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Quantum Field Theories with Shadow States. I. Soluble Models

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To construct a finite local relativistic quantum field theory we may introduce an indefinite-metric vector space, but then to avoid conflict with unitarity we must consider only a selected subset of states to be physical. The remaining states participate in the dynamics but are not among the complete set of physical states as far as probability interpretation is concerned. These states are called *shadow states*. The S matrix should be unitary when restricted to the physical states. In this paper we formulate and solve several simple models of field theories with shadow states and demonstrate the manner in which shadow states influence the dynamics and the structure of the scattering amplitude. The choice of a standing-wave boundary condition for the shadow states is shown to be completely consistent with the physical description of the scattering process in terms of wave packets. These methods are adapted to the study of low-energy pion-nucleon scattering in the following paper.

I. INTRODUCTION

In the quantum-field-theory approach to particle physics, the quanta of the fields play two roles: On the one hand the real quanta are the physical particles which constitute the initial and final states of any process, and through unitarity they generate kinematical constraints on the scattering amplitudes. On the other hand the virtual quanta supply the basic mechanism of the interaction between the particles and therefore determine the dynamical processes.[1] These ideas have formed the central theme of S-matrix theory of particle reactions,[2] the geometry of the singularities of the analytic functions describing the scattering being entirely determined by the spectrum of particles, while the strength of the singularities is determined by the coupling schemes. The principle of crossing symmetry, along with the interrelationship between singularities and spectra of particles, serves to specify the scattering amplitude as an analytic function. It is the expressed hope of the proponents of S-matrix theory that crossing symmetry and analyticity together with unitarity delineate the amplitude and perhaps even determine the scattering amplitude completely.

Despite these hopes, quantum field theories have tended to yield enigmatic answers to several fundamental questions; in particular, the question of the existence of interesting nontrivial interacting-field theories is still undecided. It is therefore worthwhile to examine the degree to which we are free to explore and develop the basic structure of the theory. In particular, we are interested in the possibility of having fields which contribute to the dynamics and serve to determine the scattering amplitude but which do not have any physical particles associated with them. This would mean that not all vectors of the linear vector space on which the field operators are defined are to be considered to be physical states.[3] There is a preselected subset of these states which are identified as physical states; the remaining states are called "shadow states."[4] A shadow state, then, is a state which contributes to the dynamics and hence may be associated with certain singularities of the scattering amplitude, but which does not enter the unitarity relation. In this and subsequent papers we would like to call attention to this possible generalization of quantum field theory and to explore the experimental consequences by studying several relevant models.

Much of the interest in shadow states arises from the possibility of providing a satisfactory probability interpretation of relativistic quantum field theories with indefinite metric. It is becoming increasingly clear that to construct a finite field theory we may have to use an indefinite metric, but then to avoid any conflict with unitarity we must consider only a selected subset of states to be physical states.[3,5] The metric of the nonphysical states is of no real interest with regard to probability interpretations, and hence the shadow states may have, but not necessarily, an indefinite metric. The use of an indefinite metric is particularly welcome to provide the requisite degree of convergence in any perturbation-theoretic calculations.

In this paper we shall study nonperturbative models involving shadow states. All the models that we are studying are exactly soluble in the sectors in which we are interested. We shall, in general, display exact solutions both for the wave functions and the S matrix in these models. These models are also of interest in that they all possess negative-energy mesons and hence a "left-hand cut" in the scattering amplitude.[6] These models are of intrinsic interest as examples of quantum-mechanical systems with shadow states, but they also serve as prototypes to the study of interesting physical processes. The next paper of this series will apply some of these results to low-energy pion-nucleon interactions.

The plan of the paper is as follows: In the next section we give a general discussion of the problem of coupling a scalar field with a source in such a manner that we get a "crossing-symmetric one-meson approximation." The role of positive- and negative-frequency components of the field should be carefully examined. Section III deals with the two-channel problem, with one physical channel and one shadow channel. We note the analytic structure of the scattering amplitudes. In Sec. IV it is shown that the choice of a standing-wave boundary condition for the shadow states of this model is completely consistent with the physical description of the scattering process in terms of wave packets. Section V deals with two distinct three-channel models and Sec. VI discusses the role of the substitution law[7] in the presence of shadow states. Section VII deals with a multichannel generalization of the program. The concluding Sec. VIII discusses the general features of the amplitude and reviews some questions like the role of boundary conditions and the analytic structure of the amplitude.

II. THE ONE-MESON APPROXIMATION IN THE COUPLING OF A KLEIN-GORDON FIELD TO A SOURCE

We consider here the coupling of mesons to nucleons in such a manner that we consider only states with one meson or no meson at all. Nevertheless, we would like to have the interaction in such a manner that the scattering amplitude is defined for both positive and negative energies and the substitution law which is characteristic of quantum field theory hold. This is accomplished in a Lagrangian scheme by associating the entire meson field with both positive and negative frequencies with annihilation operators for quanta with positive and negative energies, respectively. A conjugate field with positive and negative frequencies should be introduced for the inverse transition. We shall, therefore make use of a space of states which contain both positive- and negative-energy mesons.[6,7]

