Note also that σ, τ, and μ are related by

$$\tau^+F^+\tau^-F^- = \frac{i\lambda}{4\pi\xi^+\xi^-}\left(\sigma^+ - \sigma^- - \frac{(\tau^+ - \tau^-)^2}{\mu^+ - \mu^-}\right).$$

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[16]See, e.g., C. A. Nelson, Ref. 10.

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Quantum Field Theories with Shadow States. II. Low-Energy Pion-Nucleon Scattering

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By adapting the methods of the preceding paper, we use the concepts of shadow states and indefinite metric to construct a simple static theory of low-energy pion-nucleon scattering. This theory of s- and p-wave scattering so constructed is both finite and exactly soluble. Scattering at negative energy also is a well-defined process. The calculated scattering amplitude is found to satisfy the substitution law and to be covariant, unitary, and analytic in almost all neighborhoods of physical scattering energies. When given the masses and coupling constants as input, this theory predicts the scattering phase shifts in agreement with experiment. The present limits on the s-wave pion-nucleon scattering total cross sections are compatible with the induced cusps from the opening up of the pseudothreshold of the shadow states.

I. INTRODUCTION

The unfinished quantum field theory of low-energy pion-nucleon scattering has had a particularly long history. After all, it was over three decades ago that the meson theory of nuclear forces with Yukawa couplings was created by analogy with electrodynamics[1] and it was as early as 1942 when, on the basis of strong-coupling theory, the suggestion was first made that an isospin-$\frac{3}{2}$ resonance might exist.[2] In spite of many different theoretical attempts since these earliest beginnings, the basic challenge has remained: Construction of a *convergent, divergence-free* theory which when solved exactly predicts the observed experimental parameters of low-energy pion-nucleon scattering approximately. A true quantum field theory for this physical phenomena has not been constructed. As discussed in the preceding paper,[3] the concepts of indefinite metric[4] and shadow states[5] are valuable tools in the construction of a finite relativistic quantum field theory, and therefore we wish to employ them here in our consideration of low-energy pion-nucleon scattering. Before introducing these ideas it is useful to review briefly some relevant aspects of theoretical approaches to this problem in the past.

In the early fifties[6] in the quantum-field-theory approach to this problem a fundamental question concerned the proper field-theoretic interaction to be used. The pseudoscalar interaction was generally preferred over the pseudovector interaction because the pseudoscalar interaction can be made renormalizable by adding a meson-meson interaction to it.[7] A problem still remained though, as to how to carry out calculations when the coupling constant is large. Another aspect in the choice between these couplings was that in lowest-order perturbation theory both types of interaction were found to yield identical results[8] provided the coupling constants satisfied the relation $G/2M = f/\mu$; but then both were wrong in predicting that s-wave scattering should dominate pion-nucleon scattering at low energies.[9] Yet about the same time it was recognized from the analysis of nucleon-nucleon interactions that some additional s-wave interaction was needed, specifically, a term

$$\int d\vec{r}\, \bar{\psi}\psi\phi^2 \qquad (1.1)$$

corresponding to a short-range repulsive force.[9,10]

Discussion concerning the correct coupling continued, for while the pseudoscalar and pseudovector interactions are in some ways equivalent, they were found to be quite different for the description of virtual processes in the static limit: For example, the pseudovector coupling of pion and nucleon fields leads in this limit to a $(\vec{\sigma}\cdot\vec{\nabla})\phi$ coupling corresponding to p-wave emission of pions as had been observed at moderate energies. Instead, the pseudoscalar coupling directly described the annihilation of a nucleon-antinucleon pair and the creation of an s-wave pion, and although the Tamm-Dancoff approximation, among other methods, was used extensively to study higher-order processes for the pseudoscalar coupling, the dynamical description remained incorrect. As is well known, a description for p-wave scattering which had some contact with experiment was finally provided by the Chew-Low theory[11] which was based on the unrenormalizable $(\vec{\sigma}\cdot\vec{\nabla})\phi$ interaction. It not only predicted the (3, 3) resonance but also was the first serious attempt towards an extensive understanding of nuclear forces, photoproduction, the static properties of nucleons, as well as low-energy pion-nucleon scattering. In the one-meson approximation the solution was nonunique,[12] approximately obtained, and cutoff-dependent.

In this paper we wish to rekindle the theoretical search for a finite field-theoretic description of low-energy pion-nucleon scattering. To do this we shall adopt some of the methods of I, in particular, indefinite metric and shadow states in order that the static theory so obtained is finite with scattering at negative energies a well-defined process. In accordance with the above discussion concerning the choice of static interactions, for s-wave scattering we use the $\vec{\phi}\cdot\vec{\phi}$ coupling and for p-wave scattering the $(\vec{\sigma}\cdot\vec{\nabla})\vec{\tau}\cdot\vec{\phi}$ coupling. To make the theory soluble in closed form, the coupling to mesons is handled in such a manner that only one- and no-meson states are considered, i.e., we make the one-meson approximation. The substitution law is still valid in this theory because we follow the quantization procedure discussed in I for the coupling of a Klein-Gordon field to a source. That is, in order that the scattering amplitude is defined for both positive and negative energies, we will associate the entire meson field of both positive and negative frequencies with annihilation operators corresponding to quanta with these respective energies. Such a formulation of quantum field theory has been given previously[13] and has been very useful in both fundamental investigations in field theory and in the quantization of tachyon and infinite-component fields. By this method of quantization, scattering at negative energies is a well-defined process, and we achieve in our approach a simple one-meson approximation which satisfies the substitution law with the scattering amplitude having a left-hand cut.

Because of this the pole terms, say for low-energy p-wave pion-nucleon scattering, are of two distinct types: On the one hand there are the conventional direct-channel poles having positive metric. But on the other hand, the exchange contribution which corresponds to a resonance for the negative-energy mesons is described by a second set of poles which may or may not have a negative metric. In the case of p-wave scattering the resulting contribution of these poles in the Born limit is the same as making the nucleon-pole approximation in lowest-order perturbation theory.

The organization of this paper is as follows: In the next section, we study s-wave scattering using a direct-channel coupling of the second type to represent the exchange in the crossed channel. Comparison is made with experiment. In Sec. III the direct and exchange contributions to p-wave scattering are given, and a number of properties of the complete amplitude are examined. In Sec. IV we show that the amplitude satisfies the substitution law and study its analytic properties. Section V then continues with a comparison of our exact p-wave results with experiment. The paper concludes in Sec. VI with some general remarks and a discussion concerning relativistic modification. We also examine the effects on the s-wave total cross sections due to the pseudothreshold of the shadow state.

