at which point $(dl/dt)^2$ begins to decrease with furthur decrease in r for $r \geqslant 2m$. The value of v_0 for which the transition will occur at a given value of r ($\geqslant 2m$) is

$$v_t^2 = \frac{1}{2}\left(\frac{r-4m}{r-2m}\right) \underset{r \gg 2m}{\simeq} \frac{1}{2}\left(1-\frac{2m}{r}\right). \tag{7}$$

For $v_0^2 > 0.5$, we see that $(dl/dt)^2$ decreases monotonically with decreasing r over the entire range from $r=\infty$ to $r=2m$. This behavior is lightlike since, for photons, Eq. (5) shows that $(dl/dt)^2$ also decreases monotonically over that range.

Thus, for any $r \gg 2m$, a test particle with $v_0 \geqslant c/\sqrt{2}$ will, when viewed from $r \gg m$, behave in a lightlike fashion by appearing to slow down as it nears the source.[5]

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[1] H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, 1968).

[2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Addison-Wesley, Reading, Mass., 1962), p. 271ff.

[3] Note that dt is equivalent to proper time, and hence is invariant, when applied to a distant observer as here.

[4] The method is straightforward in principle. By inverting a sufficient number of the equations that express the measured echo times in terms of m, the constants of the motion, and t, we can obtain an explicit formula for the speed of a particle solely in terms of the readings on the distant oberver's clock. Such formulas could also easily be adapted, for example, to the Eddington-Robertson generalization of the Schwarzschild metric.

[5] An experimental verification of this prediction, whether involving high-energy elementary particles or other test bodies orbiting in the solar gravitational field, does not yet appear practical.

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Gravitational Spin Interaction*

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The gravitational spin interaction is investigated by studying the deviation from geodesic motion of spinning test bodies. The force on a spinning test body at rest in the exterior field of an arbitrary stationary, rotating source is evaluated and found to be given by

$$\vec{F} = -\vec{\nabla}\left(\frac{-\vec{S}\cdot\vec{J} + 3(\hat{r}\cdot\vec{J})(\hat{r}\cdot\vec{S})}{r^3}\right) + O(1/r^5),$$

where $\vec{S}$ is the spin of the test body, $\vec{J}$ is the angular momentum of the source of the gravitational field, and geometrized units $G=c=1$ are used. Thus, the gravitational spin-spin force has the same form as the force between two dipoles in electromagnetism except that its sign is opposite, i.e., "north pole" attracts "north pole" in gravitational spin-spin interaction. The gravitational spin-spin torque, previously investigated by Schiff, Mashhoon, and Wilkins, also has opposite sign to the corresponding electromagnetic dipole-dipole torque. The sign of the spin-spin force agrees with that predicted by Hawking on the basis of the fact that less energy can be extracted from colliding black holes if their spins are parallel rather than antiparallel. Furthermore, it is shown that the spin-interaction energy quantitatively accounts for the angular momentum dependence in Hawking's formula for the upper limit for energy released from colliding black holes. The gravitational spin-orbit force is also investigated, and it is found to differ in form from the corresponding electromagnetic spin-orbit force.

I. INTRODUCTION

Although the structure and interpretation of the theory of electromagnetism and Einstein's theory of gravitation are (at least at the present time) greatly different, the basic similarity of these two theories for the interaction of two bodies is far too familiar to be discussed at length: In lowest order, two slowly moving, nonrotating, massive bodies attract each other with force $-(m_1 m_2/r^2)\hat{r}$ (in geometrized units, $G=c=1$) while two slowly moving, nonrotating, charged bodies repel each other with force $(e_1 e_2/r^2)\hat{r}$. Of course, when higher-order corrections to the motion are taken into account,[1] or when the relative velocity of the bodies is large, the gravitational and electromagnetic interactions differ. Indeed, a meaningful comparison between gravitation and electromagnetism can only be made when the gravitational field is weak. Nevertheless, the strong analogy in lowest order between gravitation and electromagnetism suggests that other effects which occur in electromagnetism may be present in the gravitational case also. For example, suppose that in the gravitational case one of the bodies is rotating. Analogy with electromagnetism suggests that the rotation will produce an effective "gravitational magnetic dipole moment" of $\frac{1}{2}\vec{J}$, where $\vec{J}$ is the angular momentum of the body. (In the slow-motion weak-field limit, $\frac{1}{2}\vec{J}$ is related to the mass distribution by the same formula as the magnetic dipole moment $\vec{\mu}$ is related to the charge distribution in electromagnetism.) To lowest order in velocity and lowest order in $1/r$ one would then expect the motion of the second body to be affected by a "gravitational magnetic acceleration" $\vec{a} \sim \vec{v} \times \vec{B}$, where $\vec{B}$ is related to $\frac{1}{2}\vec{J}$ by the same formula as $\vec{B}_{\text{dipole}}$ is related to $\vec{\mu}$ in electromagnetism, i.e.,

$$\vec{B} = \frac{3\hat{r}(\hat{r} \cdot \frac{1}{2}\vec{J}) - \frac{1}{2}\vec{J}}{r^3} . \tag{1}$$

