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Connected-Kernel Heitler Equations*

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The Heitler-type equations relating the three-particle transition and K matrices are replaced by equivalent connected-kernel equations. This constitutes the practical realization of the K-matrix formalism for the usual class of three-particle nonrelativistic scattering problems.

I. INTRODUCTION

Recently a realization of the three-body K-matrix formalism was developed in terms of Faddeev-type scattering integral equations.[1] In this formalism on-shell integral equations of the Heitler variety appear. However, these equations in their usual form do not possess kernels which become connected after iteration. This problem was handled satisfactorily for the particular application considered in detail in Ref. 1 and in the present work we will treat it in general. The result is a set of equations which are essentially on-shell connected-kernel Heitler equations and therefore may be regarded as the *practical* realization of the K-matrix formalism for the usual class of three-particle nonrelativistic scattering problems.

II. CONNECTED-KERNEL EQUATIONS

The three-particle scattering operators $\Gamma_{\beta\alpha}(z)$, which satisfy the integral equations

$$\Gamma(z) = \bar{t}(z) + \bar{t}(z)\bar{\delta}G_0(z)\Gamma(z) = \bar{t}(z) + \Gamma(z)G_0(z)\bar{\delta}\,\bar{t}(z), \tag{2.1}$$

were found to play a central role in the K-matrix formalism developed in Ref. 1. Our notation in this paper is precisely the same as that used in Ref. 1; however we will outline some aspects of it in order to make the present treatment reasonably self-contained.

In Eqs. (2.1) we have employed a matrix notation with respect to the channel indices $\alpha = 0, 1, 2, 3$. That is, $\Gamma(z)$ represents the 4×4 matrix whose elements are the operators $\Gamma_{\beta\alpha}(z)$, $\bar{t}(z)$ is a diagonal matrix whose elements are the two-particle transition operators with the discontinuity at the two-particle bound-state poles removed, $\bar{t}_\alpha(z)$, for $\alpha \neq 0$, and $\bar{t}_0(z) = 0$, and $\bar{\delta}$ is the matrix with elements $1 - \delta_{\beta\alpha}$. The $\bar{t}_\alpha(z)$ are, of course, two-particle operators defined on the three-particle space. The free three-particle propagator is denoted by $G_0(z)$, where z is the (complex) parametric energy. For $z = E \pm i\epsilon$, $\epsilon \to 0+$, we will replace (z) by $(\pm)$.

The crucial constraint upon the operators $\Gamma_{\beta\alpha}(z)$ is that on-shell, at least, they satisfy the discontinuity relations

$$\Gamma(+) - \Gamma(-) = -2i\Gamma(\pm)D_0(1+\bar{\delta})\Gamma(\mp), \tag{2.2}$$

where

$$D_0 = \pi\delta(E - H_0),$$

and H_0 is the free three-particle Hamiltonian. D_0 appears in the decomposition of $G_0(\pm)$ into its principal-value (G) and Dirac-δ-function parts:

$$G_0(\pm) = G \mp iD_0. \tag{2.3}$$

It was shown in Ref. 1 that any $\Gamma(z)$ satisfying Eqs. (2.2) on-shell will yield a set of unitary scattering amplitudes upon solving the so-called reduced K-matrix Heitler equations. The rationale, therefore, for constructing unitary theories for three-particle scattering is to find (exact) equations for $\Gamma(z)$ which permit the convenient construction of approximate solutions which satisfy (2.2) on-shell.

One such approach is to consider the following generalization of Eqs. (2.1):

$$\Gamma'(z) = \bar{t}(z) + \bar{t}(z)\bar{\delta}\,\mathcal{G}(z)\Gamma'(z) = \bar{t}(z) + \Gamma'(z)\bar{\delta}\,\mathcal{G}(z)\bar{t}(z), \tag{2.4}$$

where $\mathcal{G}(z)$ is such that

$$\mathcal{G}(+) - \mathcal{G}(-) = G_0(+) - G_0(-).$$

We suppose that $\mathcal{G}(z)$ has no matrix structure with respect to the channel indices. Obviously, $\Gamma'(z)$ satisfies Eqs. (2.2).