II. s-WAVE SCATTERING

The interaction we take to describe s-wave scattering is the two-meson-exchange term which was mentioned in the Introduction. It is needed to give a spin-independent repulsive-core contribution to nuclear forces and provides a direct-channel description of ρ-meson dominance in the crossed channel $\pi\pi\to\bar{N}N$. During the past several years, current-algebra calculations[14] have supported the idea that the universal coupling of the ρ meson to pions and nucleons[15] is responsible for the s-wave pion-nucleon scattering lengths being so small and of opposite signs. We shall accordingly treat s-wave scattering by means of a four-point-interaction model similar to that constructed in Sec. VI of I.

The static model is defined by the Hamiltonian

$$\mathsf{H} = H + H_{\text{shadow}}, \tag{2.1}$$

where $H = H_0 + H_I$ and

$$H_0=\sum_\alpha \int d\vec{x}\tfrac{1}{2}[\dot{\phi}_\alpha^\dagger(x)\dot{\phi}_\alpha(x)+\vec{\nabla}\phi_\alpha^\dagger(x)\cdot\vec{\nabla}\phi_\alpha(x)+\mu^2\phi_\alpha^\dagger(x)\phi_\alpha(x)], \tag{2.2}$$

$$H_I=g_1^{\,2}\int d\vec{x}'d\vec{x}\,\rho(|\vec{x}'|)\phi_{\alpha'}^\dagger(x')\tfrac{1}{3}\tau_{\alpha'}\tau_\alpha\phi_\alpha(x)\rho(|\vec{x}|)+g_3^{\,2}\int d\vec{x}'d\vec{x}\,\rho(|\vec{x}'|)\phi_{\alpha'}^\dagger(x')(\delta_{\alpha'\alpha}-\tfrac{1}{3}\tau_{\alpha'}\tau_\alpha)\phi_\alpha(x)\rho(|\vec{x}|). \tag{2.3}$$

The meson fields ϕ_α and $\phi_{\alpha'}^\dagger$ are given by

$$\phi_\alpha(x)=\int\frac{d\vec{k}}{(2\pi)^{3/2}(2\omega)^{1/2}}\,[a_\alpha(+,\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}+a_\alpha(-,-\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}], \tag{2.4}$$

$$\phi_\alpha^\dagger(x)=\int\frac{d\vec{k}}{(2\pi)^{3/2}(2\omega)^{1/2}}\,[a_\alpha^\dagger(+,\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}+a_\alpha^\dagger(-,-\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}], \tag{2.5}$$

with α the meson-field isospin index. The commutation relations are

$$[a_\alpha(\pm,\vec{k}),a_{\alpha'}^\dagger(\pm,\vec{k}')]=\epsilon(w)\delta(\vec{k}-\vec{k}')\delta_{\alpha\alpha'},$$
$$[a_\alpha(\pm,\vec{k}),a_{\alpha'}(\pm,\vec{k}')]=[a_\alpha(\pm,\vec{k}),a_{\alpha'}^\dagger(\mp,\vec{k}')]=0, \tag{2.6}$$

with

$$w(k)=\epsilon(w)\omega(k),$$
$$\omega(k)=(\mu^2+\vec{k}^2)^{1/2},\quad \epsilon(w)=\pm1\ \text{ as }\ w\gtrless 0. \tag{2.7}$$

Notice that the entire meson field with both positive and negative frequencies is associated with corresponding annihilation operators. This method of quantization for the above Hamiltonian yields a one-meson approximation having a left-hand cut. In the interaction Hamiltonian the nucleon fields are suppressed since we shall be concerned only with states which always contain one nucleon. Since this is a static theory, we may set the nucleon energies equal to zero. Because of Eq. (2.6) the physical space is an indefinite-metric space.

The interaction Hamiltonian is used to simulate ρ dominance in the crossed channel by means of two direct-channel couplings. It is seen from H_I that $g_1^{\,2}$ is the coupling constant for the s-wave pion-nucleon scattering states in a pure $I=\frac{1}{2}$ state and that likewise $g_3^{\,2}$ is that for a pure $I=\frac{3}{2}$ state. These coupling constants will be determined in a moment by the condition that the s-wave scattering lengths have the ρ-dominance values[15]

$$a_{1/2}=\frac{2f_\rho^{\,2}}{4\pi m_\rho^{\,2}}\,\frac{\mu}{1+\mu/m_n},$$
$$a_{3/2}=-\tfrac{1}{2}a_{1/2}. \tag{2.8}$$

Up until now, we have not introduced the concept of shadow states. We do this by means of a set of continuum channels which we take to be coupled in the same manner as the physical pion-nucleon scattering channels introduced above. To be precise,

$$H_{\text{shadow}}=H_0(\phi_s^\dagger,\phi_s)+H_I(\phi_s^\dagger,\phi_s) \tag{2.9}$$

with H_0 and H_I given by (2.2) and (2.3) where the shadow-meson fields $\phi_s^\dagger$ and ϕ_s are expressed in terms of shadow creation and annihilation operators which satisfy

$$[b_\alpha(\pm,\vec{k}),b_{\alpha'}^\dagger(\pm,\vec{k}')]=-\epsilon(w)\delta(\vec{k}-\vec{k}')\delta_{\alpha\alpha'},$$
$$[b_\alpha(\pm,\vec{k}),b_{\alpha'}(\pm,\vec{k}')]=[b_\alpha(\pm,\vec{k}),b_{\alpha'}^\dagger(\mp,\vec{k}')]=0, \tag{2.10}$$

with

$$w(k)=\epsilon(w)\omega_s(k),$$
$$\omega_s(k)=(\mu_s^{\,2}+\vec{k}^2)^{1/2},\quad \epsilon(w)=\pm1\ \text{ as }\ w\gtrless 0. \tag{2.11}$$

First notice that we have chosen in (2.10) an opposite metric for the shadow channels from that taken above for the pion-nucleon channels. A second essential difference from the normal channels is made in the choice of boundary conditions for the shadow states: For these states there are no running waves, but only standing waves. This prescription applies to the entire shadow state and not merely to the meson alone in that state. The consequences of having such states are carefully discussed and examined in I via a number of multichannel soluble models.