For wave functions obeying the Klein-Gordon equation, the natural scalar product is given by

$$(\phi_1, \phi_2) = i \int (\phi_1^* \dot{\phi}_2 - \dot{\phi}_1^* \phi_2) d^3r , \tag{2.1}$$

which is indefinite: Positive-energy wave functions have a positive (norm)^2, and negative-energy states have a negative (norm)^2. In a second-quantized theory we choose accordingly the free field of annihilation operators ϕ and the field of creation operators $\phi^\dagger$ with a commutation relation

$$[\phi(x), \phi^\dagger(y)] = 2i\Delta(x-y) . \tag{2.2}$$

Expressed in terms of creation and destruction operators, we get

$$\begin{aligned}
&[a(\vec{k}, +\omega), a^\dagger(\vec{k}', +\omega')] = +2\omega\delta(\vec{k}-\vec{k}') , \\
&[a(\vec{k}, -\omega), a^\dagger(\vec{k}', -\omega')] = -2\omega\delta(\vec{k}-\vec{k}') , \\
&[a(\vec{k}, +\omega), a^\dagger(\vec{k}', -\omega')] = 0 ,
\end{aligned} \tag{2.3}$$

or, more succinctly,

$$[a(k), a^\dagger(k')] = 2(k_0 + k_0')\delta(\vec{k}-\vec{k}') . \tag{2.4}$$

This theory then automatically contains an indefinite metric. This method of quantization has been used in the past in the context of quantization of infinite-component fields and of the quantization of tachyon fields as well as in fundamental investigations in quantum field theory.

For a model we shall choose the system of a continuum of states coupled to a single state treated by Dirac in the context of the problem of line width in atomic physics, and formulated as a field-theoretical model by Lee.[8] We shall however make use of the relativistic version of the theory with both positive- and negative-energy "meson-nucleon" continua. Since negative-energy mesons are defined we can give content to the "substitution law" as applied to scattering processes. This aspect will be dealt with in more detail in the paper dealing with the scattering of pions on nucleons.

The basic interaction therefore consists of the following: A source (excited atom or the V particle of the Lee model) is coupled to a continuum (ground state of atom plus radiation or the $N\theta$ states of the Lee model). The continuum states have positive or negative (norm)^2 depending on whether the meson energy is positive or negative. The excited state may have either positive or negative norm. We shall deal with both kinds of states.

So far we have not introduced the shadow states. We now introduce one or more such continuum channels, coupled in essentially the same manner. The basic difference is that in discussing the boundary conditions as applied to the shadow channel there are no running waves, but only standing waves. There is no flux of radiated mesons from such a state. This is brought about by the use of a time-symmetric half-retarded half-advanced propagation function for the shadow states. It is to be noted that the standing-wave prescription applies to the shadow *state* and *not* to the *meson* alone in that state.

The scattering matrix is then calculated following the standard procedure. We determine the "in" states of scattering by considering a state with plane wave with outgoing mesons in the normal channel(s) and a standing wave in the shadow channel(s). The scalar product of these states with their time-reversed "out" states yield the scattering matrix.

We could also determine the scattering amplitude by computing the singular outgoing part of the meson wave function in the physical channel(s). We show by direct computation that the same expression is obtained for the scattering amplitude from both methods.

We also show that this is the *physical S matrix* as determined by the construction of wave packets.

The remarkable result is that the scattering amplitude is nonvanishing only for the physical channels and is *unitary*. The unitarity condition on the scattering amplitude is satisfied within the physical channels alone. This is an instance of the general feature of quantum theories with shadow states.[9]

III. SOLUBLE MODEL WITH ONE PHYSICAL CHANNEL AND ONE SHADOW CHANNEL

We consider the simplest such model which has two continuum channels $N_1\theta_1$ and $N_2\theta_2$ with a discrete "bound state" V with the possible transitions

$$N_1\theta_1 \rightleftarrows V \rightleftarrows N_2\theta_2 , \tag{3.1}$$

and displayed in Fig. 1. Here $V \rightleftarrows N_2\theta_2$ is a shadow process and the model is defined by the Hamiltonian $H = H_0 + H_I$ with

$$H_0 = m^0 \int d\vec{p}\, V^\dagger(\vec{p})V(\vec{p}) + \sum_{i=1,2} \int d\vec{k}\,\omega_i(k) a_i^\dagger(\vec{k}) a_i(\vec{k}) , \tag{3.2}$$

$$H_I = \frac{1}{(4\pi)^{1/2}} \sum_{i=1,2} \int\int d\vec{p}\,d\vec{k}\, \frac{f_i(w)}{(2\omega)^{1/2}} [V^\dagger(\vec{p}) N_i(\vec{p}-\vec{k}) a_i(\vec{k}) + N_i^\dagger(\vec{p}-\vec{k}) V(\vec{p}) a_i^\dagger(\vec{k})] . \tag{3.3}$$

Since this model is exactly soluble by well-known methods, we discuss those aspects more fully in the following section when we consider its general-

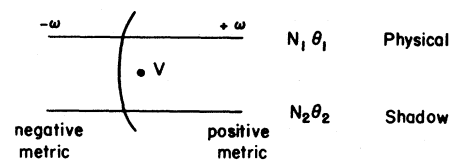

FIG. 1. Shadow diagrams having one physical channel and one shadow channel.

ization. As in the Klein-Gordon theory the meson fields contain both positive and negative frequencies so we take

$$[a_i(\vec{k}), a_i^\dagger(\vec{k}')] = \epsilon(w_i)\delta(\vec{k}-\vec{k}') \tag{3.4}$$

with

$$w_i(k) = \epsilon(w_i)\omega_i(k) , \quad \omega_i(k) = (\mu^2 + \vec{k}^2)^{1/2} , \quad \epsilon(w) = \pm 1 \text{ as } w \gtrless 0 . \tag{3.5}$$

We denote the V-particle bound-state wave function by

$$| V_\Lambda \rangle = \left| \begin{matrix} \phi_\Lambda^1(\vec{k}) \\ C \\ \phi_\Lambda^2(\vec{k}) \end{matrix} \right\rangle . \tag{3.6}$$