Let us write

$$\mathcal{G}(\pm) = B \mp iD_0,$$

where B is continuous across the unitary cut. The approximation $B = 0$ corresponds to the fully unitary impulse approximation considered in Ref. 1. For $B \neq 0$ we find that

$$\Gamma'(\pm) = \Gamma_0(\pm) + \Gamma_0(\pm)\bar{\delta}B\Gamma'(\pm)$$
$$= \Gamma_0(\pm) + \Gamma'(\pm)\bar{\delta}B\Gamma_0(\pm), \quad (2.5)$$

where

$$\Gamma_0(\pm) = \bar{t}(\pm) \mp i\bar{t}(\pm)\bar{\delta}D_0\Gamma_0(\pm)$$
$$= \bar{t}(\pm) \mp i\Gamma_0(\pm)\bar{\delta}D_0\bar{t}(\pm), \quad (2.6)$$

corresponds to the case $B=0$. It is easily verified that the off-shell matrix elements of $\Gamma_0(\pm)$ required in Eqs. (2.5) can be expressed entirely in terms of the on-shell elements since

$$\Gamma_0(\pm) = \bar{t}(\pm) \mp i\bar{t}(\pm)\bar{\delta}D_0\bar{t}(\pm)$$
$$-\bar{t}(\pm)\bar{\delta}D_0\Gamma_0(\pm)D_0\bar{\delta}\bar{t}(\pm).$$

It will be seen later that in some sense various approximations to the connected portion of the K-matrix correspond to approximating G by some other operator B. However, the K-matrix formalism of Ref. 1 is much more convenient for constructing well-defined unitary approximations than the preceding procedure appears to be.

If we introduce the two-particle K-matrices as solutions of

$$k = \bar{t}(\pm) \pm i\bar{t}(\pm)D_0k$$
$$= \bar{t}(\pm) \pm ikD_0\bar{t}(\pm), \quad (2.7)$$

we find using (2.3) that[1]

$$\Gamma(\pm) = \kappa \mp i\kappa D_0(1+\bar{\delta})\Gamma(\pm)$$
$$= \kappa \mp i\Gamma(\pm)D_0(1+\bar{\delta})\kappa, \quad (2.8)$$

where

$$\kappa = k + k\bar{\delta}G\kappa$$
$$= k + \kappa\bar{\delta}Gk. \quad (2.9)$$

The operators $\kappa_{\beta\alpha}$ are linearly related to the three-particle K matrices.

We observe that the kernels of Eqs. (2.8) do not become connected upon iteration. Thus, while Eqs. (2.8) are perfectly valid identities which imply the ordinary three-particle Heitler equations[1] they are not well-defined integral equations for determining $\Gamma(\pm)$ from a given κ. The disconnected structure arising from k is easily eliminated by summing these types of graphs resulting from k and its iterations to form two-particle transition matrices.

To do this we multiply the first of Eqs. (2.6) on the left by $[1 \mp i\bar{t}(\pm)D_0]$. Then if we decompose κ into its connected (κ^c) and disconnected (k) parts,

$$\kappa = k + \kappa^c,$$

we then obtain

$$\Gamma(\pm) = \bar{t}(\pm) + \kappa_L^c(\pm) \mp i\bar{t}(\pm)D_0\bar{\delta}\Gamma(\pm)$$
$$\mp i\kappa_L^c D_0(1+\bar{\delta})\Gamma(\pm), \quad (2.10a)$$

where

$$\kappa_L^c(\pm) \equiv [1 \mp i\bar{t}(\pm)D_0]\kappa^c$$

is a manifestly connected operator. Similarly, we obtain from the second of Eqs. (2.6)

$$\Gamma(\pm) = \bar{t}(\pm) + \kappa_R^c(\pm) \mp i\Gamma(\pm)\bar{\delta}D_0\bar{t}(\pm)$$
$$\mp i\Gamma(\pm)D_0(1+\bar{\delta})\kappa_R^c, \quad (2.10b)$$

