PHYSICAL REVIEW D VOLUME 6, NUMBER 2 15 JULY 1972

Renormalization of a Scalar Field Theory in Strong Coupling*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 22 July 1971)

The renormalization problem is solved, qualitatively, for the $\lambda_0\phi^6$ interaction of a scalar field $\phi(x)$ in two space and one time dimensions. The theory is found to be finite after mass renormalization, although perturbation theory predicts there should be ϕ^4 and ϕ^6 counterterms also. The renormalized theory is an interacting theory; it is scale-invariant at short distances. The field ϕ has canonical dimension $\frac{1}{2}$ in mass units (but this dimension may change in a more quantitative analysis) while the renormalized form of ϕ^2 has a noncanonical dimension, about 1.36. The cutoff dependence of the theory is computed by evaluating the Feynman path integral qualitatively. The analysis reduces the problem to a recursion formula for a function of one variable which acts as a representation of the renormalization group. The method of analysis is applicable to any scalar field theory.

Until now it has been impossible to give even a qualitative solution to strongly coupled quantum field theories requiring coupling-constant renormalization. Consider, for example, the $\lambda_0\phi^6$ interaction of a scalar field $\phi(x)$ in two space and one time dimensions. If this interaction is treated nonperturbatively, it is not known whether this theory has a solution (either with λ_0 fixed or with a cutoff-dependent λ_0). If it has solutions it is not known whether the solution is a field theory with interaction or not, etc. This kind of difficulty has been a major bottleneck in trying to develop a quantum field theory of strong interactions.

A new nonperturbative method for solving strongly coupled quantum field theories is presented in this paper. The method is qualitative (at best). It is, at present, limited to scalar field theories and is not applicable to theories containing Dirac fields. It is at present limited to calculations of Euclidean vacuum expectation values, i.e., vacuum expectation values calculated for imaginary times. However, the ultimate limitations on the method (if any) cannot be foreseen at present.

At the heart of the method is a representation of the renormalization group,[1-3] in the form of a recursion formula for a function of one variable. The recursion formula enables one to replace an interaction Lagrangian with cutoff Λ by an equivalent effective Lagrangian with cutoff $\frac{1}{2}\Lambda$. The effective Lagrangian is defined so that it gives the same physics as the original Lagrangian despite the lower value of the cutoff. A representation of the "renormalization group" is (in the author's definition) any set of equations which determine the variation of a Lagrangian or a set of coupling constants as one varies a cutoff or cutoff-like parameter.[4-7]

The purpose of this paper is to exhibit the recursion formula, along with a typical solution of it for the $\lambda_0\phi^6$ theory.[8] The derivation of the recursion formula will be given in exceedingly sketchy form; the interpretation of the recursion formula and its solution will be described very briefly.

The specific results on the $\lambda_0\phi^6$ theory obtained here are as follows: (1) Wave-function renormalization is absent in the qualitative analysis. (One can make arguments that show it is present in a more quantitative calculation.) (2) The theory can be renormalized with only a mass counterterm (no coupling-constant renormalization is required). (3) The renormalized form of the composite field $\phi^2(x)$ has an anomalous dimension,[9] namely, $(\text{mass})^\Delta$ with $\Delta \simeq 1.36$.

Further discussion of the analysis which gives the recursion formula will be found elsewhere, in the context of classical statistical mechanics.[10] There it is shown that the qualitative analysis of this paper is in good agreement with other more quantitative methods of calculation.

The field theory will be defined using the Feynman path integral.[11] The exact propagator will be studied explicitly. The path-integral formulation gives a closed-form expression for the propagator which is equivalent to the sum of the Feynman-diagram expansion (if the expansion converges). The path-integral formula is

$$D(x) = \frac{\langle \phi(x)\phi(0)e^{iA[\phi]}\rangle}{\langle e^{iA[\phi]}\rangle} \tag{1}$$

where

$$A[\phi] = \int d^2x \int dt \{\tfrac{1}{2}\nabla_\mu \phi(x)\nabla^\mu \phi(x) - U[\phi(x)]\} \tag{2}$$

and

$$U[\phi] = \tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^6 \tag{3}$$

(m_0 is the bare mass) and $\langle\ \rangle$ denotes the path integral over all c-number fields $\phi(x)$. The threevector x has two space components and a time component t. $A[\phi]$ is the classical action for a ϕ^6 theory. Feynman[11] has shown that Eq. (1) containing functional integrals over all classical fields ϕ gives the exact (unrenormalized) propagator of the quantized theory. A quick way of proving this will be given here. An integral $\int_{-\infty}^{\infty} f(\phi)d\phi$ over a single variable ϕ is unchanged if one translates the integration variable ϕ ($\phi \to \phi + c$ where c is a constant). Likewise a functional integral should be unchanged if one translates $\phi(x)$ at each point x: $\phi(x) \to \phi(x) + c(x)$. In particular, one should have

$$\langle \phi(0) e^{iA[\phi]} \rangle = \langle [\phi(0) + c(0)] e^{iA[\phi+c]} \rangle . \tag{4}$$

Let $c(x)$ be infinitesimal and calculate both sides of this equation in first order in c. One gets

$$0 = c(0)\langle e^{iA[\phi]}\rangle + i\int d^2x \int dt \left\langle \phi(0) \frac{\delta A}{\delta\phi(x)} e^{iA[\phi]} \right\rangle c(x), \tag{5}$$

where $\delta A/\delta\phi(x)$ is the variational derivative of A with respect to $\phi(x)$. This has to be true for any function $c(x)$, which means one has

$$\left\langle \phi(0) \frac{\delta A}{\delta\phi(x)} e^{iA[\phi]} \right\rangle = i\delta^3(x)\langle e^{iA[\phi]}\rangle . \tag{6}$$