Applying the Hamiltonian, we have $H|V_\Lambda\rangle = \Lambda |V_\Lambda\rangle$ or

$$\begin{aligned} (\Lambda - w)\phi_\Lambda^1(\vec{k}) &= \frac{C}{(4\pi)^{1/2}} \frac{f_1(w)}{(2\omega)^{1/2}} , \\ (\Lambda - m^0) C &= \frac{1}{(4\pi)^{1/2}} \int d\vec{k}\,\epsilon(w) \frac{f_1(w)}{(2\omega)^{1/2}} \phi_\Lambda^1(\vec{k}) + \frac{1}{(4\pi)^{1/2}} \int d\vec{k}\,\epsilon(w) \frac{f_2(w)}{(2\omega)^{1/2}} \phi_\Lambda^2(\vec{k}) , \\ (\Lambda - w)\phi_\Lambda^2(\vec{k}) &= \frac{C}{(4\pi)^{1/2}} \frac{f_2(w)}{(2\omega)^{1/2}} . \end{aligned} \tag{3.7}$$

According to Eq. (3.3) the interaction only takes place in S waves, so we may put

$$\phi_\Lambda^i(\vec{k}) = \frac{1}{k(4\pi)^{1/2}} \phi_\Lambda^i(w) . \tag{3.8}$$

Since $(\Lambda - w)$ can have no zeros for a bound state, its normalized wave function is

$$V_\Lambda(w) = C \begin{pmatrix} f_1(w)k/[(2\omega)^{1/2}(\Lambda - w)] \\ 1 \\ f_2(w)k/[(2\omega)^{1/2}(\Lambda - w)] \end{pmatrix} \tag{3.9}$$

with

$$|C|^2 = -\frac{1}{\sigma'(\Lambda)} \tag{3.10}$$

and

$$\sigma(\lambda) = -\lambda + m^0 + \int k^2 dk\, \frac{\epsilon(\xi)[f_1^2(\xi) + f_2^2(\xi)]}{(2\omega)(\lambda - \xi)} , \tag{3.11}$$

where the bound-state discrete eigenvalue is determined by

$$\sigma(\Lambda)=0\,,\quad 0<\Lambda<\mu\,. \tag{3.12}$$

This brings us to the crucial matter of the choice of boundary conditions for the scattering states. Up until now we have followed the conventional approach. Our point of departure with the usual formulation is to introduce into this model the idea of shadow states. Therefore, we do *not* choose the same boundary conditions for both channels; instead, while we take the first channel to contain a physical state corresponding to outgoing-wave boundary conditions, the second channel will contain a shadow state. The shadow state is defined in this model by using a principal-value Green's function for the shadow channel with a corresponding shadow ghost channel.[10] The latter contains a noninteracting standing wave whose purpose is to properly orthonormalize the physical state vector. The need for two standing-wave components and their physical significance can be simply understood in terms of wave packets (see Sec. IV):

$$\begin{aligned}&\phi^1_\lambda(w)\sim\text{plane wave plus outgoing wave},\\ &\phi^2_\lambda(w)\sim\text{principal-value Green's function},\\ &\phi^{2'}_\lambda(w)\sim\text{noninteracting standing wave},\end{aligned} \tag{3.13}$$

or explicitly

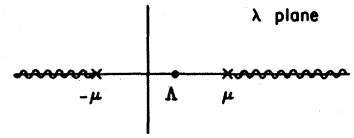

FIG. 2. Analyticity of σ and ν.

$$\begin{aligned}\phi^1_\lambda(w)&=\delta(w-\lambda)+\frac{cf_1(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)}\,,\\ \phi^2_\lambda(w)&=\mathcal{P}\frac{cf_2(w)k}{(2\omega)^{1/2}(\lambda-w)}\,,\\ \phi^{2'}_\lambda(w)&=\frac{\pi cf_2(\lambda)k}{(2\lambda)^{1/2}}\delta(\lambda-w)\,,\end{aligned} \tag{3.14}$$

where

$$\int k^2dk\,H'=\int k^2dk\begin{pmatrix}H & 0\\ 0 & \delta(w-\xi)/k\end{pmatrix} \tag{3.15}$$

and the metric matrix

$$\eta'=\begin{pmatrix}\eta & 0\\ 0 & -\epsilon(w)\delta(w-w')\,|w/k|\end{pmatrix} \tag{3.16}$$

define the shadow ghost (negative-metric) channel. Proceeding as for the bound state, we have

$$\Phi^1_\lambda(w)=\begin{bmatrix}\phi^1_\lambda(w)\\ C_\lambda\\ \phi^2_\lambda(w)\\ \phi^{2'}_\lambda(\omega)\end{bmatrix}=\begin{bmatrix}\delta(w-\lambda)-\dfrac{\epsilon f_{1\lambda}f_1(w)k}{\nu^+(2\lambda)^{1/2}(2\omega)^{1/2}(\lambda-w+i\epsilon)}\\ -\dfrac{\epsilon f_{1\lambda}}{\nu^+(2\lambda)^{1/2}}\\ -\mathcal{P}\dfrac{\epsilon f_{1\lambda}f_2(w)k}{\nu^+(2\lambda)^{1/2}(2\omega)^{1/2}(\lambda-w)}\\ -\dfrac{\epsilon\pi f_1f_2k}{2\nu^+}\delta(\lambda-w)\end{bmatrix} \tag{3.17}$$

as the properly normalized state vector with

$$\nu^\pm(\lambda)=-\lambda+m^0+\int k^2dk\frac{\epsilon(\xi)f_1{}^2(\xi)}{(2\omega)(\lambda-\xi\pm i\epsilon)}+\mathcal{P}\int k^2dk\frac{\epsilon(\xi)f_2{}^2(\xi)}{(2\omega)(\lambda-\xi)}\,. \tag{3.18}$$