If one calculates the geodesic equations of motion of a nonrotating test particle in the exterior gravitational field of an arbitrary stationary, rotating source [see Eq. (20) below] one finds that indeed an additional term due to the rotation of the source appears, giving rise to an acceleration,[2]

$$\vec{a} = -4\vec{v} \times \vec{B} \tag{2}$$

where $\vec{B}$ is given by Eq. (1). Thus, the "gravitational magnetic force" in this case is precisely (-4) times the expression one would obtain from electromagnetism by the natural substitution $q \rightarrow m$ and $\vec{\mu} \rightarrow \frac{1}{2}\vec{J}$.

Further effects occur in electromagnetism if both charged bodies are spinning. In this case there will be an additional force between the bodies resulting from the interaction of the magnetic dipoles created by the spinning charges. In addition, the rotation axes of the bodies will precess on account of the dipole-dipole torque. One may therefore ask if these effects are also present in gravitational interaction.

The precession of a spinning test body (i.e., a gyroscope) in the exterior gravitational field of a rotating body has been investigated by Schiff,[3] Mashhoon,[4] and Wilkins[5] using Papapetrou's equations of motion for the spin. For a particle at rest, the precession rate in lowest order is given by

$$\frac{d\vec{S}}{dt} = -4(\tfrac{1}{2}\vec{S}) \times \vec{B} , \tag{3}$$

where $\vec{S}$ is the spin of the test particle and $\vec{B}$ is given by Eq. (1). This is precisely (-4) times the expression obtained by replacing dipole moments in the electromagnetic precession formula by $\frac{1}{2} \times$(angular momentum).[5] In this paper we investigate the gravitational spin-spin force by studying via Papapetrou's equations of motion the initial deviation from geodesic motion of a spinning test particle released from rest in the exterior field of a rotating body. We obtain

$$\vec{F} = -4\vec{\nabla}(\tfrac{1}{2}\vec{S} \cdot \vec{B}) , \tag{4}$$

which is again (-4) times the corresponding electromagnetic formula. Both Eqs. (3) and (4) could be anticipated from Eq. (2). However, the gravitational spin-orbit force is found to have a different form than the electromagnetic spin-orbit force [see Eqs. (44) and (45) below].

Strong evidence for the probable existence of a spin-spin force in general relativity has come from the recent work of Hawking[6] on colliding black holes. Under the assumptions that (1) all singularities of gravitational collapse are hidden in black holes and (2) all black holes eventually settle down to Kerr-Newman solutions,[7] Hawking has obtained upper limits on the amount of energy which can be released when two black holes that are initially at rest and widely separated coalesce to form a single black hole. The proof runs as follows: Assumption (1) implies that the surface area of the future event horizon of a space-time can never decrease; thus in any process the final horizon surface area must be greater than the initial horizon surface area. Assumption (2) allows one to relate the horizon surface area of a black hole to its mass, angular momentum, and charge. Hence one can transform the inequality of initial and final surface area into a lower limit on the final mass of the black hole produced by the coalescence and thus an upper limit on the energy which can be radiated away. One finds from Hawking's formula that less energy can be radiated away when two rotating black holes collide with their spins parallel rather than antiparallel. This suggests the existence of a spin-spin force between the black holes which is repulsive when the spins are parallel and attractive when the spins are antiparallel. The actual existence of a gravitational spin-spin force having the properties predicted by Hawking[6] is demonstrated in this paper. Furthermore, it is shown in Sec. IV that the spin-interaction energy quantitatively accounts for the angular momentum dependence in Hawking's formula. This strongly supports the validity of Hawking's results and thus gives further support for the validity of the basic assumptions (1) and (2) which entered Hawking's analysis.

The expression for the spin-spin force, Eq. (4), derived below in Sec. III, has also been obtained independently by Tiomno[8] using correspondence and equivalence-principle arguments. In the present paper, however, no assumptions are made beyond the validity of Einstein's theory of gravitation and the approximation of a spinning test particle.

Note added in proof. Expressions for the gravitational spin-spin and spin-orbit forces have also been obtained independently by D. C. Wilkins (unpublished).