The scattering amplitude is then calculated by following the standard procedure, say as given in I. The "in" scattering states are determined by a state with a plane wave and outgoing waves in the normal channels, and standing waves (principal-value Green's functions) in the shadow channels. The striking result is that the T matrix is nonvanishing only for the physical channels. We find

$$T_{1,3}=e^{i\delta_{1,3}}\sin\delta_{1,3}, \tag{2.12}$$

where

$$\cot\delta_{1,3}=\pm\frac{12\pi wI(w)}{g_{1,3}^{\;2}k} \tag{2.13}$$

with

$$\mathrm{Re}I_{1,3}(w) = 1 \pm \frac{w}{6\pi^2}\, \mathcal{P}\int k^2 dk \left(\frac{\epsilon(\xi)(g^r_{1,3})^2}{2\xi^3(\xi - w)} - \frac{\epsilon(\xi')(g^r_{1,3})^2}{2\xi'^3(\xi' - w)}\right), \tag{2.14}$$

where $\xi = (k^2+\mu^2)^{1/2}$, $\xi' = (k^2+\mu_s^2)^{1/2}$. The subscripts "1" and "3" denote, respectively, $I=\frac{1}{2}$ and $\frac{3}{2}$, and in Eqs. (2.13) and (2.14) the plus (minus) sign is to be taken for $I=\frac{1}{2}$ ($I=\frac{3}{2}$). The Fourier transform, $\rho(k)$, of the s-wave form factor, $\rho(|\vec{x}|)$, has been set equal to one for all k since a cutoff is not necessary. Notice that the scattering amplitudes are given in terms of the renormalized coupling constants and that the direct-channel renormalized poles are at $\lambda = 0$ in the static limit.

The s-wave pion-nucleon phase shifts then can be written in the effective-range form

$$\frac{(g^r_{1,3})^2 k \cot\delta_{1,3}}{12\pi w} = 1 - w r_{1,3}(w)\,, \tag{2.15}$$

with

$$r_{1,3}(w) = \pm\frac{1}{12\pi^2}\, \mathcal{P}\int k^2 dk \left(\frac{\epsilon(\xi)(g^r_{1,3})^2}{\xi^3(\xi - w)} - \frac{\epsilon(\xi')(g^r_{1,3})^2}{\xi'^3(\xi' - w)}\right). \tag{2.16}$$

In the effective-range approximation the effective ranges are especially simple; they are

$$\begin{aligned} r_{1,3}(w=0) &= \pm\frac{1}{12\pi^2}\, \mathcal{P}\int k^2 dk \left(\frac{\epsilon(\xi)(g^r_{1,3})^2}{\xi^4} - \frac{\epsilon(\xi')(g^r_{1,3})^2}{\xi'^4}\right) \\ &= \pm\frac{(g^r_{1,3})^2}{12\pi}\left(\frac{\mu_s - \mu}{\mu_s\mu}\right). \end{aligned} \tag{2.17}$$

However, since Eq. (2.14) can be evaluated explicitly the phase shifts can be calculated exactly. We find

$$r_{1,3}(w) = \pm\frac{(g^r_{1,3})^2}{6\pi}\left[-\frac{\mu_s-\mu}{w^2} - \frac{2}{\pi}\frac{(w^2-\mu^2)^{1/2}}{w^2}\ \ln\left(\frac{w+(w^2-\mu^2)^{1/2}}{\mu}\right) + \frac{(\mu_s^{\,2}-w^2)^{1/2}}{w^2}\left(1+\frac{2}{\pi}\sin^{-1}\frac{w}{\mu_s}\right)\right] \tag{2.18}$$

for $\mu < w < \mu_s$, i.e., for physical pion-nucleon scattering below the pseudothreshold for the shadow channels. In the concluding section of this paper, the behavior of $r_{1,3}(w)$ near the shadow-meson pseudothreshold is carefully examined and discussed. To determine $(g^r_{1,3})^2$ we define the s-wave scattering lengths by

$$k\cot\delta_{1,3} = \frac{1}{a_{1,3}} + O(k^2)\,. \tag{2.19}$$

So by Eq. (2.15) we find

$$a_{1,3} = \frac{(g^r_{1,3})^2}{12\pi\mu[1-\mu r_{1,3}(\omega=\mu)]},$$

with $r_{1,3}(w=\mu)$ given by Eq. (2.18). Thus, by Eq. (2.8) $(g^r_{1,3})^2$ can be expressed in terms of $f_\rho^{\,2}$.

Using the exact result and not the effective-range approximation, we now confront Eq. (2.15) with experiment. To take some account of nucleon recoil, we take

$$w \to w^* = w + k^2/2m_n$$

in the center-of-mass system where m_n is the nucleon mass. Then for a ρ(755 MeV) width of 145 MeV as input[16] we find the scattering lengths to be

$$\begin{aligned} a_1 &= 0.166(1/\mu)\ \ [(0.176\pm0.009)(1/\mu)\text{ exper.}], \\ a_3 &= -0.083(1/\mu)\ \ [(-0.101\pm0.009)(1/\mu)\text{ exper.}] \end{aligned} \tag{2.20}$$

as compared with experiment.[17] We have taken μ_s

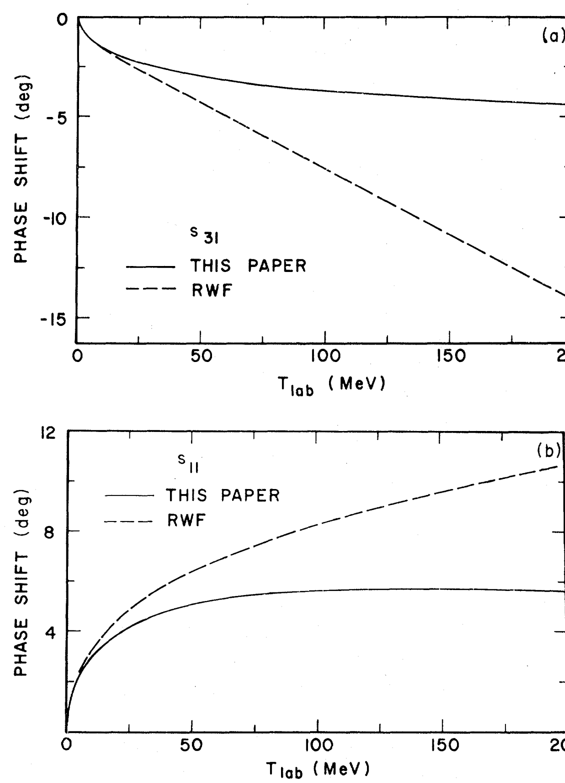

FIG. 1. s-wave phase shifts for pion-nucleon scattering. The empirical values are the best solution in the 0–350-MeV range from Ref. 18.

as determined by the threshold for the (3, 3) resonance in Sec. IV of this paper. In Fig. 1 the phase shifts are plotted as given exactly by Eq. (2.15) with respect to the "best solution" of Roper, Wright, and Feld.[18] There is agreement, and it is better at the lower energies as one expects since this is a static model. It is perhaps also interesting that for the lower energies there is little change as far as the phase shifts are concerned if one follows the same procedure and uses the effective-range approximation instead of the exact solution. In that case the effective ranges are constant, being

$$r_1 = 0.058(1/\mu)\,,\qquad r_3 = -0.032(1/\mu)\,, \tag{2.21}$$

whereas for the exact solution the corresponding effective-range functions are monotonic, averaging to about these values over the lower energies but changing sign at about 140 MeV.