When $\delta A/\delta\phi(x)$ is calculated from Eq. (2), Eq. (6) becomes

$$-(\nabla_\mu \nabla^\mu + m_0^2) D(x) - 6\lambda_0 \langle \phi^5(x)\phi(0) e^{iA[\phi]}\rangle \{\langle e^{iA[\phi]}\rangle\}^{-1} = i\delta^3(x) . \tag{7}$$

This is Schwinger's equation for the propagator $D(x)$.[12] This equation (plus equations for the vacuum expectation values of four, six, or more fields which can be derived by similar arguments) uniquely determines the perturbation expansion for $D(x)$ and gives the same result as the unrenormalized Feynman graph expansion.

In this paper the propagator will be studied only for spacelike x. It is then legitimate to convert to a Euclidean metric by replacing t by $-ix_3$. In this case $\int d^2x \int dt$ becomes $-i\int d^3x$ and $\nabla_\mu\phi(x)\nabla^\mu\phi(x)$ becomes $-[\nabla\phi(\vec{x})]^2$. Then one can write

$$D(\vec{x}) = \frac{\langle \phi(\vec{x})\phi(0) e^{A[\phi]}\rangle}{\langle e^{A[\phi]}\rangle} , \tag{8}$$

with

$$A[\phi] = -\int d^3x \{\tfrac{1}{2}[\nabla\phi(\vec{x})]^2 + U[\phi(\vec{x})]\} . \tag{9}$$

If the above analysis seems dubious one can show that $D(\vec{x})$ thus defined satisfies the Euclidean form of Schwinger's equation.[13]

A cutoff is needed to make $D(\vec{x})$ be finite, at least in perturbation theory. A simple cutoff procedure is to restrict the field $\phi(\vec{x})$ to be

$$\phi(\vec{x}) = \int_{|\vec{k}|<\Lambda} e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k}) , \tag{10}$$

where $\int_{\vec{k}}$ means $(2\pi)^{-3}\int d^3k$ and the restriction $|\vec{k}|<\Lambda$ means $\phi(\vec{x})$ cannot have Fourier components with $|\vec{k}|>\Lambda$. With the cutoff the path integral is defined as a sum only over fields of the form (10). In perturbation theory this means all virtual momenta in all Feynman graphs must be less than Λ.

The problem discussed in this paper is the dependence of the field theory on Λ and how to eliminate it in the limit of large Λ. The procedure for determining this dependence will be as follows. Suppose one has chosen a large but finite value of Λ and a given action $A[\phi]$. Define a sequence of cutoffs Λ_l:

$$\Lambda_l = 2^{-l}\Lambda . \tag{11}$$

A sequence of effective actions $A_l[\phi_l]$ will be obtained with cutoff Λ_l which are equivalent to the original action with cutoff Λ.[14] To be precise, the low-momentum behavior of the propagator can be computed with A_l and ϕ_l substituted for A and ϕ in Eq. (8). The field $\phi_l(\vec{x})$ has cutoff Λ_l:

$$\phi_l(\vec{x}) = \int_{|\vec{k}|<\Lambda_l} e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k}) . \tag{12}$$

The action $A_l[\phi_l]$ will be obtained from $A_{l-1}[\phi_{l-1}]$ by integrating out the variables $\phi(\vec{k})$ which are included in ϕ_{l-1} but not in ϕ_l, namely the variables $\phi(\vec{k})$ with $\Lambda_l < |\vec{k}| < \Lambda_{l-1}$. The result is a recursion formula giving $A_l[\phi_l]$ in terms of $A_{l-1}[\phi_{l-1}]$. The integrations will be calculated qualitatively, not quantitatively.

To determine the dependence of the field theory on Λ one chooses a reference momentum Λ_R which is held fixed while Λ is varied. One then considers cutoffs Λ which have the form $2^L\Lambda_R$. For $\Lambda = 2^L\Lambda_R$ (with L an integer), one iterates the recursion formula L times giving an effective cutoff $\Lambda_L = \Lambda_R$. The effective action A_L depends on the field $\phi_L = \phi_R$ with

$$\phi_R(\vec{x}) = \int_{|\vec{k}|<\Lambda_R} e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k}) . \tag{13}$$

This effective action will be denoted A_R. The renormalization problem is solved if one can give the unrenormalized action $A[\phi]$ a cutoff dependence such that the effective action $A_R[\phi_R]$ has no cutoff dependence in the limit $\Lambda \to \infty$.[15]

The recursion formulas defining $A_l[\phi_l]$ are as follows. (They will be derived later.) The effective action is found to have the form

$$A_l[\phi_l] = -\int d^3x \{\tfrac{1}{2}[\nabla\phi_l(\vec{x})]^2 + U_l[\phi_l(\vec{x})]\} . \tag{14}$$

The actual recursion formulas will be written for a scaled form of U_l. The following formulas are valid for arbitrary space-time dimension d. Write[16]

$$U_l[\phi_l] = \Lambda_l^{\,d} Q_l[\Lambda_l^{\,1-d/2}\phi_l]\,. \tag{15}$$

Then

$$Q_{l+1}(y) = -2^d \ln\{I_l(2\times 2^{-d/2}y)/I_l(0)\}\,, \tag{16}$$

with

$$I_l(z) = \int_{-\infty}^{\infty} dy \exp\{-y^2 - \tfrac{1}{2}Q_l(z+y) - \tfrac{1}{2}Q_l(z-y)\}\,. \tag{17}$$