In Sec. II we discuss the equations of motion of a spinning body in the presence of a gravitational field. The gravitational spin-spin interaction is analyzed in Sec. III. The spin interaction energy is compared with Hawking's formula in Sec. IV, and the spin-orbit interaction is treated in Sec. V.

II. EQUATIONS OF MOTION OF A SPINNING BODY

It is well known that the Einstein field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{5}$$

determine the motion of bodies, i.e., equations of motion do not have to be separately imposed. This was first shown by Einstein, Infeld, and Hoffmann,[9] who, starting from a linearized vacuum solution with monopole singularities, found that higher-order corrections to this solution could not exist unless certain consistency conditions were satisfied. These consistency conditions give the equations of motion of the singularities. A different approach to the equations of motion, more suitable for treating the motion of test bodies, has been developed by Fock[10] and Papapetrou.[11] In this approach one uses directly the conservation law

$$T^{\mu\nu}{}_{;\nu} = 0\,, \tag{6}$$

which follows from Einstein's Eqs. (5) and the Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$. Papapetrou's method for determining the motion of test bodies consists of choosing a "representative point" (or more precisely, a representative world line) in the body and taking the moments of Eq. (6) about that point. If one then assumes that the body is sufficiently small so that all the moments beyond the nth are negligible (the "n-pole particle" approximation), the equations of motion can be found. The equations of motion of a single-pole particle are just the geodesic equations. The equations of motion of a spinning test particle (i.e., a pole-dipole particle) were explicitly derived by Papapetrou[11] by the above procedure.

An arbitrary step in the above procedure is the choosing of a representative point in the body. This arbitrary choice should make no difference in the limit that the size of the test body tends to zero. However, Møller[12] has shown in the context of special relativity that a classical body with intrinsic angular momentum S and mass M and having positive energy density in all frames of reference (i.e., $T_{\mu\nu}\xi^\mu\xi^\nu > 0$ for all timelike vectors ξ) must have a size r_0 as measured in the proper center-of-mass frame given by

$$r_0 \gtrsim S/M\,. \tag{7}$$

(One can see the necessity of such a limit from the fact that $S \sim M r_0 v_{\max} \lesssim M r_0$ since $v_{\max} < 1$.) Thus, although one may mathematically treat the motion of point particles,[13] these particles cannot be considered the limit of any physically reasonable matter distribution (i.e., a matter distribution having positive energy density everywhere) unless $S/M \to 0$ in the limiting process. To treat classical, physically reasonable spinning bodies, one must deal with bodies of finite size given by Eq. (7). It is essential, therefore, to carefully select a representative world line through the body. It can be expected that if such a world line is not uniquely specified, the motion of the representative point obtained from Papapetrou's equations will not be fully determined. The proper specification of such a world line gives a "supplementary condition" to Papapetrou's equations which determines the motion.

The problem of determining a unique center-of-mass world line of a body in curved space-time has been treated recently by Beiglböck,[14] Madore,[15] Dixon,[16] and others. The prescription for obtaining this world line may be described as follows. (See Beiglböck[14] for the precise listing of technical assumptions needed for this construction.) At each point x in the body and for each timelike unit vector n at the point x, we define $\Sigma(x, n)$ to be the spacelike hypersurface generated by all geodesics through x orthogonal to n. We define

$$p^\mu(x, n) = \int_{\Sigma(x,n)} T^{\mu\nu}\, d\Sigma_\nu\,. \tag{8}$$

Then $p^\mu(x, n)$ has the interpretation of the momentum of the body as measured by an observer at x moving in the direction n. Beiglböck[14] has shown that at each point x there exists a unique timelike vector u such that u and $p(x, u)$ are collinear, i.e.,

$$u^{[\mu} p^{\nu]}(x, u) = 0\,, \tag{9}$$

where [ ] denotes antisymmetrization. We write

$p^\mu(x)=p^\mu(x,u)$. We define the spin tensor $S^{\mu\nu}(x)$ at the point x by

$$S^{\mu\nu}(x)=2\int_{\Sigma(x,u)}(y-x)^{[\mu}T^{\nu]\sigma}\,d\Sigma_\sigma(y)\,. \tag{10}$$

Then Beiglböck[14] has shown that there exists a unique timelike world line $z^\mu(s)$ satisfying

$$p_\mu(z)S^{\mu\nu}(z)=0\,. \tag{11}$$

We call this world line the center of mass of the body. In flat space this definition agrees with the center of gravity (or proper center of mass) defined by Møller.[12]

