon(p_i, p_j, n)[n\cdot p_i p_j^2 - p_i\cdot p_j n\cdot p_j]$$
$$+p_j\wedge\epsilon(p_i, p_j, n)[n\cdot p_j p_i^2 - p_i\cdot p_j n\cdot p_i] + (p_i\wedge p_j)^d\epsilon^2(p_i, p_j, n)\}\,, \tag{4.21}$$

which shows that the second term of (3.29) is the sum of the $\sum x_i \wedge [e_i A(x_i) + g_i B(x_i)]$ contribution from the particles plus the angular momentum of the asymptotic electromagnetic field. Finally, the gauge term is found to be identical with the phase arbitrariness upon identifying

$$\Lambda(p_i, p_j) = \chi(p_i, p_j) = -\Lambda(p_j, p_i), \tag{4.22}$$

so χ and Λ must be equal antisymmetric functions. The group-theoretical calculation is very well founded and must be regarded as giving support to the string method which is *a priori* physically less compelling, though more general. Within the context of the present calculation we see in fact that the string method amounts to no more than choosing an axis of quantization.

The method of Sec. III assures us that the $M_{\mu\nu}$ we have found are a representation of the Lie algebra of the Poincaré group. It also shows that, provided $\mu_{ij} = (e_i g_j - g_i e_j)/4\pi$ takes on integral or half-integral values, the representation of the Lie algebra integrates to a representation of the finite (covering group of the) Poincaré group. It is somewhat surprising to find half-integral quantization here because the string function we have used, Eq. (4.9), is antisymmetric, corresponding to a pair of semi-infinite solenoids each bringing in half the unit flux. Physically, one expects the solenoid to be unobservable only when it carries a full unit flux.[1,11] By choosing the antisymmetric string function one would expect to obtain only integral quantization, as has been argued in the field-theoretic case.[11,14] However, our result shows that half-integral quantization provides a consistent transformation law for the asymptotic states. This may be related to the possibility of making a gauge transformation from a semi-infinite solenoid $(n\cdot\partial)^{-1}(x) = \int_{-\infty}^{\infty} \theta(s)\delta^4(x-ns)\,ds$, to the antisymmetric form, (4.9), $(n\cdot\partial)^{-1}(x) = \frac{1}{2}\int_{-\infty}^{\infty} \epsilon(s)\delta^4(x-ns)\,ds$. In Sec. V we will find, however, that half-integral quantization leads to an abnormal connection between spin and statistics.

V. APPLICATION TO CHARGE-POLE REACTIONS

In this section we will find a selection rule and angular distributions which follow from our expressions for the angular momentum and transformation law of scattering states. Consider first the decay of a state at rest with momentum $p_1 = (m_1, 0, 0, 0)$ into a pair of particles with momenta p_2 and p_3,

$$p_2 = ((m_2{}^2 + \vec{q}^2)^{1/2}, -\vec{q}), \quad p_3 = ((m_3{}^2 + \vec{q}^2)^{1/2}, \vec{q}). \tag{5.1}$$

In Sec. II the angular momentum in the electromagnetic field was calculated, Eq. (2.16), and gives for these momenta

$$J_i = \tfrac{1}{2}\epsilon_{ijk} M^{jk} = \mu_{23}\hat{q}, \quad M^{0i} = 0. \tag{5.2}$$

If we balance angular momentum along $\vec{q}$, we observe that the electromagnetic contribution $\mu_{23}\hat{q}$ must be balanced by spin angular momentum because the orbital angular momentum is perpendicular to $\vec{q}$. Thus, if $\vec{S}_i$ $(i = 1, 2, 3)$ are the spin operators for each particle participating in the decay, $\vec{S}_i \cdot \vec{S}_i = s_i(s_i+1)$, we have from conservation of angular momentum along $\vec{q}$

$$\hat{q}\cdot\vec{S}_1 = \hat{q}\cdot\vec{S}_2 + \hat{q}\cdot\vec{S}_3 + \mu_{23}. \tag{5.3}$$

This immediately yields the selection rule

$$s_1 + s_2 + s_3 \geq |\mu_{23}| = |(e_2 g_3 - g_2 e_3)/4\pi|\,. \tag{5.4}$$

Because electric and magnetic charge are separately conserved $(e_1, g_1) = (e_2, g_2) + (e_3, g_3)$, we have $\mu_{23} = \mu_{13} = \mu_{21}$, so this selection rule relates the three particles symmetrically. Such a selection rule relating spins and coupling constants is peculiar to charge-pole theory. For example, it forbids a spinless dyon[22] from decaying into a pair of spinless dyons, an otherwise allowed reaction.