The initial formulas are

$$U[\phi] = \Lambda^d Q[\Lambda^{1-d/2}\phi]\,, \tag{18}$$

$$Q_1(y) = -2^d \ln\{I(2\times 2^{-d/2}y)/I(0)\}\,, \tag{19}$$

$$I(z) = \int_{-\infty}^{\infty} dy \exp\{-y^2 - \tfrac{1}{2}Q(z+y) - \tfrac{1}{2}Q(z-y)\}\,. \tag{20}$$

The recursion formulas will be made plausible by developing the formulas connecting $A_1[\phi_1]$ to $A[\phi]$.[17] Write

$$\phi(\vec{x}) = \phi_0(\vec{x}) + \phi_1(\vec{x})\,, \tag{21}$$

where

$$\phi_0(\vec{x}) = \int_{0.5\Lambda<|\vec{k}|<\Lambda} e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\,. \tag{22}$$

The field $\phi_0(\vec{x})$ cannot vary independently at each point $\vec{x}$ due to the limited range of momenta contained in the field. In particular, if $\vec{x}_1$ and $\vec{x}_2$ are two points with separation much smaller than Λ^{-1} then $\phi_0(\vec{x}_1) \simeq \phi_0(\vec{x}_2)$, since the field $\phi_0(\vec{x})$ does not contain short enough wavelengths to change much between $\vec{x}_1$ and $\vec{x}_2$. In contrast, if $\vec{x}_1$ and $\vec{x}_2$ are separated by much more than Λ^{-1}, there is very little correlation between $\phi_0(\vec{x}_1)$ and $\phi_0(\vec{x}_2)$: One has a spread in wavelengths of order Λ^{-1} available and a much smaller change in the average wavelength will change $\phi(\vec{x}_2)$ enormously in relation to $\phi(\vec{x}_1)$. As a very crude representation of this situation, let the independent variables of the field $\phi_0(\vec{x})$ be its values on a lattice of points with lattice spacing Λ^{-1}. Then the independent variables are

$$\phi_{0\vec{n}} = \phi_0(\vec{n}\Lambda^{-1})\,, \tag{23}$$

where $\vec{n}$ is a vector with integral components. One can divide space into boxes surrounding each of the lattice sites $\vec{n}$. Within the box containing the lattice site $\vec{n}$, the field $\phi_0(\vec{x})$ behaves like a wave packet with amplitude $\phi_{0\vec{n}}$ and mean momentum of order Λ.

The field $\phi_1(\vec{x})$ has only smaller momenta, since $|\vec{k}| < \frac{1}{2}\Lambda$ in $\phi_1(\vec{x})$, and therefore changes with $\vec{x}$ more slowly than $\phi_0(\vec{x})$. For qualitative purposes it will be assumed that $\phi_1(\vec{x})$ is constant over a given box.

With these assumptions the original action $A[\phi]$ reduces as follows:

The $(\nabla\phi)^2$ integral can be written

$$\begin{aligned}\int_{\vec{x}} [\nabla\phi(\vec{x})]^2 &= \int_{|\vec{k}|<\Lambda/2} |\vec{k}|^2|\phi(\vec{k})|^2 \\ &\quad + \int_{\Lambda/2<|\vec{k}|<\Lambda} |\vec{k}|^2|\phi^2(\vec{k})| \\ &= \int_{\vec{x}} [\nabla\phi_1(\vec{x})]^2 + \int_{\vec{x}} [\nabla\phi_0(\vec{x})]^2 \end{aligned} \tag{24}$$

(where $\int_{\vec{x}}$ means $\int d^3x$). The integral over ϕ_0 can be written as $-\int_{\vec{x}} \phi_0(\vec{x})\nabla^2\phi_0(\vec{x})$. Because of the limited range of wavelengths in $\phi_0(\vec{x})$, $\nabla^2\phi_0(\vec{x})$ is of order $\Lambda^2\phi_0(\vec{x})$. Finally $\int_{\vec{x}}\phi_0^{\,2}(\vec{x})$ is of order $\Lambda^{-d}\sum_{\vec{n}}\phi_{0\vec{n}}^{\,2}$. This is because $\phi_0^{\,2}(\vec{x})$ is of order $\phi_{0\vec{n}}^{\,2}$ when $\vec{x}$ is in the box surrounding the lattice site $\vec{n}$, and the box has volume Λ^{-d}. So[18]

$$\tfrac{1}{2}\int_{\vec{x}}[\nabla\phi(\vec{x})]^2 \sim \Lambda^{2-d}\sum_{\vec{n}}\phi_{0\vec{n}}^{\,2} + \tfrac{1}{2}\int_{\vec{x}}[\nabla\phi_1(\vec{x})]^2\,. \tag{25}$$

