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Lightlike Behavior of Particles in a Schwarzschild Field

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Above a transition velocity, $v_t \simeq c/\sqrt{2}$, test particles moving in a Schwarzschild field behave in a lightlike manner by appearing to slow down as they approach the source of the field.

I. INTRODUCTION

Slowly moving test particles in a Schwarzschild field speed up as they approach the source, whereas photons slow down. Above a transition velocity, however, particles appear to slow down and thus to behave in a lightlike fashion. Our primary purpose here is to quantify this statement and to derive an expression for the transition velocity.

II. MOTION OF PARTICLES AND LIGHT

The Schwarzschild metric, appropriate for the region exterior to a spherically symmetric mass distribution, is given in standard form ($c = G \equiv 1$) as

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1 - 2m/r} - r^2(d\phi^2 + \sin^2\phi\, d\theta^2), \tag{1}$$

where m is the gravitational radius of the body.

The geodesic equations for a test particle show the motion to be confined to a plane (chosen as $\phi = \frac{1}{2}\pi$) and yield[1]

$$\left(1 - \frac{2m}{r}\right)\frac{dt}{ds} = (1 - v_0^2)^{-1/2}, \tag{2}$$

where v_0 is the speed of the particle at spatial infinity. We are interested in the speed of a particle at any point, as measured by an observer far from the source. It is reasonable to define this speed in an invariant manner in terms of dl, an invariant infinitesimal element of spatial displacement,[2] and dt, a distant observer's proper time.[3] In the Schwarzschild field

$$dl^2 = \frac{dr^2}{1 - 2m/r} + r^2 d\theta^2, \tag{3}$$

whence, in terms of v_0, this invariant spatial speed is given by

$$\left(\frac{dl}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right) - \left(1 - \frac{2m}{r}\right)^2(1 - v_0^2)$$
$$= v_0^2 + \frac{2m}{r}(1 - 2v_0^2) - \left(\frac{2m}{r}\right)^2(1 - v_0^2). \tag{4}$$

This result can be given an operational validity, independent of any coordinate system, by consideration of a test particle whose speed is measured in terms of the echo times of light signals reflected by the particle and transmitted and received by an observer very far from the source.[4] Similarly, for light ($ds^2 = 0$), this invariant spatial speed is given by

$$\left(\frac{dl}{dt}\right)^2 = 1 - \frac{2m}{r}. \tag{5}$$

III. TRANSITION VELOCITY

For a particle with $v_0^2 < 0.5$, we see from Eq. (4) that $(dl/dt)^2$ increases with decreasing r, as we expect classically, until r reaches the transition position

$$r_t = \frac{4m(1 - v_0^2)}{(1 - 2v_0^2)} \underset{v_0^2 \ll 0.5}{\simeq} 4m(1 + v_0^2), \tag{6}$$

at which point $(dl/dt)^2$ begins to decrease with furthur decrease in r for $r \geq 2m$. The value of v_0 for which the transition will occur at a given value of r ($\geq 2m$) is

$$v_t^2 = \frac{1}{2}\left(\frac{r-4m}{r-2m}\right) \underset{r \gg 2m}{\simeq} \frac{1}{2}\left(1-\frac{2m}{r}\right). \quad (7)$$

For $v_0^2 > 0.5$, we see that $(dl/dt)^2$ decreases monotonically with decreasing r over the entire range from $r=\infty$ to $r=2m$. This behavior is lightlike since, for photons, Eq. (5) shows that $(dl/dt)^2$ also decreases monotonically over that range.

Thus, for any $r \gg 2m$, a test particle with $v_0 \geq c/\sqrt{2}$ will, when viewed from $r \gg m$, behave in a lightlike fashion by appearing to slow down as it nears the source.[5]

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[1]H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, 1968).

[2]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Addison-Wesley, Reading, Mass., 1962), p. 271ff.

[3]Note that dt is equivalent to proper time, and hence is invariant, when applied to a distant observer as here.

[4]The method is straightforward in principle. By inverting a sufficient number of the equations that express the measured echo times in terms of m, the constants of the motion, and t, we can obtain an explicit formula for the speed of a particle solely in terms of the readings on the distant oberver's clock. Such formulas could also easily be adapted, for example, to the Eddington-Robertson generalization of the Schwarzschild metric.

[5]An experimental verification of this prediction, whether involving high-energy elementary particles or other test bodies orbiting in the solar gravitational field, does not yet appear practical.

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Gravitational Spin Interaction*

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The gravitational spin interaction is investigated by studying the deviation from geodesic motion of spinning test bodies. The force on a spinning test body at rest in the exterior field of an arbitrary stationary, rotating source is evaluated and found to be given by

$$\vec{F} = -\vec{\nabla}\left(\frac{-\vec{S}\cdot\vec{J} + 3(\hat{r}\cdot\vec{J})(\hat{r}\cdot\vec{S})}{r^3}\right) + O(1/r^5),$$

where $\vec{S}$ is the spin of the test body, $\vec{J}$ is the angular momentum of the source of the gravitational field, and geometrized units $G=c=1$ are used. Thus, the gravitational spin-spin force has the same form as the force between two dipoles in electromagnetism except that its sign is opposite, i.e., "north pole" attracts "north pole" in gravitational spin-spin interaction. The gravitational spin-spin torque, previously investigated by Schiff, Mashhoon, and Wilkins, also has opposite sign to the corresponding electromagnetic dipole-dipole torque. The sign of the spin-spin force agrees with that predicted by Hawking on the basis of the fact that less energy can be extracted from colliding black holes if their spins are parallel rather than antiparallel. Furthermore, it is shown that the spin-interaction energy quantitatively accounts for the angular momentum dependence in Hawking's formula for the upper limit for energy released from colliding black holes. The gravitational spin-orbit force is also investigated, and it is found to differ in form from the corresponding electromagnetic spin-orbit force.

I. INTRODUCTION

Although the structure and interpretation of the theory of electromagnetism and Einstein's theory of gravitation are (at least at the present time) greatly different, the basic similarity of these two theories for the interaction of two bodies is far too familiar to be discussed at length: In lowest order, two slowly moving, nonrotating, massive bodies attract each other with force $-(m_1 m_2/r^2)\hat{r}$ (in geometrized units, $G=c=1$) while two slowly moving, nonrotating, charged bodies repel each other with force $(e_1 e_2/r^2)\hat{r}$. Of course, when higher-order corrections to the motion are taken