To calculate angular distributions we must translate the transformation law of the in and out states into a condition on the scattering amplitude. For the scattering matrix we use the notation

$$S(\cdots p_f \cdots p_g \cdots; \cdots p_i \cdots p_j \cdots) = \langle \cdots p_f \cdots p_g \cdots^{\text{out}} | \cdots p_i \cdots p_j \cdots^{\text{in}} \rangle, \tag{5.5}$$

where $i, j, \ldots$ labels incoming particles and $f, g, \ldots$ labels outgoing particles. The Lorentz transformation law for in and out states, Eq. (3.12), yields the Lorentz transformation law for the S matrix (omitting the dots):

$$S(p_f, p_g; p_i, p_j) = \langle p_f, p_g{}^{\text{out}} | U^\dagger(\Lambda) U(\Lambda) | p_i, p_j{}^{\text{in}} \rangle = \langle \Lambda p_f, \Lambda p_g{}^{\text{out}} | \Lambda p_i, \Lambda p_j{}^{\text{in}} \rangle \exp\Big[i \sum_{i>j} \mu_{ij}\, \phi(p_i, p_j, \Lambda) + i \sum_{f>g} \mu_{fg} \phi(p_f, p_g, \Lambda) \Big]$$

or

$$S(\Lambda p_f, \Lambda p_g; \Lambda p_i, \Lambda p_j) = S(p_f, p_g; p_i, p_j) \exp\Big[-i \sum_{i>j} \mu_{ij} \phi(p_i, p_j, \Lambda) - i \sum_{f>g} \mu_{fg} \phi(p_f, p_g, \Lambda) \Big], \tag{5.6}$$

with $\phi(p_i, p_j, \Lambda)$ given explicitly in Eqs. (3.16). Here the sum extends separately over pairs of particles in the initial state and pairs in the final state, so the transformation law is not crossing symmetric. The above transformation law is written for spinless particles. Otherwise the spin index for a particle of spin s and momentum p would be multiplied by a factor of $\mathfrak{D}^s(\Lambda_{p'}{}^{-1}\Lambda\Lambda_p)$, as usual.

We must now construct a scattering amplitude which satisfies condition (5.6). To do this we will make use of the 4-vector[19]

$$a_\pm^\mu(p_i,p_j,n) \equiv \frac{\epsilon^\mu(p_i,p_j,n) \pm i\,\dfrac{\epsilon^\mu(p_i,p_j,\epsilon(p_i,p_j,n))}{[(p_i\cdot p_j)^2 - p_i^2p_j^2]^{1/2}}}{|\epsilon(p_i,p_j,n)|}\,, \tag{5.7}$$

which possess two remarkable properties:

$$a_\pm^\mu(\Lambda p_i,\Lambda p_j,n) = \Lambda^\mu{}_\nu\, a_\pm^\nu(p_i,p_j,\Lambda^{-1}n) \tag{5.8a}$$

$$= \Lambda^\mu{}_\nu\, a_\pm^\nu(p_i,p_j,n)\exp[\pm i\phi(p_i,p_j,\Lambda)] \tag{5.8b}$$

and

$$a_\pm^\mu(p_i,p_j,n)a_\pm^{\nu*}(p_i,p_j,n) = -g_{\mu\nu} + \frac{p_i\cdot p_j(p_i^\mu p_j^\nu + p_j^\mu p_i^\nu) - p_j^2p_i^\mu p_i^\nu - p_i^2 p_j^\mu p_j^\nu}{(p_i\cdot p_j)^2 - p_i^2p_j^2} \mp i\,\frac{\epsilon^{\mu\nu}{}_{\kappa\lambda}p_i^\kappa p_j^\lambda}{[(p_i\cdot p_j)^2 - p_i^2p_j^2]^{1/2}}. \tag{5.9}$$