The integral over U reduces to

$$\begin{aligned}\int_{\vec{x}} U[\phi(\vec{x})] \sim \Lambda^{-d}\sum_{\vec{n}}\{&\tfrac{1}{2}U[\phi_{0\vec{n}} + \phi_{1\vec{n}}] \\ &+ \tfrac{1}{2}U[-\phi_{0\vec{n}} + \phi_{1\vec{n}}]\}\,,\end{aligned} \tag{26}$$

where[19]

$$\phi_{1\vec{n}} = \phi_1(\vec{n}\Lambda^{-1})\,. \tag{27}$$

In writing the approximate form for $\int_{\vec{x}} U[\phi(\vec{x})]$ the wavelike nature of $\phi_0(\vec{x})$ within the box about site $\vec{n}$ has been crudely represented by putting $\phi_0(\vec{x}) = \phi_{0\vec{n}}$ in half the box and $\phi_0(\vec{x}) = -\phi_{0\vec{n}}$ in the other half of the box. This square wave form for $\phi_0(\vec{x})$ is the simplest form one can construct which satisfies the restriction that $\phi_0(\vec{x})$ have no Fourier component with $\vec{k}=0$. The $\vec{k}=0$ Fourier component of $\phi_0(\vec{x})$ is $\int_{\vec{x}}\phi_0(\vec{x})$, and this integral vanishes for all choices of $\phi_{0\vec{n}}$, given the square wave form of $\phi_0(\vec{x})$ used here. The field $\phi_1(\vec{x})$ is replaced by a constant $\phi_{1\vec{n}}$ inside the box, where

$$\phi_{1\vec{n}} = \phi_1(\vec{n}\Lambda^{-1})\,. \tag{28}$$

The action A now has the form

$$\begin{aligned}A[\phi] \sim -\sum_{\vec{n}}\{&\Lambda^{2-d}\phi_{0\vec{n}}^{\,2} + \tfrac{1}{2}\Lambda^{-d}U[\phi_{0\vec{n}} + \phi_{1\vec{n}}] \\ &+ \tfrac{1}{2}\Lambda^{-d}U[-\phi_{0\vec{n}} + \phi_{1\vec{n}}]\}\,.\end{aligned} \tag{29}$$

The functional integration of $e^{A[\phi]}$ over $\phi_0(\vec{x})$ thus reduces to an infinite set of independent integrals over each of the $\phi_{0\vec{n}}$. The $\phi_{1\vec{n}}$ are held fixed and act as parameters in these integrals. These integrals all have the form of Eq. (20) once one introduces the scaled function $Q(y)$ of Eq. (18).

Having computed the integrals over $\phi_{0\vec{n}}$, the functional integral $\langle \exp A[\phi]\rangle$ becomes $\langle \exp A_1[\phi_1]\rangle$, with

$$A_1[\phi_1] = \sum_{\vec{n}} \ln I[\Lambda^{1-d/2}\phi_{1\vec{n}}] - \tfrac{1}{2}\int_{\vec{x}} [\nabla\phi_1(\vec{x})]^2 . \quad (30)$$

The sum over $\vec{n}$ is now approximated by an integral, resulting in Eq. (14) with

$$U_1[\phi_1] = -\Lambda^d \ln\{I[\Lambda^{1-d/2}\phi_1]/I(0)\} . \quad (31)$$

The factor $I(0)$ only changes $U_1[\phi_1]$ by a constant; this constant cancels out in the ratio defining $D(x)$. It is introduced for convenience in calculation [otherwise the functions $Q_l(y)$ have large constants when l is large – this becomes evident when one does actual calculation]. When $U_1[\phi_1]$ is converted to $Q_1(y)$ according to Eq. (15) one gets Eq. (19). The factor 2^d multiplying $\ln I$ arises because Q_1 is proportional to Λ_1^{-d} times U_1, not Λ^{-d} times U_1.

This completes the derivation of the recursion formulas, except to note the following. The recursion formula was defined in terms of a calculation of $\langle e^{A[\phi]}\rangle$, not $\langle\phi(\vec{x})\phi(0)e^{A[\phi]}\rangle$. However if one is interested in $D(\vec{k})$ for small $\vec{k}$, say $|\vec{k}| < \Lambda_R$, where

$$D(\vec{k}) = \int_{\vec{x}} e^{-i\vec{k}\cdot\vec{x}} D(\vec{x}) , \quad (32)$$

then only the Fourier components $\phi(\vec{k})$ with $|\vec{k}| < \Lambda_R$ are important in $\phi(\vec{x})$ and $\phi(0)$. Hence one can replace $\phi(\vec{x})$ and $\phi(0)$ by $\phi_R(\vec{x})$ and $\phi_R(0)$, which are held fixed when one integrates the components $\phi(\vec{k})$ with $|\vec{k}| > \Lambda_R$. Hence one can use the recursion formula for $\langle\phi(\vec{x})\phi(0)e^{A[\phi]}\rangle$ as well, provided that $\Lambda_l \geq \Lambda_R$ and one is interested in $D(\vec{k})$ only for $|\vec{k}| < \Lambda_R$.

Now the renormalization problem will be discussed in terms of explicit numerical solutions of the recursion formulas. Let the cutoff interaction $U[\phi]$ have the form

$$U[\phi] = \tfrac{1}{2}[m_0(\Lambda)]^2\phi^2 + \lambda_0\phi^6 . \quad (33)$$

(As will be seen, it will not be necessary to let λ_0 depend on Λ, nor is it necessary to include a counterterm proportional to ϕ^4.) Let the cutoff Λ have the form

$$\Lambda = 2^L \Lambda_R , \quad (34)$$

where L is an integer. Consider now dimension $d=3$. The invariant function $Q(y)$ corresponding to $U[\phi]$ has the form

$$Q(y, L) = r_L y^2 + \lambda_0 y^6 , \quad (35)$$

with

$$r_L = \tfrac{1}{2} m_0^2(2^L\Lambda_R)/(2^L\Lambda_R)^2 . \quad (36)$$

The function $Q(y, L)$ is to be used as the initial function for the recursion formulas (16), (17), (19), and (20). To obtain the effective action A_R, one must iterate the recursion formulas L times to give an effective invariant interaction $Q_L(y, L)$, from which the effective interaction $U_R(\phi_R)$ is obtained by

$$U_R[\phi_R] = \Lambda_R{}^3 Q_L(\Lambda_R{}^{-1/2}\phi_R, L) . \quad (37)$$

If U_R is to have a finite limit when $\Lambda \to \infty$, then $Q_L(y, L)$ must also have a finite limit when $L \to \infty$. So the renormalization problem, technically speaking, is to choose a sequence of constants $\{r_L\}$ such that $Q_L(y, L)$ has a finite limit for $L \to \infty$.