With the center-of-mass world line defined, one may now repeat the original derivation of Papapetrou[11] for the motion of a spinning test body, choosing the representative world line to be the center of mass and choosing the direction denoted as "4" by Papapetrou to be the direction of $p^\mu(z)$. Writing $p^\mu(s)=p^\mu(z(s))$, $S^{\mu\nu}(s)=S^{\mu\nu}(z(s))$, and defining $v^\mu(s)$ to be the unit tangent vector to the world line $z^\mu(s)$, one obtains

$$\frac{Dp^\mu}{Ds}=-\tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma}\,, \tag{12}$$

$$\frac{DS^{\mu\nu}}{Ds}=p^\mu v^\nu-p^\nu v^\mu\,, \tag{13}$$

where D/Ds denotes covariant derivative along the world line $z^\mu(s)$ and

$$R^\mu{}_{\nu\rho\sigma}=\frac{\partial\Gamma^\mu_{\nu\sigma}}{\partial x^\rho}-\frac{\partial\Gamma^\mu_{\nu\rho}}{\partial x^\sigma}+\Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma}-\Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho}$$

is the Riemann tensor. Equations (12) and (13) together with the "supplementary condition"

$$p_\mu S^{\mu\nu}=0 \tag{14}$$

determine the motion of the spinning test body. (Note that the Pirani condition[17] $v_\mu S^{\mu\nu}=0$ does not uniquely specify a world line through the body. Thus, with this supplementary condition the motion is not uniquely determined, and even in flat space one obtains solutions with helical motion and precessing spin[18] in addition to the solution of straight-line motion with constant spin. See Møller[12] for the interpretation of this effect in terms of nonunique choice of world line.)

Contracting $S_{\mu\nu}$ into Eq. (13) and using Eq. (14) one immediately obtains that the magnitude S of the spin, defined by

$$S^2=\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}\,, \tag{15}$$

is a constant of the motion. A somewhat longer calculation shows that

$$M^2=p_\mu p^\mu \tag{16}$$

is also a constant of the motion. M has the interpretation of the mass of the body.

Note that p^μ and v^μ are *not* in general collinear. In fact, Dixon[16] has shown that Eqs. (12)–(14) imply that

$$p^{[\mu}v^{\nu]}=-\frac{1}{4M}\sqrt{-g}\;\epsilon^{\mu\nu\lambda\rho}R_{\lambda\alpha\beta\gamma}v^\alpha S^{\beta\gamma}S_\rho\,, \tag{17}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor density with $\epsilon_{0123}=1$ and where S_ρ is the spin vector defined by

$$S_\rho=\frac{1}{2M}\sqrt{-g}\;\epsilon_{\mu\nu\lambda\rho}p^\mu S^{\nu\lambda}\,. \tag{18}$$

In flat space, however, p^μ and v^μ are collinear and the only solution of Eqs. (12)–(14) is straight-line motion with constant spin.

The right-hand side of Eq. (12) gives the rate of change of momentum of the body as measured by freely falling observer initially comoving with the center of mass. We define this to be the force exerted on the spinning test body due to its spin,

$$F^\mu\equiv\frac{Dp^\mu}{Ds}=-\tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma}\,. \tag{19}$$

Another possibly reasonable definition of force would be to define it as

$$\tilde{F}^\mu=\frac{D}{Ds}(Mv^\mu)=M\frac{Dv^\mu}{Ds}\,.$$

Since p^μ and Mv^μ differ in general according to Eq. (17), this definition of force will differ in general from Eq. (19). However, it is easy to verify using Eq. (17) that in the force calculation of Secs. III and V (where only the lowest nonvanishing contribution to the force is calculated) it makes no difference which definition one uses. In fact, one may argue that whenever the minimal size of the body, given by Eq. (7), is much smaller than the radius of curvature of the space-time – a criterion that certainly must be satisfied by a test body – the difference between the two proposed definitions of force will always be negligible compared with the force itself.

In the next section we calculate the spin force, Eq. (19), on a spinning test body initially at rest in the exterior field of a rotating source.