III. p-WAVE SCATTERING

In quantum field theory in the perturbative approach to p-wave pion-nucleon scattering there are both direct and exchange contributions in the static limit from the $(\vec{\sigma}\cdot\vec{\nabla})\vec{\tau}\cdot\vec{\phi}$ interaction. In our approach, as discussed in the Introduction, the exchange contribution is described by an equivalent set of poles in the direct channel. In the case of p-wave scattering in the static limit the direct and exchange $I=\frac{1}{2}$, $J=\frac{1}{2}$ poles are treated as one. This is possible because in this limit they have the same energies.

Thus, the static model Hamiltonian is defined by

$$\mathsf{H} = H + H_{\text{shadow}}\,, \tag{3.1}$$

where $H = H_{\text{meson}} + H_I$ and the interaction Hamiltonian is of Yukawa type

$$\begin{aligned} H_I = \frac{2\sqrt{2}}{3}\,\frac{f_0}{\mu} \int d\vec{x}\rho(\vec{x}) N^\dagger (\vec{\sigma}\cdot\vec{\nabla})\vec{\tau}\cdot\vec{\phi}N \\ + \frac{2}{3}\,\frac{f_0}{\mu}\int d\vec{x}\rho(\vec{x})[N'^\dagger s_i\,\vec{\tau}\cdot\partial_i\vec{\phi}N + \Delta^\dagger t_\alpha(\vec{\sigma}\cdot\vec{\nabla})\phi_\alpha N] \\ + \frac{2}{3}\,\frac{f_0}{\mu}\int d\vec{x}\rho(\vec{x})\Delta'^\dagger s_i t_\alpha \partial_i\phi_\alpha N + \text{H.c.}\,, \end{aligned} \tag{3.2}$$

where N, N', Δ, and Δ' are (1, 1), (1, 3), (3, 1), and (3, 3) states, respectively, in the notation $(2I, 2J)$. The matrices s_i and t_α, labeled by spinor (isospinor) and vector (isovector) indices in the nucleon and pion spaces, are the generalizations of σ_i and τ_i, respectively. For example, the nonrelativistic s_i couples the $J=\frac{1}{2}$ nucleon to a p-wave pion to form an object which transforms as a $J=\frac{3}{2}$ state in the N' spin space. Notice that if there were only the direct-channel contribution, there would only be the first term with $\frac{2}{3}\sqrt{2}$ replaced by unity.

The meson fields ϕ_α and $\phi_\alpha^\dagger$ are again quantized in such a manner that scattering at negative energy will be a well-defined process. They are given by

$$\phi_\alpha(x) = \int \frac{d\vec{k}}{(2\pi)^{3/2}(2\omega)^{1/2}}[a_\alpha(+,\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + a_\alpha(-,-\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)}]\,, \tag{3.3}$$

$$\phi_\alpha^\dagger(x) = \int \frac{d\vec{k}}{(2\pi)^{3/2}(2\omega)^{1/2}}[a_\alpha^\dagger(+,\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega t)} + a_\alpha^\dagger(-,-\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}]\,. \tag{3.4}$$

Notice that $a_\alpha(+,\vec{k})$ is the annihilation operator for a positive-energy meson having isospin α and will be used to define the physical scattering amplitude for scattering of positive-energy mesons. The corresponding positive-energy creation operator is $a_\alpha^\dagger(+,\vec{k})$, which occurs in the field $\phi_\alpha^\dagger(x)$. The latter field, $\phi_\alpha^\dagger(x)$, appears in the "H.c." part of the Hamiltonian in Eq. (3.2). If only these positive-energy parts were included, the standard soluble static theory having only direct-channel contributions from positive-energy poles would be the result. The negative-energy parts of the fields, i.e., the contributions from $a_\alpha(-,-\vec{k})$ and $a_\alpha^\dagger(-,-\vec{k})$, are a direct consequence of the quantization method[13] adopted here. These components are essential for the inclusion of the exchange contribution in p-wave scattering and allow us to give content to the "substitution law" as applied to scattering processes. This aspect is discussed in the next section. The commutation relations for the meson creation and annihilation operators are the same as for the s-wave case,

$$\begin{aligned} [a_\alpha(\pm,\vec{k}), a_{\alpha'}^\dagger(\pm,\vec{k}')] &= \epsilon(w)\delta(\vec{k}-\vec{k}')\delta_{\alpha\alpha'}\,, \\ [a_\alpha(\pm,\vec{k}), a_{\alpha'}(\pm,\vec{k}')] &= [a_\alpha(\pm,\vec{k}), a_{\alpha'}^\dagger(\mp,\vec{k}')] \\ &= 0\,, \end{aligned} \tag{3.5}$$

with

$$w(k) = \epsilon(w)\omega(k)\,,\qquad \omega(k) = (\mu^2 + \vec{k}^2)^{1/2}\,,\quad \epsilon(w) = \pm 1 \text{ as } w \gtrless 0\,. \tag{3.6}$$

The nucleonlike fields satisfy the commutation relations

$$[N, N^\dagger] = [N_i, N_i^\dagger] = [\Delta, \Delta^\dagger] = -[\Delta_i, \Delta_i^\dagger] = 1\,, \tag{3.7}$$

and all others vanish.

The shadow part of the Hamiltonian is given by

$$H_{\text{shadow}} = H_{\text{meson}}(\phi_s^\dagger, \phi_s) + H_I(\phi_s^\dagger, \phi_s) \tag{3.8}$$

with the shadow fields quantized as in Eqs. (3.3) and (3.4) with their quanta, $b_\alpha(\pm,\vec{k})$ and $b_\alpha^\dagger(\pm,\vec{k})$, satisfying the commutation relations (3.5) with a_α replaced by $b_\alpha^\dagger$, etc., as for the s-wave case.

From the full Hamiltonian, by following the methods in I the unitary nonvanishing scattering amplitudes for positive energies are found to be

$$T_i^> = e^{i\delta_i}\sin\delta_i \,, \tag{3.9}$$

where we have introduced the labels $i=1,2,3$ for the three phase shifts δ_{11}, $\delta_{13}=\delta_{31}$, δ_{33} in terms of the notation $\delta_{2I,2J}$. The phase shifts are given by

$$\cot\delta_i = \frac{w\,\mathrm{Re}D_i(w)}{\lambda_i k^3}\,, \tag{3.10}$$

where in pionic mass units

$$\lambda_i = \frac{f^2}{4\pi}\left(-\tfrac{8}{3},\, -\tfrac{2}{3},\, \tfrac{4}{3}\right), \tag{3.11}$$

$$\mathrm{Re}D_i(w) = 1 - w\lambda_i\frac{1}{\pi}\,\mathcal{P}\int k^2dk\left(\frac{\epsilon(\xi)k^2}{2\xi^3(\xi-w)} - \frac{\epsilon(\xi')k^2}{2\xi'^3(\xi'-w)}\right), \tag{3.12}$$

with $\xi=(k^2+\mu^2)^{1/2}$, $\xi'=(k^2+\mu_s^2)^{1/2}$, and $\rho(k)\equiv 1$. Note that f^2 is the renormalized coupling constant.