The recursion formulas have been solved numerically for the initial condition[20]

$$Q(y) = ry^2 + 0.1y^6 \quad (38)$$

for a number of values of r. The results were as follows. There is a critical value r_c for r with the property that if $r = r_c$ then $Q_l(y)$ has a finite limit for $l \to \infty$:

$$Q_l(y) \to Q_c(y) \;\text{ for }\; l \to \infty . \quad (39)$$

The value of r_c is about -1.09. In practice one cannot calculate $Q_l(y)$ for $l \to \infty$: what one sees numerically is that for $r \simeq r_c$, $Q_l(y)$ is approximately independent of l over a large range of l. See Table I for an example of this. The function $Q_c(y)$ is shown in Fig. 1. For $r \simeq r_c$ but not equal to r_c and for sufficiently large l, one finds that[21]

$$Q_l(y) \simeq Q_c(y) + 2^{l\alpha}\epsilon R_c(y) , \quad (40)$$

where ϵ is $r - r_c$, α is a constant ($\alpha \simeq 1.64$) and $R_c(y)$ is a function of y independent of l. It is doubtful that either $Q_c(y)$ or $R_c(y)$ have simple analytic forms, but this has not been checked.[22] The form (40) is what one might expect if one considers a linearized form of the recursion formulas in which terms of second order in the difference $Q_l(y) - Q_c(y)$ are neglected. The constant α was

TABLE I. The functions $Q_l(y)$ for[a] $Q(y) = -1.09y^2 + 0.1y^6$.

l	$y=0$	0.45	1.35	2.25
0[b]	0	−0.221	−1.388	7.439
3	0	−0.053	−0.591	0.807
6	0	−0.0782	−0.584	0.696
9	0	−0.0783	−0.582	0.688
12	0	−0.0783	−0.582	0.688
15	0	−0.0773	−0.571	0.722
18	0	−0.0469	−0.251	1.77

[a] To be exact, r was $-1.0934813...$ in this calculation, but to reproduce the results of this table with this precise value for r one would have to use the same numerical approximations to the recursion integral used by the author. See Ref. 10.

[b] $Q_0(y)$ is the same as $Q(y)$.

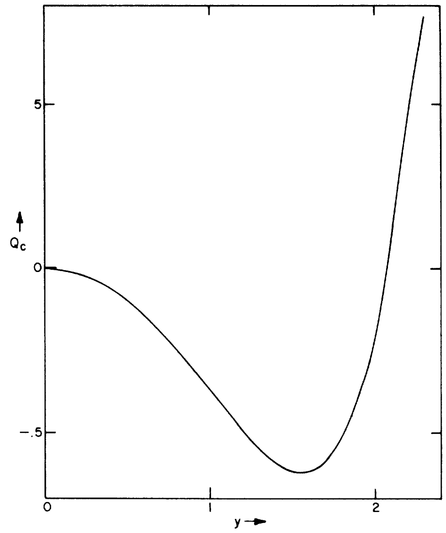

FIG. 1. The function $Q_c(y)$ vs y. Beyond the range of y shown, $Q_c(y)$ becomes large and positive (it is estimated to be proportional to y^6 for large y).

not known *a priori*; the function $R_c(y)$ and the constant α were found by fitting the form (40) to numerical calculations of $Q_l(y)$ for small ϵ.

With these results one can construct the required sequence r_L. If L is large and if $\epsilon_L = r_L - r_c$ is small enough then

$$Q_L(y, L) \simeq Q_c(y) + 2^{L\alpha}\epsilon_L R_c(y)\,. \tag{41}$$

For $Q_L(y, L)$ to approach a limit $Q_R(y)$ for $L\to\infty$, one must have $2^{L\alpha}\epsilon_L$ approach a limit, i.e.,

$$\epsilon_L = 2^{-L\alpha}\eta_R\,, \tag{42}$$

where η_R is a constant. For Eq. (41) to hold (which neglects terms of order $[Q_L(y, L) - Q_c(y)]^2$) one must have η_R small.[23]

So an answer to the renormalization problem is that if

$$r_L = r_c + 2^{-L\alpha}\eta_R\,, \tag{43}$$

where η_R is small, and if

$$\lambda_0 = 0.1\,, \tag{44}$$

then the effective interaction function $Q_R(y)$ corresponding to the fixed cutoff Λ_R is

$$Q_R(y) \simeq Q_c(y) + \eta_R R_c(y)\,. \tag{45}$$

This is independent of the original cutoff Λ. The original interaction $U[\phi]$ has the form

$$U[\phi] = \{\Lambda^2 r_c + \Lambda^{2-\alpha}\eta_R\Lambda_R{}^{\alpha}\}\phi^2 + \lambda_0\phi^6\,. \tag{46}$$

The parameter η_R is a free parameter of the renormalized theory since it appears in the renormalized interaction $Q_R(y)$.[24]

If one makes a small change in η_R, i.e., $\eta_R \to \eta_R + \delta\eta_R$, the interaction changes by

$$U[\phi] \to U[\phi] + \delta U[\phi]\,, \tag{47}$$

with

$$\delta U[\phi] = \delta\eta_R\Lambda_R{}^{\alpha}\{\Lambda^{2-\alpha}\phi^2\}\,.$$