III. GRAVITATIONAL SPIN-SPIN INTERACTION

The metric of an arbitrary stationary, asymptotically flat space-time can be put in the following form[19] for large r:

$$dS^2 = -\left(1 - \frac{2m}{r} + \frac{2m^2}{r^2}\right) dt^2 - 4\epsilon_{ijk}\frac{J^j x^k}{r^3} dt dx^i + \left(1 + \frac{2m}{r} + \frac{3m^2}{2r^2}\right)\delta_{jk} dx^j dx^k + O(1/r^3) dx^\mu dx^\nu . \tag{20}$$

Here Roman indices run from 1 to 3, Greek indices run from 0 to 3, and ϵ_{ijk} is the completely antisymmetric tensor. The parameters m and $\vec{J} = (J_1, J_2, J_3)$ are, respectively, the total mass and total angular momentum of the space-time.[19] Note that it is not assumed here that the gravitational field is weak in the interior, i.e., the expansion (20) and the interpretation of m and $\vec{J}$ would apply, for example, to a black hole. We now calculate the lowest nonvanishing contribution to the spin force, Eq. (19), on a spinning test particle initially at rest in this exterior gravitational field, Eq. (20). (In this approximation it makes no difference whether at rest is taken to mean p^μ or v^μ points in the direction of the stationary Killing vector of the space-time; the difference between p^μ and Mv^μ is entirely negligible.) To lowest nonvanishing order in $1/r$ we have $v^\mu = (1, 0, 0, 0)$, $S^{0i} = -S^{i0} = 0$, and $S_{jk} = \epsilon_{jkl} S^l$, where $\vec{S}$ is the spin vector, Eq. (18). It also makes no difference in lowest order if spatial indices are up or down. Hence from Eq. (19) we have

$$F^i = -\tfrac{1}{2} R^i{}_{0jk}\epsilon_{jkl} S^l . \tag{21}$$

A direct computation of $R^i{}_{0jk}$ from the metric of Eq. (20) yields

$$R^i{}_{0jk} = \frac{\partial}{\partial x^i}\left[\epsilon_{kmn} J^m \frac{\partial}{\partial x^j}\left(\frac{x^n}{r^3}\right) - \epsilon_{jmn} J^m \frac{\partial}{\partial x^k}\left(\frac{x^n}{r^3}\right)\right] + O(1/r^5) . \tag{22}$$

Hence we get

$$F^i = -\frac{\partial}{\partial x^i}\left[\epsilon_{jkl}\epsilon_{kmn} S^l J^m \left(\frac{\delta_{jn}}{r^3} - \frac{3x^j x^n}{r^5}\right)\right] + O(1/r^5) . \tag{23}$$

In vector notation, we have

$$\vec{F} = -\vec{\nabla}\left[\frac{2\vec{S}\cdot\vec{J}}{r^3} - \frac{3(\vec{S}\times\vec{r})\cdot(\vec{J}\times\vec{r})}{r^5}\right] + O(1/r^5) = -\vec{\nabla}\left[\frac{-\vec{S}\cdot\vec{J} + 3(\vec{S}\cdot\hat{r})(\vec{J}\cdot\hat{r})}{r^3}\right] + O(1/r^5) . \tag{24}$$

For comparison, the electromagnetic force between two dipoles $\vec{\mu}_1$ and $\vec{\mu}_2$ is

$$\vec{F} = -\vec{\nabla}\left[\frac{+\vec{\mu}_1\cdot\vec{\mu}_2 - 3(\vec{\mu}_1\cdot\hat{r})(\vec{\mu}_2\cdot\hat{r})}{r^3}\right] . \tag{25}$$

Thus, the gravitational spin-spin force has the same spatial dependence but opposite sign to the electromagnetic dipole-dipole force, i.e., in gravitation "north pole" attracts "north pole." Note that if the background metric is static (as opposed to merely stationary) then $R^i{}_{0jk}$ vanishes identically. Hence, for a spinning test body at rest, the spin force $\vec{F}$ is zero to all orders in $1/r$, as would be expected from analogy with electromagnetism.

The precession rate of the spin, which can be calculated from Eq. (13), has been investigated by Schiff,[3] Mashhoon,[4] and Wilkins.[5] For a spinning body at rest, they obtain[3-5]

$$\frac{d\vec{S}}{dt} = \left(\frac{-\vec{J} + 3\hat{r}(\vec{J}\cdot\hat{r})}{r^3}\right)\times\vec{S} . \tag{26}$$

For comparison the corresponding dipole-dipole torque in electromagnetism is

$$\vec{\tau} = \vec{\mu}\times\vec{B} = -\left(\frac{-\vec{P} + 3\hat{r}(\vec{P}\cdot\hat{r})}{r^3}\right)\times\vec{\mu} , \tag{27}$$

where $\vec{\mu}$ is the dipole moment of the test body and $\vec{P}$ is the dipole moment of the source of the dipole field. Thus, the gravitational spin-spin torque also has the same spatial dependence but opposite sign to the electromagnetic dipole-dipole torque.