The phase of the shadow ghost component is independent of the phases of the other components. These state vectors are also orthogonal with the scalar product given by

$$\langle\Phi^1_\lambda|\Phi^1_{\lambda'}\rangle=c^*c'+\sum_{i=1,2}\int_{-\infty}^{+\infty}\left|\frac{w}{k}\right|dw\,\epsilon(w)\phi^i_\lambda(w)^*\phi^i_{\lambda'}(w)-\int_{-\infty}^{\infty}\left|\frac{w}{k}\right|dw\,\epsilon(w)\phi^{2'}_\lambda(w)^*\phi^{2'}_{\lambda'}(w) \tag{3.19}$$

and are found to be orthogonal to the bound state by using $\sigma(\Lambda)=0$ and

$$\nu^\pm(\lambda)=\sigma^\pm(\lambda)\pm\epsilon(\lambda)i\pi f_2(\lambda)^2\tfrac{1}{2}k \tag{3.20}$$

for $\lambda \in \{[-\infty, -\mu], [\mu, \infty]\}$. The analytic properties of σ and ν are shown in Fig. 2.

To fully understand this model, it is necessary to construct the state vectors corresponding to the shadow channel and the shadow ghost channel. Supposing an initial standing wave in the shadow channel 2, we find

$$\Phi^S_{\lambda_S}(w) = \begin{bmatrix} \phi^1_{\lambda_S}(w) \\ C_{\lambda_S} \\ \phi^2_{\lambda_S}(w) \\ \phi^{2'}_{\lambda_S}(w) \end{bmatrix} = \left(\frac{N_{11}}{N_{22}}\right)^{1/2} \begin{bmatrix} -\dfrac{\epsilon f_2 \lambda}{N_{11}\nu^+\nu^-(2\lambda)^{1/2}} \operatorname{Re}\left(\dfrac{\sigma^- f_1(w)k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)}\right) \\ -\dfrac{\epsilon f_2 \lambda \nu^P}{N_{11}\nu^+\nu^-(2\lambda)^{1/2}} \\ \delta(\lambda - w) - \dfrac{\epsilon f_2 \lambda \nu^P}{N_{11}\nu^+\nu^-(2\lambda)^{1/2}} \mathcal{P}\left(\dfrac{f_2(w)k}{(2\omega)^{1/2}(\lambda - w)}\right) \\ 0 \end{bmatrix}, \tag{3.21}$$

which is *purely real*, as well as *orthogonal* to the physical scattering states of channel 1 and also the V-particle bound state. Here

$$N_{11} = 1 + \frac{\pi^2 f_1{}^2 f_2{}^2 k^2}{4\nu^+\nu^-} \tag{3.22}$$

and

$$N_{22} = 1 + \frac{\pi^2 f_2{}^2 (f_2{}^2 + 2 f_1{}^2) k^2}{4\nu^+\nu^-}. \tag{3.23}$$

The orthonormal, *real* shadow-ghost-channel wave function is found to be

$$\Phi^G_{\lambda_G}(w) = \begin{bmatrix} \phi^1_{\lambda_G}(w) \\ C_{\lambda_G} \\ \phi^2_{\lambda_G}(w) \\ \phi^{2'}_{\lambda_G}(w) \end{bmatrix} = \epsilon\left(\frac{1}{N_{11}}\right)^{1/2} \begin{bmatrix} \operatorname{Re}\left(\dfrac{i f_2 \lambda f_1(w) k}{\nu^+(2\lambda)^{1/2}(2\omega)^{1/2}(\lambda - w + i\epsilon)}\right) \\ \operatorname{Re}\left(\dfrac{i f_2 \lambda}{\nu^+(2\lambda)^{1/2}}\right) \\ \operatorname{Re}\left(\dfrac{i f_2 \lambda}{\nu^+(2\lambda)^{1/2}}\right)\mathcal{P}\left(\dfrac{f_2(w)k}{(2\omega)^{1/2}(\lambda - w)}\right) \\ -\epsilon N_{11}\delta(w - \lambda) \end{bmatrix}. \tag{3.24}$$

Computing the generalized S matrix for both the physical and shadow channels, we find a diagonal matrix

$$S = \operatorname{diag}(\nu^-/\nu^+, 1, 1), \tag{3.25}$$

where

$$S_{I,I'} = \langle \text{out state}, I | \text{in state}, I' \rangle = \int_{-\infty}^{\infty} dw\, \phi^I(w)\eta(w)\phi^{I'}(w),$$

as discussed in Secs. II and IV, and it is obviously fully unitary in the physical channel 1.

Having demonstrated the unitarity, it is now natural to demonstrate the related relationship of completeness. To do this, it is convenient to introduce a matrix A, called the generalized Møller matrix, which is defined by

$$A = \| A_{\lambda w} \|, \quad A_{\lambda w} = \phi_\lambda(w) \tag{3.26}$$

with indices λ and w such that λ is an index with both a discrete part, Λ, and a continuous part, $\lambda \in \{[-\infty, -\mu], [\mu, \infty]\}$, and w is similarly an index with both a discrete part and a continuous part, $w \in \{[-\infty, -\mu], [\mu, \infty]\}$. For instance, for the two-channel model we have