The projection operators, $P_{IJ}^>$, for positive-energy states of definite isospin and angular momentum can be used to express the observed pion-nucleon states having positive energy in terms of these amplitudes. We label the meson indices by a single index $q=(\alpha,\vec{q})$, $p=(\alpha',\vec{p})$ for the initial and final mesons, respectively. Then for

$$T_{qp}^>(w) = \tfrac{1}{3}\sum_{I,J} P_{IJ}^>(p,q)\,T_{IJ}^>(w)\,, \tag{3.13}$$

the appropriate projection operators are (e.g., see Refs. 6 or 10)

$$\begin{aligned}
P_{11}^>(p,q) &= \tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_{\alpha'}^*\,\vec{\tau}\cdot\hat{\phi}_\alpha(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{q})\,,\\
P_{13}^>(p,q) &= \tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_{\alpha'}^*\,\vec{\tau}\cdot\hat{\phi}_\alpha\,[3\vec{p}\cdot\vec{q} - (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{q})]\,,\\
P_{31}^>(p,q) &= (\hat{\phi}_{\alpha'}^*\cdot\hat{\phi}_\alpha - \tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_{\alpha'}^*\,\vec{\tau}\cdot\hat{\phi}_\alpha)(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{q})\,,\\
P_{33}^>(p,q) &= (\hat{\phi}_{\alpha'}^*\cdot\hat{\phi}_\alpha - \tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_{\alpha'}^*\,\vec{\tau}\cdot\hat{\phi}_\alpha)\\
&\quad\times[3\vec{p}\cdot\vec{q} - (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{q})]\,.
\end{aligned} \tag{3.14}$$

Note that the isotopic wave functions, $\hat{\phi}$ or $\hat{\phi}^*$, for the pion in the initial or final state are given by

$$\hat{\phi}_{\pi^\pm} = \frac{1}{\sqrt{2}}(1,\pm i,0)\,,$$

$$\hat{\phi}_{\pi^0} = (0,0,1)\,.$$

Restricting our attention for the moment to the Born terms, we see that it is

$$T^B = \frac{\lambda_i k^3}{w}\,. \tag{3.15}$$

The projection operators, $P_{IJ}^>$, then yield the conventional nucleon Born term contributions from the direct and exchange diagrams.

Scattering at negative energy is a well-defined process in our theory because of the quantization procedure we have employed above in which the meson field contains contributions from $a_\alpha(-,-\vec{k})$. Consequently, content can be given to the substitution law and this will be done in the following section; first, though, it is of interest to examine the analytic properties of the amplitude, $T_i^>$, particularly in the region near the real energies at which physical scattering takes place. For positive energies, $w>\mu$, we have already found

$$T_i^>(w+i\epsilon) = \frac{\lambda_i k^3}{w\,D_i(w+i\epsilon)}\,, \tag{3.16}$$

where

$$D_i(w+i\epsilon) = 1 - w\lambda_i\frac{1}{\pi}\left(\int k^2dk\,\frac{\epsilon(\zeta)k^2}{2\zeta^3(\zeta-w-i\epsilon)} - \mathcal{P}\int k^2dk\,\frac{\epsilon(\zeta')k^2}{2\zeta'^3(\zeta'-w)}\right). \tag{3.17}$$

Notice that along the real axis for $w>\mu$ this amplitude is continuous as a function of energy and that it is indeed analytic in almost all neighborhoods of real physical scattering energies. Because of the shadow states in the theory, $T_i^>$ of Eq. (3.16) is however a piecewise-analytic function with points of nonanalyticity at the positive- and negative-energy thresholds for the shadow mesons. At such a junction between two analytic functions, say at the shadow state's pseudothreshold at positive energy, the explicit behavior of the amplitude and its corresponding experimental observables, such as total cross sections and polarizations, merits special study. In Sec. V we show by explicit calculation what happens to the total s-wave pion-nucleon cross section at the shadow meson's pseudothreshold.

IV. p-WAVE SCATTERING: THE SUBSTITUTION LAW

This theory of p-wave pion-nucleon scattering contains the meson field, considered in the one-meson approximation, which contains both positive- and negative-frequency mesons. It therefore has a left-hand cut. In this section we show that the theory is also crossing-symmetric even though this crossing symmetry is obtained by *substitution* rather than by *analytic continuation*.

Consider the p-wave Hamiltonian and meson field $\phi_\alpha(x)$ as defined in the preceding section by Eqs. (3.2) and (3.3). Since a static theory is invariant under $I\longleftrightarrow J$, with no loss in generality we can show the crossing symmetry for each separately. Let us pick a definite J value, say $J=1$. Notice that like in the Klein-Gordon theory of charged mesons the corresponding charges for positive- and negative-frequency mesons in the field $\phi_\alpha(x)$ must be opposite. This follows by gauge invariance of the first kind and is analogous to the correspondence of negative metric for negative-energy mesons. So in the p-wave Hamiltonian the "π^+ field" (coupled to an "n" to get a "p," by the first term of the Hamiltonian, and coupled to a "p" to get a "Δ^{++}," by the third term of the Hamiltonian) has positive-energy π^+ quanta and negative-energy π^- quanta. Hence the $\pi^- n$ with a negative-energy pion is coupled to the "p" but not to the $\pi^0 n$ negative-energy channel. Similar comments apply to the other meson channels. For example, for positive energies the $\pi^+ p$ amplitude is proportional to the $T^>_{I=3/2}$ amplitude and entirely independent of the $T^>_{I=1/2}$ amplitude whereas for the negative-energy scattering processes it is the $\pi^- p$ amplitude which has this property.