Equation (24) for the spin force is valid for a spinning test particle at rest in any stationary, asymptotically flat space-time, but, of course, is only useful at large r. As an interesting example of spin force in the strong-field region, we consider the following problem: A spinning test particle is located on the symmetry axis of the Kerr black-hole solution[20]

$$ds^2 = -\frac{r^2 + a^2 - 2mr}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}[a\,dt - (r^2 + a^2)d\phi]^2 + (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2 + a^2 - 2mr} + d\theta^2\right) \tag{28}$$

(where $|a| \leq m$) with the spin of the test particle pointing along the axis in the same direction as the angular momentum of the black hole. We calculate the ratio, S/M, of spin to mass of the test particle required for the spin force repulsion to balance the gravitational attraction, so that the test particle remains at rest.

For the particle to remain at rest we must have

$$0 = \frac{dp^\mu}{ds} = \frac{Dp^\mu}{Ds} - \Gamma^\mu_{\alpha\beta} p^\alpha v^\beta . \tag{29}$$

Using Eq. (12), this yields

$$\tfrac{1}{2}R^{\mu}{}_{\nu\rho\sigma}v^{\nu}S^{\rho\sigma} = -M\Gamma^{\mu}_{\alpha\beta}v^{\alpha}v^{\beta}\,, \tag{30}$$

and a straightforward computation from the Kerr metric, Eq. (28), gives

$$\frac{S}{M}(r) = \frac{(r^2-a^2)(r^2+a^2)^{3/2}}{2a(3r^2-a^2)(r^2+a^2-2mr)^{1/2}}\,. \tag{31}$$

From Eq. (31) it is seen that [for $|a| \leq m$ and $r > r_+ = m + (m^2-a^2)^{1/2}$] the value of S/M needed to maintain equilibrium is so large that the minimum size of the body, given by Eq. (7), is of the order of magnitude or greater than the background curvature, so that the spinning test particle approximation should no longer be valid. Thus, in all cases where the spinning test particle approximation should apply, equilibrium cannot be maintained.

In the next section we examine Hawking's formula[6] for the maximum energy which can be extracted from the coalescence of two black holes for the case in which a small, slowly rotating black hole falls along the z axis into a large one, with the spins of the two black holes being either parallel or antiparallel to the direction of approach. We show that the dependence of this energy upper limit upon angular momentum is precisely accounted for by the effective spin interaction energy, as conjectured by Hawking.[6]

IV. SPIN INTERACTION ENERGY AND HAWKING'S ENERGY LIMITS

As described above (see Sec. I), Hawking has shown that if two rotating black holes coalesce to form a third one, the mass m_3 and angular momentum $J_3 = m_3 a_3$ of the final black hole is related to the mass and angular momenta of the initial black holes by the inequality[6]

$$m_3{}^2 + (m_3{}^4 - J_3{}^2)^{1/2} > m_1{}^2 + (m_1{}^4 - J_1{}^2)^{1/2} + m_2{}^2 + (m_2{}^4 - J_2{}^2)^{1/2}\,. \tag{32}$$

Equation (32) gives an upper limit, $E_{\max}$, given by

$$E_{\max} = m_1 + m_2 - (m_3)_{\min}\,, \tag{33}$$

for the amount of energy released via gravitational radiation in the coalescence process. In the case where the initial black holes have their rotation axes aligned parallel or antiparallel to their direction of approach, the configuration is axisymmetric and no angular momentum can be radiated away; hence, $J_3 = J_1 + J_2$. (In the nonaxisymmetric case, J_3 is not known, so one cannot solve for $E_{\max}$ in terms of m_1, J_1, m_2, and J_2 as required in the analysis given below.) We now consider the case where the first black hole is much smaller than the second ($m_1 \ll m_2$) and is slowly rotating ($J_1 \ll m_1{}^2$), so that its quadrupole and higher multipole moments are negligible. If we vary J_1, holding m_1, m_2, and J_2 fixed, we obtain from Eq. (32)

$$\frac{\partial E_{\max}}{\partial J_1} = -\frac{J}{2m[m^2+(m^4-J^2)^{1/2}]} + O(J_1)\,, \tag{34}$$

where, in this approximation, m and J denote the mass and angular momentum of either the second or the final black hole. Equation (34) suggests that there is an effective spin-interaction energy

$$\begin{aligned} E_s &= +\frac{J}{2m[m^2+(m^4-J^2)^{1/2}]}J_1 \\ &= \frac{a}{2m[m+(m^2-a^2)^{1/2}]}J_1 \end{aligned} \tag{35}$$

between the black holes; namely, Eq. (34) shows that, when the spins of the black holes are parallel, the upper limit in the amount of energy which can be released by gravitational radiation in the coalescence process is reduced by E_s over what it would be if J_1 were zero. Presumably, then, E_s must be the energy which is expended in doing work against the spin repulsive force, and for this reason the energy E_s is not available to be released via gravitational radiation. Similarly, when the spins are antiparallel, one gains the energy E_s from the spin interaction. *We now show that E_s is indeed the spin interaction energy.*