$$A=\begin{bmatrix} C_\Lambda & \phi^1_\Lambda(w) & \phi^2_\Lambda(w) & 0 \\ C_\lambda & \phi^1_\lambda(w) & \phi^2_\lambda & \phi^{2'}_\lambda(w) \\ C_{\lambda_S} & \phi^1_{\lambda_S}(w) & \phi^2_{\lambda_S}(w) & \phi^{2'}_{\lambda_S}(w) \\ C_{\lambda_G} & \phi^1_{\lambda_G} & \phi^2_{\lambda_G}(w) & \phi^{2'}_{\lambda_G}(w) \end{bmatrix}$$

$$=\begin{bmatrix} C & \frac{Cf_1(w)k}{(2\omega)^{1/2}(\Lambda-w)} & \frac{Cf_2(w)k}{(2\omega)^{1/2}(\Lambda-w)} & 0 \\ -\frac{\epsilon f_1\lambda}{\nu^+(2\lambda)^{1/2}} & \delta(w-\lambda)-\frac{\epsilon f_1\lambda \mathcal{G}_1^+}{\nu^+(2\lambda)^{1/2}} & -\mathcal{P}\,\frac{\epsilon f_1\lambda\mathcal{G}_2}{\nu^+(2\lambda)^{1/2}} & -\frac{\epsilon\pi f_1 f_2 k}{2\nu^+}\delta(\lambda-w) \\ -\frac{\epsilon f_2\lambda\nu^P}{N_{11}\nu^+\nu^-(2\lambda)^{1/2}} & -\frac{\epsilon f_2\lambda}{(2\lambda N_{11}N_{22})^{1/2}}\,\frac{\mathrm{Re}(\sigma^-\mathcal{G}_1^+)}{\nu^+\nu^-} & \left(\frac{N_{11}}{N_{22}}\right)^{1/2}\delta(\lambda-w)-\frac{\epsilon f_2\lambda\nu^P\mathcal{P}(\mathcal{G}_2)}{(2\lambda N_{11}N_{22})^{1/2}\nu^+\nu^-} & 0 \\ \mathrm{Re}\left(\frac{if_2\lambda}{\nu^+(2\lambda)^{1/2}}\right) & \mathrm{Re}\left(\frac{if_2\lambda\mathcal{G}_1^+}{(2\lambda N_{11})^{1/2}\nu^+}\right) & \mathrm{Re}\left(\frac{if_2\lambda}{(2\lambda N_{11})^{1/2}\nu^+}\right)\mathcal{P}(\mathcal{G}_2) & -\epsilon\sqrt{N_{11}}\,\delta(\lambda-w) \end{bmatrix}, \tag{3.27}$$

where

$$\mathcal{G}_1^+=\frac{f_1(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)},$$

$$\mathcal{G}_2=\frac{f_2(w)k}{(2\omega)^{1/2}(\lambda-w)}.$$

Now, if we use the summation convention with the understanding that summation over a continuous index corresponds to integration, the completeness relation is $A^\dagger A=1$ or

$$\sum_\lambda \phi_\lambda(w')\eta(\lambda)\phi_\lambda(w)^*=\eta_{ww'}. \tag{3.28}$$

It is straightforward to verify that this is true for the A given explicitly above.

For comparison with the diagrammatic approach of quantum field theory we need the T amplitude. From Eq. (3.25) it can be found for physical λ via

$$S=1+i2T \tag{3.29}$$

and is ($\lambda>0$)

$$T(\lambda)=\frac{\pi f_1(\lambda)^2 k}{2\nu^+(\lambda)}, \tag{3.30}$$

where $k(\lambda)=(\lambda^2-\mu^2)^{1/2}$. The analytic extension of this piecewise-analytic function from domain $\lambda\in\{[-\infty,-\mu],[\mu,\infty]\}$ for complex λ is

$$T(\lambda)=\frac{\pi f_1(\lambda)^2 k(\lambda)}{2\nu(\lambda)}, \tag{3.31}$$

where $\nu(\lambda)$ is real-analytic. Notice that the original Hamiltonian contains bare coupling constants and the bare V mass, m^0. Mass renormalization can be carried out in the usual manner, so from Eq. (3.12) we have

$$m_V-m^0=\int k^2dk\,\frac{\epsilon(\xi)[f_1(\xi)^2+f_2(\xi)^2]}{(2\omega)(m_V-\xi)} \tag{3.32}$$

as an implicit equation for the physical V mass, m_V. Thus from Eq. (3.31) we have

$$T(\lambda)=\frac{\pi f_1(\lambda)^2 k}{2(m_V-\lambda)\left(1+\int k^2dk\,\frac{\epsilon(\xi)[f_1(\xi)^2+f_2(\xi)^2]}{(2\omega)(\xi-m_V)(\xi-\lambda)}\right)}. \tag{3.33}$$

Hence, if as $\lambda \to m_V$ we require

$$T(\lambda) \underset{\lambda \to m_V}{\sim} \frac{\pi f_1^r(m_V)^2 k}{2(m_V - \lambda)}, \tag{3.34}$$

it is natural to define the renormalized coupling constant by

$$f_1^r(z)^2 = \frac{f_1(z)^2}{1 + \int k^2 dk \frac{\epsilon(\xi)[f_1(\xi)^2 + f_2(\xi)^2]}{(2\omega)(\xi - m_V)^2}}, \tag{3.35}$$

and similarly for $f_2^r(z)^2$ and $f_2(z)^2$. If we assume a universal form factor, $\rho(z)$, i.e., that $f_{1,2}(z) = f_{1,2}\rho(z)$, this equation can be inverted

$$f_{1,2}{}^2 = \frac{f_{1,2}^r{}^2}{1 - (f_1^{r2} + f_2^{r2}) \int k^2 dk \frac{\epsilon \rho(\xi)^2}{(2\omega)(\xi - m_V)^2}} \tag{3.36}$$

and the T amplitude can be explicitly displayed in terms of the renormalized quantities

$$T(\lambda) = \frac{\pi f_1^{r2} \rho(\lambda)^2 k_0}{2(m_V - \lambda)\left(1 - (f_1^{r2} + f_2^{r2})(m_V - \lambda) \int k^2 dk \frac{\epsilon \rho(\xi)^2}{(2\omega)(\xi - m_V)^2(\xi - \lambda)}\right)}. \tag{3.37}$$