The projection operators, $P^<_{IJ}$, for negative states of definite isospin and angular momentum are therefore different from those used at positive energy. For negative energy for

$$T^<_{qp}(w)=\tfrac{1}{3}\sum_{IJ} P^<_{IJ}(p,q)T^<_{IJ}(w)\,, \tag{4.1}$$

the appropriate projection operators are

$$\begin{aligned} P^<_{11}(p,q)&=\tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_\alpha\vec{\tau}\cdot\hat{\phi}^*_{\alpha'}(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})\,,\\ P^<_{13}(p,q)&=\tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_\alpha\vec{\tau}\cdot\hat{\phi}^*_{\alpha'}[3\vec{q}\cdot\vec{p}-(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})]\,,\\ P^<_{31}(p,q)&=(\hat{\phi}_\alpha\cdot\hat{\phi}^*_{\alpha'}-\tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_\alpha\vec{\tau}\cdot\hat{\phi}^*_{\alpha'})(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})\,,\\ P^<_{33}(p,q)&=(\hat{\phi}_\alpha\cdot\hat{\phi}^*_{\alpha'}-\tfrac{1}{3}\vec{\tau}\cdot\hat{\phi}_\alpha\vec{\tau}\cdot\hat{\phi}^*_{\alpha'})\\ &\quad\times[3\vec{q}\cdot\vec{p}-(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})]\,. \end{aligned} \tag{4.2}$$

The mesons are labeled by the single index $q=(\alpha,\vec{q})$ $p=(\alpha',\vec{p})$ for the initial and final mesons, respectively.

We can now proceed to do the explicit calculations at negative energies by the methods used in paper I. It is crucial to recognize that at negative energies the pions have negative norm and that the Klein-Gordon propagator dictates that the physical scattering there is at $-|w|-i\epsilon$ just as it is at $|w|+i\epsilon$ for positive-energy processes. The resulting I,J amplitudes at negative energy, $w=-u<0$, are

$$T^<_i(w=-u<0)=\frac{\xi\lambda_i k^3}{w\left[1-w\,\xi\lambda_i\,\frac{1}{\pi}\left(\int k^2dk\,\frac{\epsilon(\xi)k^2}{2\xi^3[\xi-w-i\epsilon\epsilon(w)]}-\mathcal{P}\int k^2dk\,\frac{\epsilon(\xi')k^2}{2\xi'^3(\xi'-w)}\right)\right]}\,, \tag{4.3}$$

or, since the metric is negative ($\xi=-1$),

$$T^<_i(-u-i\epsilon)=\frac{\lambda_i k^3}{u\left[1-u\lambda_i\,\frac{1}{\pi}\left(\int k^2dk\,\frac{\epsilon(\xi)k^2}{2\xi^3(\xi-u-i\epsilon)}-\mathcal{P}\int k^2dk\,\frac{\epsilon(\xi')k^2}{2\xi'^3(\xi'-u)}\right)\right]}\,. \tag{4.4}$$

Then from Eq. (4.1) it is seen that

$$\begin{aligned} T^<_{p,q}(-|w|)&=\tfrac{1}{3}\sum_{IJ}P^<_{IJ}(q,p)T^<_{IJ}(-|w|)\\ &=T^>_{q,p}(|w|)\,. \end{aligned} \tag{4.5}$$

This is the substitution law. Hence for the calculated transition amplitude at negative energy, the negative-energy meson in the initial state is to be identified with its positive-energy antiparticle in the final state. Notice that we have shown here only that crossing symmetry is meaningful for transition *amplitudes*, rather than being a property of the negative-energy *states* themselves.

Now it is natural to consider the question of the analytic continuation of the positive-energy amplitude, $T^>_i(w)$, as given by Eq. (3.16) as w is continued from $w>\mu$. It is natural to take the amplitude in the domain $0<w<\mu$ to be the analytic continuation of $T^>_i(w)$, and similarly to take for the domain $-\mu<w<0$ the analytic continuation of $T^<_i(w)$ from the negative-energy domain. Their leading terms (the Born poles at the point $w=0$) will be the *same*.

V. p-WAVE SCATTERING: RESULTS

Given this simple, finite theory of low-energy p-wave pion-nucleon scattering we must now confront its exact quantitative predictions with experiment. The p-wave pion-nucleon scattering lengths and phase shifts can be computed from

$$\frac{\lambda_i k^3\cot\delta_i}{w^*}=1-w^*r_i(w^*)\,, \tag{5.1}$$

TABLE I. p-wave scattering lengths.

Type	Predicted length $(1/\mu)^3$	Hamilton-Woolcock[a] $(1/\mu)^3$
a_{11}	−0.121	−0.101 ± 0.007
a_{13}	−0.045	−0.029 ± 0.005
a_{31}	−0.045	−0.038 ± 0.005
a_{33}	0.178	0.215 ± 0.005

[a] Taken from Ref. 19.

where we find for $\mu < w^* < \mu_s$

$$r_i(w^*) = \frac{\lambda_i}{2\pi}\,\mathcal{P}\int k^2 dk\left(\frac{\epsilon(\xi)k^2}{\xi^3(\xi - w^*)} - \frac{\epsilon(\xi')k^2}{\xi'^3(\xi' - w^*)}\right)$$

$$= \lambda_i\left[\left(\frac{\mu_s^3-\mu^3}{w^{*2}}\right) + \frac{2}{\pi}\left(\frac{\mu_s^2-\mu^2}{w^*}\right) + \frac{2}{\pi}w^*\ln(\mu_s/\mu) - \frac{2}{\pi}\frac{(w^{*2}-\mu^2)^{3/2}}{w^{*2}}\ln\left(\frac{w^*+(w^{*2}-\mu^2)^{1/2}}{\mu}\right)\right.$$

$$\left. - \frac{(\mu_s^2-w^{*2})^{3/2}}{w^{*2}}\left(1+\frac{2}{\pi}\sin^{-1}\frac{w^*}{\mu_s}\right)\right] \qquad (5.2)$$

with $w^* = w + k^2/2m_n$ in the center-of-mass scattering frame. In comparison, evaluation in the effective-range approximation gives

$$r_i(w^*=0) = \frac{\lambda_i}{2\pi}\int k^2 dk\left(\frac{\epsilon(\xi)k^2}{\xi^4} - \frac{\epsilon(\xi')k^2}{\xi'^4}\right)$$

$$= \tfrac{3}{2}\lambda_i(\mu_s - \mu). \qquad (5.3)$$

Using this, from Eqs. (5.1) and (5.2) one can see that the (3, 3) resonance is predicted to exist.