The corresponding change in $D(\vec{x})$ is

$$D(\vec{x}) \to D(\vec{x}) + \delta D(\vec{x})\,, \tag{48}$$

with

$$\delta D(\vec{x}) = -\int_{\vec{y}}\left\{\frac{\langle\phi(\vec{x})\phi(0)\delta U[\phi(\vec{y})]e^{A[\phi]}\rangle}{\langle e^{A[\phi]}\rangle} - D(\vec{x})\frac{\langle\delta U[\phi(\vec{y})]e^{A[\phi]}\rangle}{\langle e^{A[\phi]}\rangle}\right\}. \tag{49}$$

Since changing the renormalized parameter η_R produces a finite (not infinite) change in $D(\vec{x})$, it follows that the field $\delta U[\phi(\vec{y})]$ has finite matrix elements [provided one subtracts its vacuum expectation value, as is done in Eq. (49)]. This means that $\Lambda^{2-\alpha}\phi^2(\vec{x})$ is a finite operator in the limit $\Lambda\to\infty$.

One can learn further consequences of the renormalization analysis by performing dimensional analysis. Consider the function $D(\vec{k})$. By techniques similar to those discussed here, the function $D(\vec{k})$ can be computed directly.[10] These computations are complicated; therefore only the dimensional analysis is discussed here. $D(\vec{k})$ depends on $\vec{k}$, Λ, and the interaction U. Dimensional analysis, in mass units, gives the following dimensions (for $d=3$): ϕ has dimensions $m^{1/2}$, U has dimensions m^3, and $D(\vec{k})$ has dimension m^{-2}. The dimensions of ϕ and U are determined by the requirement that the action A be dimensionless. The variable y and function Q are easily seen to be dimensionless. So one can write[25]

$$D(\vec{k}) = |\vec{k}|^{-2}D(\vec{k}/\Lambda, Q)\,. \tag{50}$$

One can also calculate $D(\vec{k})$ using the effective interaction Q_l and cutoff Λ_l, provided that $\vec{k}$ is less than Λ_l:

$$|\vec{k}|^{-2}D(\vec{k}/\Lambda, Q) = |\vec{k}|^{-2}D(\vec{k}/\Lambda_l, Q_l)\,. \tag{51}$$

In the case that $Q_l = Q_c$ independent of l (i.e., $\eta_R = 0$), this gives

$$D(\vec{k}/\Lambda, Q_c) = D(\vec{k}/\Lambda_l, Q_c)\,, \tag{52}$$

which means $D(\vec{k}/\Lambda, Q_c)$ is a constant independent of both $\vec{k}$ and Λ.[26] This means $D(\vec{k}) \propto |\vec{k}|^{-2}$, which is a scale-invariant propagator for a field of dimension 0.5.[27] Suppose one now considers functions $Q(y)$ of the form $Q_c(y) + \epsilon R_c(y)$ with ϵ small; write

$$D(\vec{k}/\Lambda, Q_c + \epsilon R_c) = G(\vec{k}/\Lambda, \epsilon)\,. \tag{53}$$

Then

$$G(\vec{k}/\Lambda, \epsilon) = G(\vec{k}/\Lambda_l, 2^{l\alpha}\epsilon), \tag{54}$$

which means G depends only on the single variable $(\vec{k}/\Lambda)\epsilon^{-1/\alpha}$, i.e.,

$$D(\vec{k}) = |\vec{k}|^{-2}G(\epsilon^{-1/\alpha}\vec{k}/\Lambda). \tag{55}$$

Now with the substitutions $\eta_R = 2^{L\alpha}\epsilon$, $\Lambda_R = 2^{-L}\Lambda$, one gets for the renormalized theory

$$D(\vec{k}) = |\vec{k}|^{-2}G(\vec{k}/m_R), \tag{56}$$

where

$$m_R = \Lambda_R \eta_R^{1/\alpha}. \tag{57}$$

So if $\eta_R \neq 0$, one has an effective-mass parameter m_R on which $D(\vec{k})$ depends and $D(\vec{k})$ is no longer scale-invariant.

In the scale-invariant theory ($\eta_R = 0$) the finite field which is the limit of $\Lambda^{2-\alpha}\phi^2(\vec{x})$ for $\Lambda \to \infty$ has an anomalous dimension, namely, $(\text{mass})^{3-\alpha}$. With $\alpha = 1.64$, this field has dimension 1.36.

The results obtained above are rather contrary to what one would anticipate in perturbation theory. In perturbation theory one must have λ_0 depend logarithmically on Λ, and a linearly divergent ϕ^4 term and a logarithmically divergent $\nabla\phi^2$ term in the original action, if the renormalized theory is to be finite. If one studies the recursion formula in perturbation theory [i.e., if one computes $Q_l(y)$ as an expansion in λ_0] one indeed finds that λ_0 must be logarithmically divergent and a linearly divergent ϕ^4 term is needed to make $Q_R(y)$ be finite order by order in perturbation theory. No divergent $\nabla\phi^2$ counterterm is needed (such a term is required only if a more accurate calculation is made than the qualitative analysis of this paper[28]). The reason that λ_0 can be held fixed and no ϕ^4 counterterm is required in the nonperturbative analysis reported here is that the self-regulating effects of strong coupling damp out the divergences which appear in perturbation theory, except for the self-mass divergence which must still be compensated for.