The physical significance of this result can be seen by expanding it in powers of f_1^{r2} and f_2^{r2}. We get

$$T(\lambda) = \frac{\pi f_1^{r2} \rho^2(m_V) k}{2(m_V - \lambda)} \left(1 + (f_1^{r2} + f_2^{r2})(m_V - \lambda) \int k^2 dk \frac{\epsilon \rho(\xi)^2}{(2\omega)(\xi - m_V)^2(\xi - \lambda)} + \cdots \right), \tag{3.38}$$

corresponding to the field-theoretic expansions[10a] in Fig. 3(a), which should be compared with the conventional expansion, Fig. 3(b),

$$T_0(\lambda) = \frac{\pi f_1^{r2} \rho^2(m_V)}{2(m_V - \lambda)} \left(1 + f_1^{r2}(m_V - \lambda) \int k^2 dk \frac{\epsilon \rho^2(\xi)}{(2\omega)(\xi - m_V)^2(\xi - \lambda)} + \cdots \right), \tag{3.39}$$

which is also the limit of (3.38) when $(f_2^r)^2 \to 0$. This property illustrates the fact that the shadow states are dynamical states, for they cease to exist when the coupling constant is allowed to go to zero. Also it is obvious from the diagrams that the dynamics involves both the physical and shadow states in spite of the fact that the unitarity of the S matrix involves only the physical states.

IV. THE SCATTERING OF A WAVE PACKET IN THE PRESENCE OF SHADOW STATES

We employed the standard results of the formal theory of scattering in our treatment in the preceding section of a simple soluble model with one physical channel and one shadow channel. As is well known, when no shadow states are present, this formalism has been justified on physical grounds by the construction of wave packets. The question can be raised, however, as to what happens to this justification when shadow states are present. In particular, is the formal definition of the S matrix as used above still correct and, if so, what is the physical significance of the two standing-wave components of the physical wave function for the normal channel?

The formal definition of the S matrix remains correct: Consider the physical wave function for the normal scattering channel. It was derived above in the energy representation, Eq. (3.17), as a solution of

$$(\lambda - w)\phi_\lambda^1(\vec{k}) = \frac{C_\lambda}{(4\pi)^{1/2}} \frac{f_1(w)}{(2\omega)^{1/2}},$$
$$(\lambda - w)\phi_\lambda^2(\vec{k}) = \frac{C_\lambda}{(4\pi)^{1/2}} \frac{f_2(w)}{(2\omega)^{1/2}}, \tag{4.1}$$
$$(\lambda - w)\phi_\lambda^{2\prime}(\vec{k}) = 0,$$

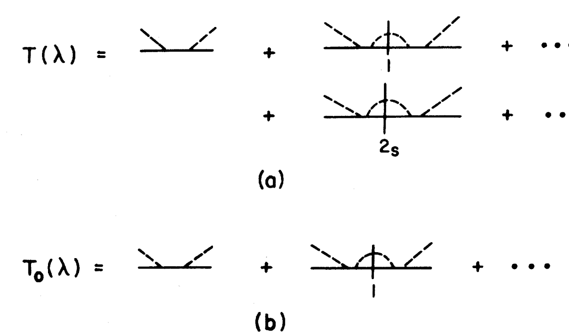

FIG. 3. (a) Field-theoretic equivalents to shadow diagrams. (b) Conventional diagrams.

with C_λ determined by

$$(\lambda - m_0)C_\lambda = \frac{1}{(4\pi)^{1/2}} \int d\vec{k}\epsilon(\xi) \frac{f_1(\xi)}{(2\omega)^{1/2}} \phi_\lambda^1(\vec{k}) + \frac{1}{(4\pi)^{1/2}} \int d\vec{k}\epsilon(\xi) \frac{f_2(\xi)}{(2\omega)^{1/2}} \phi_\lambda^2(\vec{k}), \quad (4.2)$$

with the boundary conditions that the normal component $\phi_\lambda^1(\vec{k})$ is an incoming plane wave and outgoing spherical wave at large distances. For the shadow components, $\phi_\lambda^2(\vec{k})$ and $\phi_\lambda^{2'}(\vec{k})$, the boundary conditions respectively are that they are the time-symmetric solutions of the inhomogeneous differential equation (principal-value Green's function) and of the homogeneous differential equation (a regular solution). Equivalently, in the nonrelativistic limit, $\omega \simeq \mu + \vec{k}^2/2\mu + \cdots$, $\lambda \simeq \mu + \vec{p}^2/2\mu + \cdots$,

$$(\vec{p}^2 - \vec{k}^2)\phi_{\vec{p}}^1(\vec{k}) = \frac{2\mu C_\lambda}{(4\pi)^{1/2}} \frac{f_1(w)}{(2\omega)^{1/2}},$$

$$(\vec{p}^2 - \vec{k}^2)\phi_{\vec{p}}^2(\vec{k}) = \frac{2\mu C_\lambda}{(4\pi)^{1/2}} \frac{f_2(w)}{(2\omega)^{1/2}}, \quad (4.3)$$