The first property will allow us to construct amplitudes satisfying the correct transformation laws, and the second property shows that the corresponding cross sections are independent of the direction n. Equation (5.8a) is obvious, and Eq. (5.8b) is a consequence of the argument which follows Eq. (3.17), or it may be proved directly as follows. We denote the Lorentz transformation defined in Eqs. (3.14) and (3.15) by $\Lambda(p_i,p_j,n)$, and recall that $\phi(p_i,p_j,\Lambda)$ is defined by

$$\Lambda^{-1}(p_i',p_j',n)\Lambda\Lambda(p_i,p_j,n) = R_z(\phi(p_i,p_j,\Lambda)),$$

with $p_i' = \Lambda p_i$, $p_j' = \Lambda p_j$, so

$$\Lambda\Lambda(p_i,p_j,n) = \Lambda(p_i',p_j',n)R_z(\phi(p_i,p_j,\Lambda)).$$

From its definition, Eqs. (3.14) and (3.15), $\Lambda(p_i,p_j,n)$ satisfies the identity

$$\Lambda(p_i',p_j',n) = \Lambda\Lambda(p_i,p_j,\Lambda^{-1}n),$$

so we find

$$\Lambda(p_i,p_j,n) = \Lambda(p_i,p_j,\Lambda^{-1}n)R_z(\phi(p_i,p_j,n)). \tag{5.10}$$

However, according to Eqs. (5.7) and (3.15), $a(p_i,p_j,n)$ and $\Lambda(p_i,p_j,n)$ are related by

$$a_\pm^\mu(p_i,p_j,n) = \mp i[\Lambda^\mu{}_1(p_i,p_j,n) \pm i\Lambda^\mu{}_2(p_i,p_j,n)]. \tag{5.11}$$

Equation (5.8b) is then established by writing out the matrix multiplication appearing in Eq. (5.10) in terms of components. To prove Eq. (5.9) we observe that by Eq. (5.8) the left-hand side is a covariant tensor function of p_i and p_j, so it is independent of n and may be evaluated in any frame with any n. A quick way to find it is provided by Eq. (5.11), which gives (with the arguments suppressed)

$$a_\pm^\mu a_\pm^{\nu*} = \Lambda^\mu{}_1\Lambda^\nu{}_1 + \Lambda^\mu{}_2\Lambda^\nu{}_2 \mp i(\Lambda^\mu{}_1\Lambda^\nu{}_2 - \Lambda^\mu{}_2\Lambda^\nu{}_1)$$

$$= -g^{\mu\nu} - \Lambda^\mu{}_3\Lambda^\nu{}_3 + \Lambda^\mu{}_0\Lambda^\nu{}_0 \mp i(\Lambda^\mu{}_1\Lambda^\nu{}_2 - \Lambda^\mu{}_2\Lambda^\nu{}_1).$$

The real part is obtained from Eqs. (3.14) and the imaginary part is easily found by observing that it is an antisymmetric tensor orthogonal to p_i and p_j and hence proportional to $\epsilon^{\mu\nu}{}_{\kappa\lambda}p_i^\kappa p_j^\lambda$. This establishes Eq. (5.9)

To describe a decay process $1 \to 2+3$, with $p_1 = p_2 + p_3$, that satisfies the transformation law (5.6), with μ_{23} integer $\mu_{23} = \pm|\mu_{23}|$ contract the vector $a_\mp^\mu(p_2,p_3,n)$ $|\mu_{23}|$ times. Since $a_\pm^\mu(p_2,p_3 n)$ is orthogonal to p_1, p_2, p_3 and to itself, it must be contracted with spin vectors, which accords with the selection rule (5.4). For example, suppose $\mu_{23} = 1$ and one of the particles has spin 1 with polarization vector ϵ and the others are spinless, then the unique decay amplitude is

$$C\epsilon\cdot a_-(p_2,p_3,n). \tag{5.12}$$

If $\mu_{23} = 1$ and two particles have spin $\frac{1}{2}$ and are represented by Dirac spinors while the third is spinless, the unique decay amplitude is

$$C\gamma\cdot a_-(p_2,p_3,n), \tag{5.13}$$

where γ^μ are the Dirac matrices. [The amplitude $\gamma^5\gamma\cdot a_-(p_2,p_3,n)$ may be eliminated using the Dirac equation and $a_-(p_2,p_3,n) = a_-(p_1,p_3,n) = a_-(p_2,p_1,n)$ if necessary.] Finally, for an example with $\mu_{23} = 2$. let two of the particles be spin-$\frac{1}{2}$ Dirac particles, and let the third have spin 1. The unique decay amplitude is then

$$C\gamma\cdot a_-(p_2,p_3,n)\epsilon\cdot a_-(p_2,p_3,n). \tag{5.14}$$

The general case may be treated systematically by using a covariant spinorial basis in spin space[23] and combining spins into a total spin s. The decay amplitude using an undotted representation is then a sum of terms of the form

$$F_m^s(p_2,p_3)=c_s\mathfrak{D}^{s,0}_{m,-\mu_{23}}[\Lambda(p_2,p_3,n)]\,. \qquad (5.15)$$