where

$$\kappa_R^c(\pm) \equiv \kappa^c[1 \mp iD_0\bar{t}(\pm)]$$

is also a manifestly connected operator. Equations (2.10) can be derived directly from Eqs. (2.1) using Eqs. (2.7) and (2.9) without going through the intermediate step of the disconnected-kernel equations (2.8). In component form Eq. (2.10a), for example, becomes

$$\Gamma_{\beta\alpha}(\pm) = \bar{t}_\beta(\pm)\delta_{\beta\alpha} + [\kappa_L^c(\pm)]_{\beta\alpha}$$
$$\mp i\sum_\gamma\{\bar{t}_\beta(\pm)\bar{\delta}_{\beta\gamma} + \sum_\lambda[\kappa_L^c(\pm)]_{\beta\lambda}\}D_0\Gamma_{\gamma\alpha}(\pm) \quad (2.11)$$

which possesses a kernel which is manifestly connected after iteration. It follows from Eqs. (2.10) that for any κ^c such that

$$\kappa^{c\dagger} = \kappa^c, \quad (2.12)$$

the solution $\Gamma(\pm)$ of Eqs. (2.10) will satisfy the discontinuity Eqs. (2.2). It is assumed, of course, that the $\bar{t}(\pm)$ satisfy the correct two-particle discontinuity relations.[1]

Actually Eqs. (2.10) are not very convenient except for formal considerations. This is because the operators of physical interest are special linear combinations of the $\Gamma_{\beta\alpha}(+)$ and only these need be considered. A computational program based upon (2.10) for the on-shell matrix elements $\langle\phi_\beta|\Gamma_{\lambda\gamma}(+)|\phi_\alpha\rangle$ would generate superfluous information. Here the channel states $|\phi_\alpha\rangle$ for $\alpha = 1, 2, 3$ refer to a noninteracting two-particle state comprised of a particle α moving freely and a bound state of other two; $|\phi_0\rangle$ corresponds to a three-particle plane-wave state.

It is much more appropriate[1] to consider the operator

$$\xi(z) \equiv \bar{\delta}\Gamma(z)\bar{\delta}.$$

The on-shell matrix elements $\langle\phi_\beta|\xi_{\beta\alpha}(+)|\phi_\alpha\rangle$ are essentially the reduced K-matrix elements.[1] The on-shell Heitler integral equations relating the latter quantities to the physical scattering amplitudes are of the standard two-particle multichannel form and their solution may be regarded as trivi-

al.[1,2]

We find from Eqs. (2.10) that

$$\zeta_{\beta\alpha}(\pm) = \{\bar{\delta}[\bar{t}(\pm) + \kappa_L^c(\pm)]\bar{\delta}\}_{\beta\alpha} \mp i\sum_\gamma \{\bar{\delta}_{\beta\gamma}\bar{t}_\gamma(\pm) + [\bar{\delta}\kappa_L^c(\pm)\bar{\delta}]_{\beta\gamma}\delta_{\gamma 0}\} D_0\zeta_{\gamma\alpha}(\pm) \tag{2.13a}$$

and

$$\zeta_{\beta\alpha}(\pm) = \{\bar{\delta}[\bar{t}(\pm) + \kappa_R^c(\pm)]\bar{\delta}\}_{\beta\alpha} \mp i\sum_\gamma \zeta_{\beta\gamma}(\pm) D_0\{\bar{t}_\gamma(\pm)\bar{\delta}_{\gamma\alpha} + \delta_{\gamma 0}[\bar{\delta}\kappa_R^c(\pm)\bar{\delta}]_{\gamma\alpha}\}. \tag{2.13b}$$

One verifies that for any κ^c satisfying (2.12) that

$$\zeta_{\beta\alpha}(+) - \zeta_{\beta\alpha}(-) = -2i\zeta_{\beta 0}(\pm) D_0\zeta_{0\alpha}(\mp). \tag{2.14}$$

Obviously (2.14) follows from (2.2) and conversely. Equations (2.13) are our connected-kernel Heitler equations. They constitute along with the reduced K-matrix formalism of Ref. 1 the practical form of the complete K-matrix formalism for the nonrelativistic three-particle scattering problem. They also represent the solution of the disconnected-kernel problem in this particular case of the multiparticle Heitler equations.