For comparison with experiment[16] we take $f^2/4\pi = 0.081$ and use the exact result as given by Eq. (5.2). Requiring δ_{33} to pass through 90° at the position of the (3, 3) as determined by the Roper-Wright-Feld (RWF) phase-shift analysis,[18] we find

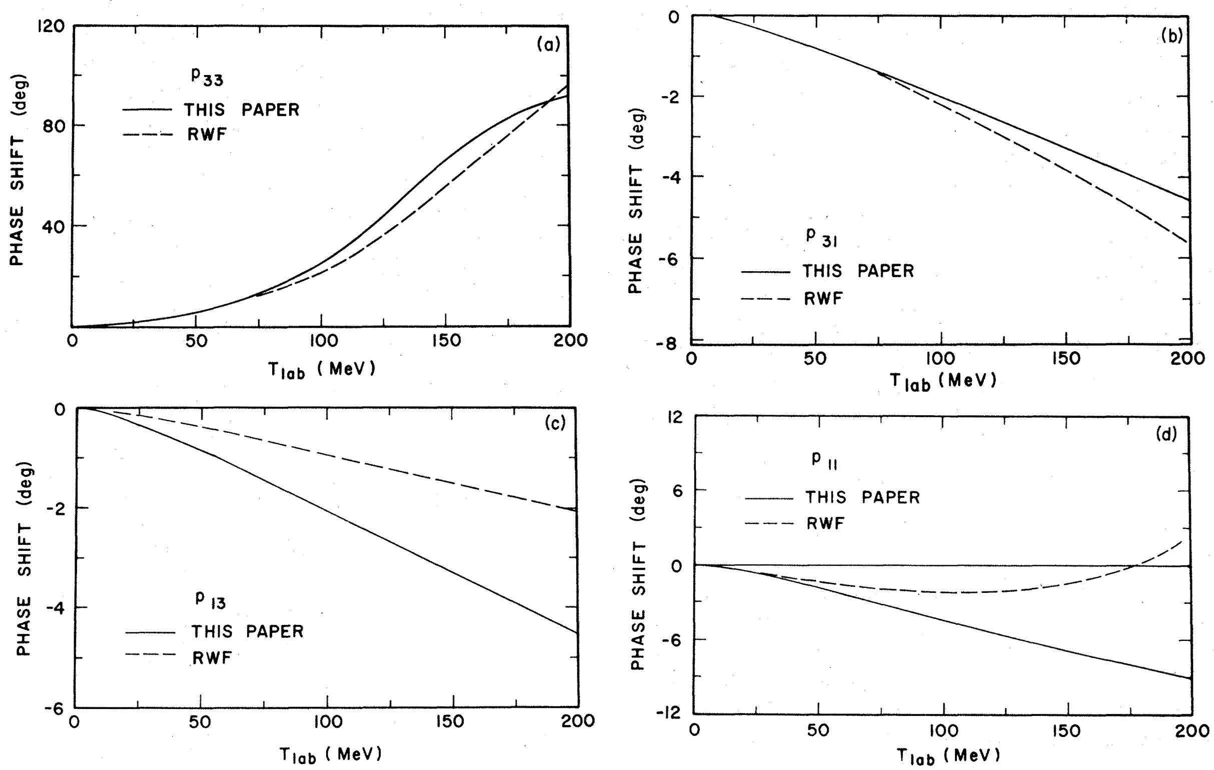

FIG. 2. p-wave phase shifts for pion-nucleon scattering. The empirical values are the best solution in the 0–350-MeV range from Ref. 18.

the mass of the mesons in the shadow states to be 2.7μ. Notice that the value for the corresponding "shadow threshold" is $T_\pi = 288$ MeV to be compared with the N^* threshold of $T_\pi = 193$ MeV and the lower inelastic $\pi\pi N$ threshold at $T_\pi = 170$ MeV. The precise mass value of the "shadow meson" should therefore not be taken too seriously as the effects of these and other intermediate states have not been included in our present analysis. (In the effective-range approximation the shadow mass is shifted to a higher value.) The p-wave pion-nucleon scattering lengths are defined by

$$k^3 \cot\delta_i = \frac{1}{a_i} + O(k^2), \qquad (5.4)$$

and we find the numerical values given in Table I. They are in agreement with the Hamilton-Woolcock values.[19]

The exact theoretical predictions for the p-wave phase shifts can be determined from Eqs. (5.1) and (5.2) and are plotted in Fig. 2 against the "best solutions" of the RWF analysis. One can make several remarks: First notice that a standard by which to judge the success of the theory is provided by the experimental difference between the δ_{13} and δ_{31} phase shifts because this theory, as a static theory, predicts $\delta_{13} = \delta_{31}$. We are pleased by the fact that the theoretical $\delta_{13}^t = \delta_{31}^t$ falls about halfway between the δ_{13}^e and δ_{31}^e phase shifts. A better fit to experiment could be obtained by altering the coupling of the (1, 3) and (3, 1) poles but this will introduce a new parameter. We resist the temptation to do this. The p_{11} empirical phase shift indicates that an attractive contribution is needed as a next order correction, i.e., the attractive interaction due to N^* exchange. By following the same procedure but making the effective-range approximation, one obtains almost identical p-wave phase shifts with constant effective ranges of

$$r = (-0.94, -0.23, 0.47)(1/\mu). \qquad (5.5)$$

For the exact solution the corresponding effective-range functions maintain the same signs and average out to about these constant values over the energies below the (3, 3) resonance.

VI. DISCUSSION

Armed with the concepts of indefinite metric and shadow states we have constructed in this paper an extremely simple static theory of low-energy pion-nucleon scattering. This theory, both finite and exactly soluble, applies to both s- and p-wave scattering, and when given the masses and coupling constants as input, it successfully predicts the scattering phase shifts in agreement with experiment. The most striking facts are that the scattering amplitude so obtained is unitary, satisfies the substitution law, and is analytic in almost all neighborhoods of real physical scattering energies.

Nevertheless, since we have worked here completely in the static limit, there remains questions concerning the relativistic generalization of this theory. That is, "What is the connection of this simple static theory with a fully relativistic quantum field theory?" As our viewpoint stresses the importance of the concepts of indefinite metric and shadow states for construction of a finite and consistent theory, it is especially imperative for us to inquire as to the correct treatment of the shadow states, needed for the finiteness but which should not contribute to the probability, i.e., not enter the unitarity sum. This question has already been investigated elsewhere[20,21] so we make only a few brief remarks here in review. The crucial point regarding the shadow states in both the static theory as well as in the relativistic theory is to use a standing-wave propagator for the complete state and not merely for the shadow principle alone. If it consists exclusively of physical quanta, then the usual forward propagator is used and these physical states by themselves lead to imaginary parts for real momenta. For states involving shadow quanta there thus should be no imaginary part. The shadow quanta affect the scattering amplitude only through its real part and in this manner contribute to the dynamics so as to have a finite theory. For example, consider the single-loop diagram, say, for the field-theoretic equivalents, to the second order, to our model for p-wave pion-nucleon scattering. This is shown in Fig. 3. For the physical state, (a) the propagating Green's function is given by the product of the two causal propagators:

$$G^p(q,p) = \frac{-i}{q^2 - \mu^2 + i\epsilon}\,\frac{-i}{p^2 - m^2 + i\epsilon}. \qquad (6.1)$$