In the center-of-mass frame Eq. (5.1) the Lorentz transformation $\Lambda(p_2,p_3,n)$, Eqs. (3.14) and (3.15), may be written as the product of a rotation $R(\hat{q},\vec{n})$, where $n^\mu=(n^0,\vec{n})$ and $R(\hat{q},\vec{n})$ has the columns

$$R_1^m=-\frac{\hat{q}\times(\hat{q}\times\vec{n})}{|\hat{q}\times\vec{n}|},\quad R_2^m=\frac{\hat{q}\times\vec{n}}{|\hat{q}\times\vec{n}|},\quad R_3^m=\hat{q}\;, \qquad (5.16)$$

followed by a Lorentz transformation along the z axis going from the rest frame of particle 2 to the center-of-mass frame. The effect of this Lorentz transformation may be absorbed in the definition of c_s, so the decay amplitude (5.15) becomes

$$F_m^s(\hat{q})=c_s\mathfrak{D}^{s}_{m,-\mu_{23}}[R(\hat{q},\vec{n})]\,. \qquad (5.17)$$

The dependence on μ_{23} of the angular distribution is very striking. It yields strong parity violation. For example, the angular distribution corresponding to (5.12) is, according to Eq. (5.9),

$$|c|^2|\epsilon\cdot a_-(p_2,p_3,n)|^2$$
$$=|c|^2\left[-\epsilon\cdot\epsilon^*+\frac{p_2\cdot p_3(\epsilon\cdot p_2\epsilon^*\cdot p_3+\epsilon\cdot p_3\epsilon^*\cdot p_2)-p_2{}^2\epsilon\cdot p_3\epsilon^*\cdot p_3-p_3{}^2\epsilon\cdot p_2\epsilon^*\cdot p_2)}{(p_2\cdot p_3)^2-p_2{}^2p_3{}^2}-i\,\frac{\epsilon_{\kappa\lambda\mu\nu}\epsilon^\kappa\epsilon^{*\lambda}p_2^\mu p_3^\nu}{[(p_2\cdot p_3)^2-p_2{}^2p_3{}^2]^{1/2}}\right]. \qquad (5.18)$$

The n dependence has disappeared and the distribution is a characteristic sum of scalar and pseudoscalar. In the center-of-mass frame, Eq. (5.1), with $\epsilon^\mu=(0,\vec{\epsilon})$, this is easily evaluated replacing p_2 by $p_1=(m_1,0)$ [because $a_-(p_2,p_3,n)=a_-(p_1,p_3,n)$] which gives

$$I(\hat{q})=|c|^2\{\vec{\epsilon}\cdot\vec{\epsilon}^*-\vec{\epsilon}\cdot\hat{q}\vec{\epsilon}^*\cdot\hat{q}-i\vec{\epsilon}\times\vec{\epsilon}^*\cdot\hat{q}\}\,. \qquad (5.19)$$

If the initial spin state is upward $\vec{\epsilon}=2^{-1/2}(1,i,0)$, the resulting angular distribution

$$I(\cos\theta)=|c|^2\tfrac{1}{2}(1-\cos\theta)^2 \qquad (5.20)$$

shows strong up-down asymmetry, with particle 3 emitted preferentially antiparallel to the spin. The amplitude (5.12) is also time-reversal violating,[24] both violations already occurring at the classical level. The theory of charges and poles thus provides a very natural mechanism for observed parity and time-reversal violating decays.