$$(\vec{p}^2 - \vec{k}^2)\phi_{\vec{p}}^{2'}(\vec{k}) = 0,$$

or, defining

$$\phi^i(\vec{p}, \vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{p}}^i(\vec{k}),$$

$$f_i(\vec{r}) = (4\pi)\left(\frac{1}{2\pi}\right)^3 \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \frac{f_i(w)}{(2\omega)^{1/2}}, \quad (4.4)$$

we have ($E = \vec{p}^2/2\mu$)

$$\phi^1(\vec{p}, \vec{r}) = \frac{(\mu p)^{1/2}}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{r}} + \left(\frac{p}{\mu}\right)^{1/2} \frac{C_\lambda}{2^{5/2}\pi^{1/2}} \int d\vec{r}'\, G^+(E; \vec{r}, \vec{r}') f_1(\vec{r}'), \quad (4.5)$$

$$\phi^2(\vec{p}, \vec{r}) = \left(\frac{p}{\mu}\right)^{1/2} \frac{C_\lambda}{2^{5/2}\pi^{1/2}} \int d\vec{r}'\, G^p(E; \vec{r}, \vec{r}') f_1(\vec{r}'),$$

with

$$G^+(E; \vec{r}, \vec{r}') = -\frac{\mu}{2\pi} \frac{\exp(ip|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|},$$

$$G^p(E; \vec{r}, \vec{r}') = -\frac{\mu}{2\pi} \frac{\cos(p|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|},$$

and the wave-function renormalization is with respect to the measure $dEd\Omega_{\vec{p}}$.[11] Partial-wave reduction using

$$G^{+,p}(E; \vec{r}, \vec{r}') = 2\mu \sum_{lm} Y_l^m(\hat{r}) Y_l^{m*}(\hat{r}') \times \frac{1}{rr'} G_l^{+,p}(p; r, r'),$$

$$\phi^i(\vec{p}, \vec{r}) = \left(\frac{2\mu p}{\pi}\right)^{1/2} \frac{1}{pr} \sum_{lm} Y_l^m(\hat{r}) Y_l^{m*}(\hat{p}) \times i^l \phi_l^i(p, r),$$

then gives ($l = 0$, s wave)

$$\phi_0^1(p, r) = \sin(pr) + \pi p C_\lambda \int_0^\infty r'\, dr'\, G_0^+(p; r, r') f_1(|r'|),$$

$$\phi_0^2(p, r) = \pi p C_\lambda \int_0^\infty r'\, dr'\, G_0^p(p; r, r') f_2(|r'|), \quad (4.6)$$

$$\phi_0^{2'}(p, r) = \sin(pr)\pi p C_\lambda \frac{f_2(\lambda)}{(2\lambda)^{1/2}},$$

with

$$G_0^+(p, r, r') = -\frac{1}{p} \sin(pr_<) e^{ipr_>},$$

$$G_0^p(p, r, r') = -\frac{1}{p} \sin(pr_<) \cos(pr_>),$$

where $r_<$ ($r_>$) is the lesser (greater) of r and r'.

Hence at very large distances, $r \to \infty$, we obtain the simple and transparent form (s wave)

$$\phi_0^1(p, r) \simeq \tfrac{1}{2} i (e^{-ikr} - S_\lambda e^{ikr}),$$

$$\phi_0^2(p, r) \simeq \cos(pr) \frac{\pi p f_1 f_2}{2\nu^+}, \quad (4.7)$$

$$\phi_0^{2'}(p, r) \simeq -\sin(pr) \frac{\pi p f_1 f_2}{2\nu^+}$$

for the wave function which describes the physical scattering state. Notice that we have identified $S_\lambda = \nu^-(\lambda)/\nu^+(\lambda)$ which we interpreted formally in the preceding section as the S matrix for the physical scattering channel. This should be compared with the normal asymptotic solution when no shadow states are present (lth angular wave),

$$\phi_l^1(p, r) \simeq \tfrac{1}{2} e^{i\pi(l+1)/2} (e^{-ikr} - e^{-i\pi l} S_l e^{ikr}), \quad (4.8)$$

where S_l *is* the partial-wave S matrix as interpreted by construction of wave packets. Thus, here in Eqs. (4.7) the only change asymptotically, except for the naturally different S matrix, is the presence of the extra standing-wave components which have a common *amplitude proportional to the strength of the source*. Consequently, when one averages over many *shadow* wavelengths in r space, the two standing-wave components cancel. Recall that this is just the essential assumption employed in the construction of wave packets, i.e., that the wave packets are supposed *large* on the microscopic scale of the particle wavelength h/p. Note, too, that the largest shadow wavelength will occur for the lightest shadow mass, which may be quite large in nature. Hence, the standing-wave components are irrelevant for the macroscopic justification of the S matrix. The usual wave-packet construction for the remaining component,

$$\phi_0^1(p, r) \simeq \tfrac{1}{2} i (e^{-ikr} - S_\lambda e^{ikr}),$$

assures one that S_λ is indeed the physical S matrix.

Some additional physical comments are also appropriate. The asymptotic physical interpolating field for the scattering state, unlike in normal scattering theory, is not only composed of free-field components from normal mesons but also contains standing-wave parts from the shadow mesons with amplitudes proportional to the strength of the source. This action at a distance is, as we have shown, a small effect in terms of the wave packet but a large one in ridding field theory of its infinities – by solving the problem of interpretation of the negative-metric states in indefinite-metric theories.