Are the simplifications of this paper reliable? The crucial question is not how accurate are numerical results like the dimension 1.36 for ϕ_R^2. The most important question is the validity of qualitative results such as the renormalizability of the ϕ^6 theory without coupling-constant renormalization. This question has to be answered by experience; the simplifications of this paper are too great to permit a credible formal analysis of errors generated by the simplifications. (See, however, Sec. V of Ref. 10.)

At present most of the applications of the recursion formula have been to problems in critical phenomena in classical statistical mechanics.

The functional integral of Eq.(8) can be interpreted as giving the spin-spin correlation function of a generalized Ising model.[10,29] The recursion formula of this paper gives a remarkable good description of the critical behavior of the Ising model.[10] The Ising model has been extensively studied by other methods; in particular, high-temperature expansions combined with Padé approximant summation techniques appear to give very accurate values for critical exponents.[30] The numbers calculated from the recursion formula can be compared with numbers calculated from the high-temperature expansion. When the results are translated into field-theoretic terms, the high-temperature expansion gives[31] 1.44 for the dimension of ϕ^2 with an error of about 0.01.

Further experience has been gained by studying the recursion formula in $4-\epsilon$ space dimensions with ϵ small.[32] It turns out that one can set up exact calculations of critical behavior for ϵ small; one finds that the recursion formula is exact to order ϵ and not so good in order ϵ^2. The exact calculation for $d(\phi^2)$ gives $2-\frac{2}{3}\epsilon+0.117\epsilon^2$ while the recursion formula gives $2-\frac{2}{3}\epsilon+0.026\epsilon^2$.

The crucial question, whether the ϕ^6 theory is renormalizable using only mass (and perhaps wave-function) renormalization, cannot be answered as yet. The trouble is that, when the hypothesis of renormalizability is translated into predictions for critical phenomena, one predicts the Widom-Kadanoff scaling laws, some of which are still controversial.[30] In particular there is a prediction that $\alpha = 2 - d\nu$, where α and ν are critical exponents. High-temperature expansion calculations indicate that there may be a small violation of this prediction, by about 0.05. One will have to understand whether this prediction for α is correct before one can claim that the ϕ^6 theory is renormalizable.

This paper leaves many questions unanswered. However, enough has been said about the recursion formulas so that one can do further practice calculations with it, such as perturbation expansions or further numerical calculations. The work of this paper may be easier to follow if one studies Refs. 4 – 7 and 10 also.

There has been much more rigorous work done on superrenormalizable theories[33]; the importance of the analysis of this paper is that it is a start on understanding theories which in perturbation theory require coupling-constant renormalization. The results of this paper suggest that perturbation theory is not a good guide to the strong-coupling behavior of these theories, so nonperturbative methods are crucial to understanding them.

The calculations described here were performed on the PDP-10 computer of the Laboratory of

Nuclear Studies. I wish to thank Professor B. Widom and Professor P. Carruthers for encouragement. I have benefitted from conversations with Professor M. E. Fisher, Professor D. Jasnow, Professor J. Wilkins, and participants in the Midwinter Conference on Solid State Physics (Irvine, 1970).

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*Supported in part by the National Science Foundation.

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[13]For an extensive study of Euclidean quantum field theory, see K. Symanzik in *Proceedings of the International School of Physics "Enrico Fermi"*, Course XLV, edited by R. Jost (Academic, New York, 1969), p. 152.

[14]A procedure of this type was suggested long ago by F. J. Dyson, Proc. Roy. Soc. (London) A207, 395 (1951).

[15]For another example of this approach to renormalization see Refs. 4 and 5.

[16]Q_l is *not* a Legendre function.

[17]For a more careful (but still qualitative) discussion see Ref. 10.

[18]In a more careful analysis some constants of proportionality occur, for example, one constant multiplies $\Lambda^{2-d}\sum_{\vec{n}}\phi_{0\vec{n}}^2$. These constants do not appear in the recursion formulas for $Q_l(y)$ [Eqs. (16) and (17)] and do not change the conclusions of this paper. See Ref. 10 for details.

[19]Note that the quantities $\phi_{1\vec{n}}$ are not being defined as independent variables; they are simply values of the field $\phi_1(\vec{x})$.

[20]The choice $\lambda_0=0.1$ was arbitrary. The same results should occur for a range of values of λ_0. See Ref. 22.

[21]For small l there are extra transient terms in $Q_l(y)$; these terms go rapidly to zero as l increases. For very large l (for which $2^{l\alpha}\epsilon$ is of order 1 or larger), $Q_l(y)$ becomes nonlinear in ϵ and Eq. (40) again breaks down.

[22]The functions $Q_c(y)$ and $R_c(y)$ are determined by the recursion formulas (16) and (17) and not by the initial function $Q(y)$. For example, $Q_c(y)$ is determined by the fact that it gives an l-independent solution of the recursion formulas. In other words, $Q_c(y)$ represents a fixed point of the renormalization group. See Ref. 6 for a discussion of fixed points of the renormalization group and their consequences. To check that $Q_c(y)$ and $R_c(y)$ are independent of the detailed choice of the initial function $Q(y)$, the recursion formulas were solved with $Q(y)=ry^2+0.5y^4$; in this case r_c was found to be approximately -1.67, and for $r\simeq r_c$ the same behavior (40) was found for large l with the same constant α and the same functions $Q_c(y)$ and $R_c(y)$ [except for a scale factor in $R_c(y)$].

[23]The restriction that η_R be small is made here simply to avoid discussing the form of $Q_l(y)$ when l is so large that $Q_l(y)$ is nonlinear in ϵ. There is no difficulty in giving such a discussion, but it would make this paper too long.