For a nonspinning (i.e., single-pole) particle in an arbitrary stationary space-time, the quantity $E = -p_0$ is the constant of the motion resulting from the time translation symmetry. (Here 0 denotes the direction of the timelike Killing vector.) E has the interpretation of the energy of the particle (as measured from infinity). For a spinning test body, however, E is not, in general, constant. We can interpret the change in E as the energy lost or gained by the particle due to its spin interaction. We now calculate the change in E of a spinning test particle falling along the symmetry axis into a Kerr black hole, with the spin of the particle aligned along the axis. From Eq. (12) we obtain

$$\begin{aligned} \frac{dE}{ds} &= -\frac{dp_0}{ds} \\ &= -\frac{Dp_0}{Ds} \\ &= \tfrac{1}{2}R_{0\nu\rho\sigma}v^{\nu}S^{\rho\sigma}\,. \end{aligned} \tag{36}$$

Evaluating the relevant Riemann tensor component for the Kerr metric, Eq. (28), we obtain

$$\frac{dE}{ds} = \frac{2maS(3r^2-a^2)}{(r^2+a^2)^3}\frac{dr}{ds}\,. \tag{37}$$

[Note that for $r \gg a$, Eq. (37) reduces to

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} , \quad (38)$$

where $\vec{F}$ is given by Eq. (24).] Integrating over the path of the particle from infinity to the horizon, $r = r_+ = m + (m^2 - a^2)^{1/2}$, we get

$$\begin{aligned} E_{\text{initial}} - E_{\text{final}} &= 2maS \int_{r_+}^{\infty} \frac{3r^2 - a^2}{(r^2+a^2)^3} dr \\ &= 2maS\left(-\frac{r}{(r^2+a^2)^2}\right)\Bigg|_{r_+}^{\infty} \\ &= \frac{a}{2m[m + (m^2-a^2)^{1/2}]} S . \end{aligned} \quad (39)$$

Equations (35) and (39) are identical. Thus, we see that spin interaction energy quantitatively accounts for the angular momentum dependence in Hawking's formula.

V. SPIN-ORBIT INTERACTION

The spin-orbit force and torque, i.e., the additional force and torque on a spinning test body due to its motion around a nonrotating source, may be evaluated in the same manner as for the spin-spin force and torque discussed in Sec. III. The spin-orbit precession has been calculated to lowest nonvanishing order in $\vec{v}$ and $1/r$ by Schiff[3] and Wilkins,[5] who obtained

$$\frac{d\vec{S}}{dt} = \frac{3m}{2r^3} (\vec{r} \times \vec{v}) \times \vec{S} , \quad (40)$$

where $\vec{S}$ denotes the spatial components of the spin vector S^μ in the gyroscope rest frame. The above precession rate includes the Thomas precession[21] effect. If one subtracts the Thomas precession,

$$\vec{\omega}_T = -\frac{1}{2}\frac{m}{r^3}(\vec{r} \times \vec{v}) , \quad (41)$$

one obtains the effective gravitational spin-orbit torque,

$$\vec{\tau}_G = \frac{2m}{r^3}(\vec{r} \times \vec{v}) \times \vec{S} . \quad (42)$$

By comparison, the corresponding electromagnetic spin-orbit torque (uncorrected for the Thomas precession) is

$$\begin{aligned} \vec{\tau}_E &= \vec{\mu} \times \vec{B} \\ &= -\vec{\mu} \times (\vec{v} \times \vec{E}) \\ &= -\frac{q}{r^3}(\vec{r} \times \vec{v}) \times \vec{\mu} . \end{aligned} \quad (43)$$

Thus, the gravitational spin-orbit torque, like the spin-spin force and torque, is -4 times the expression obtained from electromagnetism by the natural replacement $m \to q$ and $\vec{\mu} \to \frac{1}{2}\vec{S}$.[5] However, if one computes the gravitational spin-orbit force from Eq. (19) using the metric of Eq. (20) with $\vec{J} = 0$, one obtains to lowest order in $\vec{v}$ and $1/r$

$$\vec{F}_G = -\frac{3m}{r^3}\{\vec{v} \times \vec{S} + 2\hat{r}[\vec{v} \cdot (\hat{r} \times \vec{S})] - (\hat{r} \cdot \vec{v})(\hat{r} \times \vec{S})\} . \quad (44)$$