It is interesting to note that Eqs. (2.13) can be written explicitly in terms of the connected part of the three-particle K matrix since[1]

$$K_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0(+)^{-1} + (\bar{\delta}\kappa\bar{\delta})_{\beta\alpha}$$

so that

$$K_{\beta\alpha}^c = (\bar{\delta}\kappa^c\bar{\delta})_{\beta\alpha}.$$

But

$$\kappa_L^c(\pm) = [1 \mp i\bar{t}(\pm)D_0]\kappa^c$$

so

$$\bar{\delta}\kappa_L^c(\pm)\bar{\delta} = K^c \mp i[\bar{\delta}\bar{t}(\pm)\bar{\delta}^{-1}] D_0 K^c,$$

where K^c denotes the matrix $(K_{\beta\alpha}^c)$. Similarly,

$$\bar{\delta}\kappa_R^c(\pm)\bar{\delta} = K^c \mp i K^c D_0[\bar{\delta}^{-1}\bar{t}(\pm)\bar{\delta}].$$

We note that in four dimensions

$$\bar{\delta}^{-1} = \tfrac{1}{3}(\bar{\delta} - 2I)$$

and that K^c satisfies

$$K^c = \bar{\delta}k\bar{\delta}Gk\bar{\delta} + \bar{\delta}kGK^c.$$

Now given any connected Hermitian K^c, which implies that

$$\langle\phi_\beta|K_{\beta\alpha}^c|\phi_\alpha\rangle = \langle\phi_\alpha|K_{\alpha\beta}^c|\phi_\beta\rangle^*,$$

we generate via Eqs. (2.13) a unitary theory. We will next outline how one goes about solving this set of on-shell equations. The special case of $K^c = 0$ was considered in Ref. 1.

Equation (2.13a) has the general form (in the + case)

$$\zeta_{\beta\alpha}(+) = A_{\beta\alpha}(+) - i\sum_\gamma R_{\beta\gamma}(+) D_0\zeta_{\gamma\alpha}(+), \tag{2.15}$$

where the definitions of A and R are obvious from Eq. (2.13a). Let us denote the channel states of energy E by $|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle$, where η_α refers to any other labels needed to specify the state. The matrix elements one wants to determine are of the form $\langle\phi_\beta(\eta_\beta, E)|\zeta_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle$. It is evident that for $\beta \neq 0$ these quantities can be found by quadrature from (2.15) once one has computed the matrix elements $\langle\phi_0(\eta_0, E)|\zeta_{\gamma\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle$ since

$$\langle\phi_\beta(\eta_\beta, E)|\zeta_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle = \langle\phi_\beta(\eta_\beta, E)|A_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle - i\pi\sum_\gamma \sum_{E',\eta'} \langle\phi_\beta(\eta_\beta, E)|R_{\beta\gamma}(+)|\phi_0(\eta', E)\rangle\!\rangle \delta(E'-E)\langle\phi_0(\eta', E')|\zeta_{\gamma\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle. \tag{2.16}$$

The computation of the matrix elements $\langle\phi_0(\eta_0, E)|\zeta_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle$ constitute the principal numerical difficulty of the entire procedure. From Eq. (2.15) we see that these matrix elements are to be determined as the solutions of the integral equations

$$\langle\phi_0(\eta_0, E)|\zeta_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle = \langle\phi_0(\eta_0, E)|A_{\beta\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle - i\pi\sum_\gamma \sum_{E',\eta'} \langle\phi_0(\eta_0, E)|R_{\beta\gamma}(+)|\phi_0(\eta', E')\rangle\!\rangle \delta(E'-E)\langle\phi_0(\eta', E')|\zeta_{\gamma\alpha}(+)|\phi_\alpha(\eta_\alpha, E)\rangle\!\rangle. \tag{2.17}$$

The (finite) range of the independent kinematic variables in these equations is over the three-body phase space appropriate to a total energy E. For $E < 0$, that is below the three-body threshold, Eq. (2.15) becomes trivial and (2.17) nonexistent. Cahill[3] has shown that an on-shell equation of the form (2.17) reduces to a one-dimensional integral equation with a finite domain of integration after a partial-wave decomposition.