But for the shadow state, (b) the standing propaga-

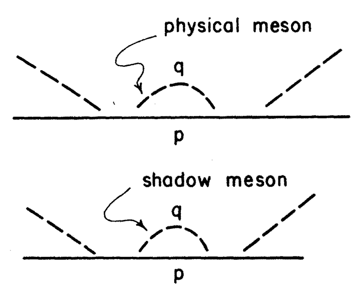

FIG. 3. Field-theoretic equivalents, to the second order, to the p-wave pion-nucleon static model.

tor is used which is obtained by taking the average of the forward and backward propagators[22]:

$$G^s(q,p)=\frac{1}{2}\left(\frac{-i}{q^2-\mu_s{}^2+i\epsilon}\,\frac{-i}{p^2-m^2+i\epsilon}-\frac{-i}{q^2-\mu_s{}^2-i\epsilon}\,\frac{-i}{p^2-m^2-i\epsilon}\right). \tag{6.2}$$

With this prescription, diagram 3(b) has no imaginary part and thus the shadow state does not contribute to the unitarity relation. For more complicated diagrams, the calculation has to be arranged such that physical and shadow intermediate states are explicitly displayed. When this is done, the net result is that all the imaginary parts are associated with the physical intermediate states and the resulting amplitude is fully unitary.

As discussed in Sec. III the onset of the shadow channel is a point of nonanalyticity as it is the point of join between two different analytic functions. We shall refer to this point as a "shadow pseudothreshold" since it may only be observed indirectly through its effects on normal channels, say for example on their total cross sections. As the cross section for a new channel at threshold is primarily s wave, we expect the anomaly in the physical channel to occur only in those partial-wave cross sections which are allowed to couple to the s-wave shadow partial wave by angular momentum and parity conservation. This is indeed the case for the theory studied in this paper as is apparent directly from the integral expressions for the s- and p-wave effective-range functions, Eqs. (2.6) and (4.2). Whereas above the shadow pseudothreshold the principal-value prescription smooths out the singularities, below it the first derivative of the s-wave effective-range function can be infinite and likewise the second derivative of the p-wave effective-range function. The precise behavior of both of these functions at the pseudothreshold is displayed in Fig. 4 where the proper functions R_s and R_p are defined by

$$r_{1,3}=\pm\frac{g^{r\;2}_{1,3}}{6\pi}R_s\,,$$

$$r_i=\tfrac{1}{2}\lambda_i R_p\,.$$

To understand the physics of this cusp it is useful to recall the well-known fact that in the usual case of a normal threshold similar cusps are expected to occur. For example, let us suppose there is only one channel open below the threshold T_0. Then, if the new channel starts to diminish the flux in the old channel at a rate having an infinite energy derivative, the old cross section will decrease above T_0 at an infinite rate. This is a classical argument based on flux conservation and

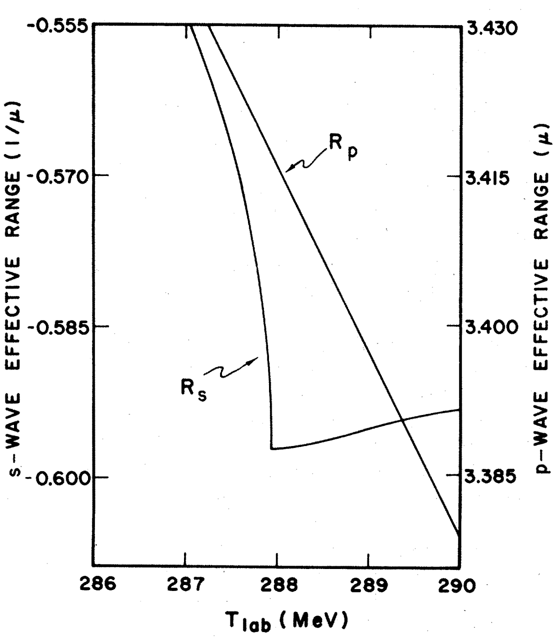

FIG. 4. Behavior of s- and p-wave "proper effective-range functions" near the shadow pseudothreshold. The relations between R_s (R_p) and $r_{1,3}$ (r_i) are given in Sec. VI of the text.

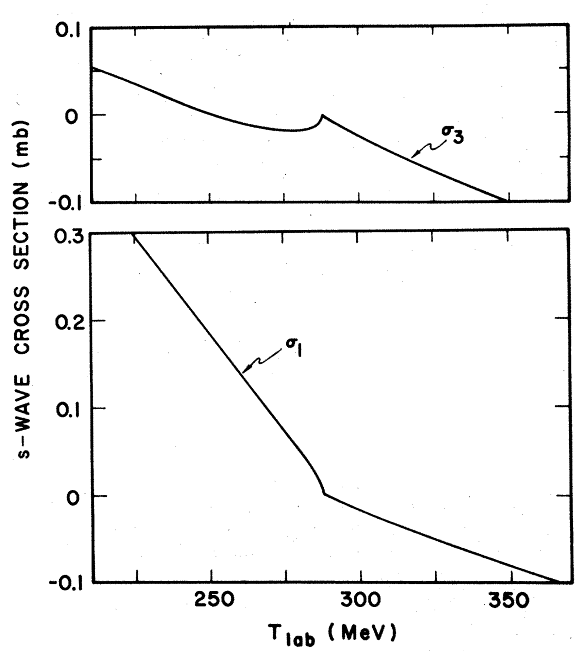

FIG. 5. Predicted s-wave cross-section behavior near the shadow pseudothreshold. Empirical observation of such an effect would indicate that a real shadow state is being dynamically excited.

so we do not expect such a cusp above a *shadow threshold*. However, *below* T_0 the *physical* and *shadow* thresholds are physically *identical*, and thus there are quantum-mechanical virtual transitions into the new channel which can produce an infinite slope below in both cases.

The important question of course concerns the shape and magnitude of the corresponding cusp in physical observables, such as the total s-wave pion-nucleon cross sections. Experimentally, the s-wave cross sections corresponding to the favored RWF solutions vary at 345 MeV by 0.5 and 1.5 mb for σ_1 and σ_3, respectively. The predictions based on our s-wave static model are given in Fig. 5. The shape and magnitude of the cusps should perhaps be noted, but one should ignore the over-all slope of the cross sections, as well as their total magnitude, since we do not expect the static theory to predict such features accurately at such high energies. Here we have assumed the shadow meson to be stable; if it were unstable we would expect instead a "woolly" cusp.

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[22]The prescription for a shadow intermediate state is to add half the amplitude T and half the conjugate amplitude T^*. Thus, for a single loop the antipropagator is multiplied by a minus sign. In general, $(-)^{k-1}$ appears where k is the number of s-channel intermediate internal lines.