On the other hand, the corresponding electromagnetic spin-orbit force (computed from $\vec{F} = \vec{\mu} \cdot \vec{\nabla}\vec{B}$ and $\vec{B} = -\vec{v} \times \vec{E}$) is

$$\begin{aligned} \vec{F}_E &= \frac{q}{r^3}[-\vec{v} \times \vec{\mu} + 3(\vec{\mu} \cdot \hat{r})(\vec{v} \times \hat{r})] \\ &= \frac{q}{r^3}\{2\vec{v} \times \vec{\mu} + 3\hat{r}[\vec{v} \cdot (\hat{r} \times \vec{\mu})] - 3(\hat{r} \cdot \vec{v})(\hat{r} \times \vec{\mu})\} . \end{aligned} \quad (45)$$

Thus, the form of the gravitational spin-orbit force differs from that of the electromagnetic spin-orbit force.

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Investigation of the Paradox of Einstein, Podolsky, and Rosen for Ultrasmall Space-Time Intervals

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It is demonstrated that the Einstein-Podolsky-Rosen paradox (EPRP) can be investigated for ultrasmall space-time intervals by a reaction called "proximity correlation." One of the examples of proximity correlation analyzed in detail is the emission from a state $J=0$ of two spin-1 particles, A and B, in a relative orbital angular momentum state $L=1$, A and B decaying subsequently into spin-0 particles. It is shown that the quantity of interest in EPRP, $C(\vec{a},\vec{b})$, the probability of particle A having polarization $M_{S_A}=0$ with respect to $\vec{a}$, and particle B having polarization $M_{S_B}=0$ with respect to $\vec{b}$, is predicted according to quantum mechanics as being $C(\vec{a},\vec{b})\equiv C(\theta)\propto \sin^2\theta$. It is further shown that this relation can be tested experimentally by the decay of a $0^-(J^P)$ meson into two vector mesons. Experimental details are discussed.

INTRODUCTION

Einstein, Podolsky, and Rosen[1] gave an example of a hypothetical experiment capable of testing certain apparently paradoxical predictions of the current quantum theory. A simplified form of the experiment is the following.[2,3] Two spin-$\frac{1}{2}$ particles, A and B, are emitted from a $J_{AB}=0$ state, and have a relative orbital angular momentum zero. The state vector for the system with respect to the z axis is

$$|\psi\rangle=(2)^{-1/2}[\,|M_{S_A}=+\tfrac{1}{2}\rangle_z|M_{S_B}=-\tfrac{1}{2}\rangle_z - |M_{S_A}=-\tfrac{1}{2}\rangle_z|M_{S_B}=+\tfrac{1}{2}\rangle_z\,]\,. \tag{1}$$

Measuring the z component of the spin of particle A, when A and B are sufficiently separated so that they are noninteracting, we can conclude that the spin component of particle B is exactly opposite to that of particle A, since $J_{AB}=0$. Assume that particle A is found to have spin up along the z axis. The state vector (1) then becomes

$$|\psi\rangle \to |\psi\rangle=|M_{S_A}=+\tfrac{1}{2}\rangle_z|M_{S_B}=-\tfrac{1}{2}\rangle_z\,. \tag{2}$$

The state vector (2) describes a particle B with a definite spin component in the z direction, but with spin components, in the x and y directions randomly fluctuating. The J_{AB} of (2), in particular, can be zero or one. For example, the state vector for $J_{AB}=1$, $M_{J_{AB}}=0$ is

$$|\psi\rangle=(2)^{-1/2}[M_{S_A}=+\tfrac{1}{2}\rangle_z|M_{S_B}=-\tfrac{1}{2}\rangle_z + |M_{S_A}=-\tfrac{1}{2}\rangle_z|M_{S_B}=+\tfrac{1}{2}\rangle_z]\,. \tag{2$'$}$$

On the other hand, (1) describes a particle B with all its spin components correlated to those of particle A such that $J_{AB}=0$. The randomization of the x and y spin components of B in (2) was accomplished with no direct interaction with B.

The indeterminacy principle has been regarded as representing the disturbance of an observed system by a measuring apparatus. This interpretation leads to no difficulty for a single particle. For example, there is no problem according to this interpretation in explaining the random fluctuation of the x and y spin components of A in (2). In general this view implies that the definiteness of any desired component of spin, as well as the indefiniteness of the other two components, is a potentiality which is realized with the aid of a spin measuring apparatus. This interpretation does not explain how particle B realizes the potentiality of the indefiniteness of the x and y components of its spin without any apparatus having acted upon it.