[24]The effective interaction $Q_R(y)$ is not a Gaussian function (see Fig. 1) so one expects the renormalized field theory to include interactions. In contrast, if one considers the $\lambda_0\phi^4$ theory in four space-time dimensions and sufficiently small λ_0 ($\lambda_0<0.3$, at least) the corresponding effective renormalized interaction is found to be Gaussian.

[25]The function $D(\vec{k}/\Lambda, Q)$ is a functional of the function Q.

[26]This statement is not a strict consequence of Eq. (52) since the set of cutoffs Λ_l are discrete rather than continuous, but it should be at least qualitatively correct.

[27]In the case $Q_l(y)=Q_c(y)$ the complete field theory is scale-invariant, not just the propagator. The proof of this is analogous to the proof given in Ref. 6 that a fixed-point solution of the renormalization group equations corresponds to a scale-invariant field theory.

[28]For example, wave-function renormalization is probably necessary if one does not treat $\phi_1(\vec{x})$ as constant over a box about a lattice site. Instead, one can expand $\phi_1(\vec{x})$ in a Taylor's series about the point $\vec{x}=\vec{n}\Lambda^{-1}$. This will generate new terms in the effective action $A_1[\phi_1]$ involving $\nabla\phi_1$. Such terms may have to be compensated for in the original action if the renormalization program is to work; hence there may have to be wave-function renormalization.

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[31]The number 1.44 was obtained indirectly from critical exponents using the Widom-Kadanoff scaling laws (see Ref. 30). The dimension $d(\phi^2)$ can be obtained from the energy-energy correlation function discussed in Kadanoff *et al.* (Ref. 30), with the conclusion that $d(\phi^2)=d-1/\nu$

where ν is the susceptibility exponent. The best estimate for ν is 0.642 (see Ref. 30).

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PHYSICAL REVIEW D VOLUME 6, NUMBER 2 15 JULY 1972

Tachyon Commutator and Invariance

J. E. Murphy
Physics Department, Louisiana State University in New Orleans, New Orleans, Louisiana 70122
(Received 30 March 1972)

Using the Klein-Gordon equation and relativistic invariance, we calculate the commutator and anticommutator for free tachyon fields. We exhibit the form of the spectral representation of the vacuum expectation value of the commutator for interacting tachyon fields.

I. INTRODUCTION

Two striking examples of the power of relativistic invariance are the calculation (up to a multiplicative constant) of the free-field commutator, and the Lehmann spectral representation for the vacuum expectation value of the commutator. We will attempt to extend these calculations to the case of tachyons (faster-than-light particles). In Sec. II we show that the requirement that the commutator (anticommutator) of free fields be an odd (even), invariant, c-number solution of the Klein-Gordon equation fixes the value of the commutator (anticommutator), apart from a multiplicative constant. For the commutator, we find the free fields are more than causal. Section III involves writing the commutator vacuum expectation value (VEV) in a spectral representation. The form which the spectral representation takes suggests that a theory of interacting tachyons would satisfy microcausality. Finally, in Sec. IV we end with some comparisons of our results with those of others.

II. COMMUTATORS AND ANTICOMMUTATORS

We assume that it is possible to describe spinless tachyons by a Hermitian field $\phi(x)$, which satisfies the Klein-Gordon equation $(\Box - \alpha^2)\phi(x) = 0$. The requirement that the commutator be an *odd, invariant, c-number solution* of the Klein-Gordon equation will be shown to fix the value of the commutator.

Denote by $F(x,y)$ the right-hand side of the commutator,

$$[\phi(x), \phi(y)] = F(x,y)\,, \tag{1}$$

where by assumption F is invariant c-number. Invariance under translations requires for arbitrary a that

$$F(x,y) = F(x+a, y+a)\,, \tag{2}$$

so that F can only be a function of $x-y$. Invariance under proper homogeneous Lorentz transformations requires that

$$F(\Lambda x) = F(x)\,. \tag{3}$$

Since $\phi(x)$ satisfies the Klein-Gordon equation, $F(x,y)$ must likewise satisfy

$$(\Box - \alpha^2)[\phi(x), \phi(y)] = 0 = (\Box - \alpha^2)F(x,y)\,. \tag{4}$$

Finally, the commutator has the property that

$$[\phi(x), \phi(y)] = -[\phi(y), \phi(x)]\,, \tag{5}$$

so F must be odd:

$$F(x-y) = -F(y-x)\,, \tag{6}$$

or equivalently

$$F(x) = -F(-x)\,. \tag{7}$$

Since $F(x)$ satisfies the Klein-Gordon equation, we will write it in the form

$$F(x) = \int d^4k\, \delta(k^2 + \alpha^2)\tilde{F}(k)e^{-ik\cdot x}\,. \tag{8}$$

Here $\tilde{F}(k)$ is only defined on the "mass shell" $k^2 = k_0{}^2 - \vec{k}^2 = -\alpha^2$. The possible values of the vector k are all spacelike. Lorentz invariance of $\tilde{F}(x)$ means

$$\begin{aligned} F(\Lambda x) &= F(x) \\ &= \int d^4k\, \delta(k^2+\alpha^2)\tilde{F}(k)e^{-ik\cdot\Lambda x} \\ &= \int d^4k\, \delta(k^2+\alpha^2)\tilde{F}(\Lambda k)e^{-ik\cdot x}\,, \end{aligned} \tag{9}$$

so that $\tilde{F}(k) = \tilde{F}(\Lambda k)$, and $\tilde{F}$ can only be a function of the invariants formed with the vector k. For