So far we have discussed two-body decay where the results are most striking because, apart from multiplicative constants, the amplitudes are determined kinematically. For other processes, where there are more than two linearly independent momentum 4-vectors, it is easy to construct amplitudes satisfying the transformation law (5.6). Suppose p_i and p_j are a pair of momentum vectors and let $\bar{p}_{ij}$ be any 4-vector which is linearly independent of p_i and p_j. The phase factor

$$\exp[\pm i\,\theta(p_i,p_j,\bar{p}_{ij})]\equiv\frac{\bar{p}_{ij}\cdot a_\pm(p_i,p_j,n)}{|\bar{p}_{ij}\cdot a_\pm(p_i,p_j,n)|} \qquad (5.21)$$

obviously satisfies

$$\exp[\pm i\,\theta(\Lambda p_i,\Lambda p_j,\Lambda\bar{p}_{ij})]=\exp[\pm i\,\theta(p_i,p_j,\bar{p}_{ij})]\exp[\pm i\phi(p_i,p_j,\Lambda)]\,, \qquad (5.22)$$

so the most general amplitude satisfying the transformation law (5.6) may be expressed as

$$S(p_f,p_g;p_i,p_j)=S_I(p_f,p_g;p_i,p_j)\exp\left[-i\sum_{i>j}\mu_{ij}\theta(p_i,p_j,\bar{p}_{ij})-i\sum_{f>g}\mu_{fg}\theta(p_f,p_g,\bar{p}_{fg})\right]. \qquad (5.23)$$

Here S_I has the ordinary transformation law for amplitudes in the absence of charge-pole pairs. For example, in the case of spinless particles it is an invariant function of the momenta. The cross section is given by $|S_I|^2$ and is independent of n, in agreement with the classical argument presented in the introduction. However, the motion of wave packets, both for scattering and decay processes depends on a phase which is the azimuthal angle in the plane perpendicular to p_i and p_j, measured by convention from the projection of n in this plane. Partial-wave expansions make use of the vertex amplitudes (5.15) or (5.16). For example, the amplitude for the scattering of spinless particles $1+2\to3+4$ has the expansion

$$S(p_3,p_4,p_1,p_2)=\sum_j(2j+1)a_j(s)\mathfrak{D}^{j,0}_{\mu_{34},-\mu_{12}}[\Lambda^{-1}(p_3,p_4,n)\Lambda(p_1,p_2,n)]\,, \qquad (5.24a)$$

which in the center-of-mass frame becomes

$$S(\vec{q}\,',\vec{q})=\sum_j(2j+1)a_j(s)\mathfrak{D}^{j}_{\mu_{34},-\mu_{12}}[R^{-1}(\hat{q}\,',\vec{n})R(\hat{q},\vec{n})]\,. \qquad (5.24b)$$

Finally, let us consider the question of whether half-integral quantization of $\mu_{ij}=(e_i g_j - g_i e_j)/4\pi$ is allowed within the simple kinematic framework presented here. The Lorentz transformation laws require that these parameters be integral or half-integral. However, half-integral quantization contradicts the generally accepted connection between spin and statistics. To see this consider the amplitude which describes the simultaneous decay of particle 1 into 2 and 3 and 1′ into 2′ and 3′, where 1 and 1′, 2 and 2′, and 3 and 3′ are identical particles. Further suppose, to be definite, that particles of type 2 and 3 have integral spin and obey normal Bose statistics. The decay amplitude for the simultaneous decay process, with particles 1 and 1′ in the same spin state is of the form

$$\begin{aligned}&[\delta^4(p_1-p_2-p_3)\delta^4(p_1'-p_2'-p_3')+\delta^4(p_1-p_2'-p_3')\delta^4(p_1'-p_2-p_3)]\\&\quad\times F(p_2,p_3)F(p_2',p_3')\exp\{-i\mu_{23}[\theta(p_2,p_3',p)+\theta(p_2',p_3,p)]\}\\&\qquad+[\delta^4(p_1-p_2'-p_3)\delta^4(p_1'-p_2-p_3')+\delta^4(p_1-p_2-p_3')\delta^4(p_1'-p_2'-p_3)]\\&\qquad\times F(p_2',p_3)F(p_2,p_3')\exp\{-i\mu_{23}[\theta(p_2,p_3,p)+\theta(p_2',p_3',p)]\},\end{aligned} \tag{5.25}$$

where p may be chosen to be $p_2+p_2'+p_3+p_3'$. This is obtained by taking the product of amplitudes $F(p_2,p_3)\times F(p_2',p_3')$ of the type (5.15) as expected, multiplying by the phases (5.21) of the 2-3′ and 2′-3 pairs, and symmetrizing on 2-2′ and 3-3′. The resulting amplitude is symmetric on interchange of particles 1 and 1′, so we see that, as usual, if a particle decays into bosons, it must itself obey Bose statistics. On the other hand, if particles 2 and 3 have integral spin, by angular momentum conservation, particle 1 will have integral or half-integral spin accordingly as μ_{23} is integral or half-integral. Hence, in order to maintain the usual connection between spin and statistics for particles of type 1, we must have μ_{23} integer.