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Connection Between Electroproduction and Annihilation

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It is shown why the ladder-graph model is special in allowing analytic continuation of the deep-inelastic electroproduction structure functions to describe e^+e^- annihilation. It is also explained why this property cannot be expected to hold for more complicated graphs that also contribute in the Bjorken limit.

Drell, Levy, and Yan[1] have considered deep-inelastic electroproduction and e^+e^- annihilation, where one final-state hadron is detected, in perturbation theory with a cutoff, and have shown that for ladder graphs the structure functions for the two processes are related by analytic continuations. It is known[2,3] that this is not the case for more complicated Feynman graphs that also contribute in the Bjorken limit, and the main purpose of this note is to explain why the ladder graphs are special in this respect. Thus one cannot expect that the complete structure functions are related by analytic continuation.[4]

It is, of course, easy to see that the continuation is not likely to be possible[1,4,5] without considering any specific model such as Feynman graphs. For either process the amplitude involved is the forward virtual Compton amplitude. Let the momentum of the virtual photon be q. Let that of the target hadron in electroproduction be p, and that of the detected hadron in the annihilation be $-p$. Define

$$\nu=\frac{p\cdot q}{M}\ ,\quad \omega=-\frac{2M\nu}{q^2}\ . \tag{1}$$

In the physical region for electroproduction

$$q^2<0\ ,\quad \nu>0\ ,\quad \omega>1\ , \tag{2}$$

and in that for annihilation

$$q^2>0\ ,\quad \nu<0\ ,\quad 0<\omega<1\ . \tag{3}$$

The trouble arises because of the cut that is expected to appear in the variable q^2 along the positive real axis; evidently the continuation from (2) to (3) will encounter this cut. Because the annihilation cross section is a sum-integral of the squared *moduli* of the amplitudes corresponding to the various possible final states, the correct prescription for calculating its structure function is as follows. Calculate the structure function in region (2) as in electroproduction, but with different squared momenta, q_1^2 and q_2^2, assigned to the two virtual photons. Continue q_1^2 from negative to positive values, passing above the branch points in that variable. Do the same for q_2^2 but pass below the branch points. Then make q_1^2 and q_2^2 numerically equal to q^2. This separation of the two variables is an essential feature, and the same effect cannot normally be achieved by analytic continuation in the single variable q^2. In terms of the variable ω, if $\omega_1=-2M\nu/q_1^2$ and $\omega_2=-2M\nu/q_2^2$, this prescription reads as follows: First calculate $F(\omega_1,\omega_2)$, where $F(\omega,\omega)$ is the electroproduction structure function $F(\omega)$. Then the annihilation structure function $\bar{F}(\omega)$ is $F(\omega+i\epsilon,\omega-i\epsilon)$. From this one can see that a necessary condition that $\bar{F}(\omega)$ is an analytic continuation of $F(\omega)$ is that $F(\omega)$ has no branch point at $\omega=1$. However, this is *not* a sufficient condition; a counterexample is provided by

$$F(\omega_1,\omega_2)=(\omega_1-1)^{1/2}(\omega_2-1)^{1/2}\ . \tag{4}$$

Thus by examining the properties of $F(\omega)$ alone one can never be sure that the continuation is possible, though it is perhaps unlikely that a situation such as (4) will arise.

Turn now to the Feynman-graph model, with a cutoff to give Bjorken scaling. The graphs that contribute[2,6] in the Bjorken limit are all those having the structure of Fig. 1, where the internal line represents the field to which the photons couple, that is, the parton field. T is any subgraph, whose amplitude depends upon k^2 and $s'=(p-k)^2$ in the integration over the loop momentum k. In the Bjorken limit the electroproduction structure function $F_2(\omega)$