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[1]P. A. M. Dirac, Proc. Roy. Soc. (London) A133, 60 (1931).

[2]Ig. Tamm, Z. Physik 71, 141 (1931).

[3]M. Fierz, Helv. Phys. Acta 17, 27 (1944).

[4]P. P. Banderet, Helv. Phys. Acta 19, 503 (1946).

[5]A. S. Goldhaber, Phys. Rev. 140, B1407 (1965).

[6]Daniel Zwanziger, Phys. Rev. 176, 1480 (1968).

[7]C. A. Hurst, Ann. Phys. (N.Y.) 50, 51 (1968).

[8]H. V. McIntosh and A. Cisneros, J. Math. Phys. 11, 896 (1970).

[9]P. A. M. Dirac, Phys. Rev. 74, 817 (1948).

[10]N. Cabibbo and E. Ferrari, Nuovo Cimento 23, 1146 (1962).

[11]J. Schwinger, Phys. Rev. 144, 1087 (1966); 151, 1048 (1966); 151, 1055 (1966); 173, 1536 (1968); Science 165, 757 (1969); 166, 690 (1969).

[12]T.-M. Yan, Phys. Rev. 150, 1349 (1966); 155, 1423 (1967); 160, 1182 (1967).

[13]B. Zumino, in *Strong and Weak Interactions*, edited by A. Zichichi (Academic, New York, 1966), p. 711.

[14]Daniel Zwanziger, Phys. Rev. 176, 1489 (1968); Phys. Rev. D 3, 880 (1971).

[15]For a recent experimental search and further references to the experimental literature, see H. H. Kohn, F. Villa, and A. Odian, Phys. Rev. D 4, 1285 (1971).

[16]E. P. Wigner, Ann. Math. 40, 149 (1939).

[17]This was presented as a conjecture in Ref. 6.

[18]Our metric is $g_{\mu\nu}=\mathrm{diag}\,(1,-1,-1,-1)$, $\epsilon_{0123}=1$.

[19]We use a matrix notation F for an antisymmetric tensor $F^{\mu\nu}$. The dual tensor F^d is defined by $F^{d\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu}{}_{\kappa\lambda}F^{\kappa\lambda}$, so $F^{dd}=-F$. For 4-vectors a^μ and b^μ we write $a\wedge b$ to represent the antisymmetric tensor $(a\wedge b)^{\mu\nu}=a^\mu b^\nu - b^\mu a^\nu$. Dot product means contraction on adjacent indices, thus $a\cdot(b\wedge c)^d=-(b\wedge c)^d\cdot a$ is the 4-vector $a_\mu\epsilon^{\mu\nu}{}_{\kappa\lambda}b^\kappa c^\lambda$. We also define $\epsilon^\mu(a,b,c)\equiv\epsilon^\mu{}_{\nu\kappa\lambda}a^\nu b^\kappa c^\lambda$, $\epsilon(a,b,c,d)\equiv\epsilon_{\mu\nu\kappa\lambda}a^\mu b^\nu c^\kappa d^\lambda$, and for any 4-vector a, $|a|\equiv|a^2|^{1/2}$.

[20]The scattering assumption includes $v_i\neq v_j$ for $i\neq j$.

[21]The p_i are labels of asymptotic states satisfying $p_i{}^2=m_i{}^2$, so the matrix $p_\mu\partial/\partial p^\nu - p_\nu\partial/\partial p^\mu$ actually has space-space components $p_m\nabla_{p_n}-p_n\nabla_{p_m}$ and time-space components $(\vec{p}^2+m^2)^{1/2}\nabla_{p_n}$.

[22]This term has been used to describe a particle with both electric and magnetic charge in the fifth entry of Ref. 11.

[23]Hans Joos, Fortsch. Phys. 11, 65 (1962); A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. 130, 442 (1963); Daniel Zwanziger, *ibid.* 133, B1036 (1964).

[24]N. F. Ramsey, Phys. Rev. 109, 225 (1958).