Concerning the physical wave functions for the shadow, $\phi_\lambda^2(w)$, and shadow ghost states, $\phi_\lambda^{2'}(w)$, recall that they are *purely real* in the energy representation. They are also *purely real* in the (partial-wave-reduced) coordinate representation, and hence, when multiplied by e^{-iEt}, they form waves whose nodes are *fixed in space*, i.e., they are standing waves. Their asymptotic form can be determined as was done above for the physical scattering state, $\phi_\lambda^1(w)$. Generalizing the concept of an S matrix to such objects, the formal S matrix used in the preceding section seems most appropriate: Their diagonal elements are unity (i.e., there is no scattering for nothing is traveling) and their off-diagonal elements vanish (i.e., they do not overlap with the physical states and do not appear in the unitarity relation for the physical states).

Finally, since the scattering amplitude can be computed by computing the singular outgoing part of the meson wave function in the physical channel(s), one can directly study formally the S matrix of shadow theories by integral equations or other means which do not explicitly involve the corresponding wave functions but only their boundary conditions. Because of the physical arguments given in this section, such a formal approach is justified.

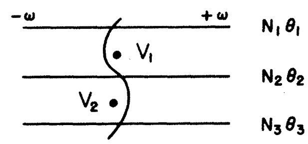

FIG. 4. Three-channel shadow diagram.

V. MODEL WITH THREE CHANNELS

We begin our analysis of the dynamical effects of shadow states in multichannel processes with a simple three-channel soluble model. The complementary roles of the shadow channel(s) and shadow ghost channel(s) are especially interesting here. Figure 4 shows the noninteraction spectrum in this model and the couplings between the fields which produce the interactions

$$N_1\theta_1 \rightleftarrows V_1 \rightleftarrows N_2\theta_2 \rightleftarrows V_2 \rightleftarrows N_3\theta_3\,. \tag{5.1}$$

The Hamiltonian is

$$H=H_0+H_I\,, \tag{5.2}$$

where, with $r=1, 2, 3$,

$$H_0=m_1^0\int d\vec{p}\,V_1^\dagger(\vec{p})V_1(\vec{p})+m_2^0\int d\vec{p}\,V_2^\dagger(\vec{p})V_2(\vec{p})+\sum_i\int d\vec{k}\,\omega_i(k)a_i^\dagger(\vec{k})a_i(\vec{k})\,, \tag{5.3}$$

$$\begin{aligned}H_I=&\int\!\!\int d\vec{p}\,d\vec{k}\,\frac{F_1(\vec{p},\vec{k})}{(2\omega_1)^{1/2}}[V_1^\dagger(\vec{p})N_1(\vec{p}-\vec{k})a_1(\vec{k})+N_1^\dagger(\vec{p}-\vec{k})V_1(\vec{p})a_1^\dagger(\vec{k})]\\&+\int\!\!\int d\vec{p}\,d\vec{k}\bigg(\frac{F_2(\vec{p},\vec{k})}{(2\omega_2)^{1/2}}[V_1^\dagger(\vec{p})N_2(\vec{p}-\vec{k})a_2(\vec{k})+N_2^\dagger(\vec{p}-\vec{k})V_1(\vec{p})a_2^\dagger(\vec{k})]\\&\qquad+\frac{G_2(\vec{p},\vec{k})}{(2\omega_2)^{1/2}}[V_2^\dagger(\vec{p})N_2(\vec{p}-\vec{k})a_2(\vec{k})+N_2^\dagger(\vec{p}-\vec{k})V_2(\vec{p})a_2^\dagger(\vec{k})]\bigg)\\&+\int\!\!\int d\vec{p}\,d\vec{k}\,\frac{G_1(\vec{p},\vec{k})}{(2\omega_3)^{1/2}}[V_2^\dagger(\vec{p})N_3(\vec{p}-\vec{k})a_3(\vec{k})+N_3^\dagger(\vec{p}-\vec{k})V_2(\vec{p})a_3^\dagger(\vec{k})]\,.\end{aligned} \tag{5.4}$$

The unrenormalized energies for the fermion V fields are $E_1(\vec{p})=m_1^0$, $E_2(\vec{p})=-m_2^0$, and for the N fields for convenience we take $E_i^N(\vec{p})=0$ with no loss in generality since this is a static model. Again we take the unrenormalized meson energies to be

$$\begin{aligned}&w_i(k)=\epsilon(w_i)\omega_i(k), \quad i=1,2,3\\&\omega_i(k)=(\mu^2+\vec{k}^2)^{1/2}\end{aligned} \tag{5.5}$$

so that w_i runs over the two ranges, $-\infty<w_i<-\mu$ and $\mu<w_i<\infty$, with $\epsilon(w_i)$ being the sign function taking the values ±1, respectively. The commutation relations for the fields are

$$\begin{aligned}&[V_1(\vec{p}),V_1^\dagger(\vec{p}\,')]_+=\delta(\vec{p}-\vec{p}\,'),\\&[V_2(\vec{p}),V_2^\dagger(\vec{p}\,')]_+=-\delta(\vec{p}-\vec{p}\,'),\\&[N_i(\vec{p}),N_j^\dagger(\vec{p}\,')]_+=\delta_{ij}\delta(\vec{p}-\vec{p}\,'),\\&[a_i(\vec{k}),a_i^\dagger(\vec{k}\,')]=\epsilon(w_i)\delta(\vec{k}-\vec{k}\,'),\end{aligned} \tag{5.6}$$

and all others vanish. The physical space is an indefinite-metric space with the metric operator

$$\eta=\exp\left[-i\pi\left(\int d\vec{p}\,V_2^\dagger(\vec{p})V_2(\vec{p})+\sum_i\int d\vec{k}\,\theta(-w_i)a_i^\dagger(\vec{k})a_i(\vec{k})\right)\right], \tag{5.7}$$

where $\theta(x)=1$ for $x>0$, zero otherwise. Notice that the negative-energy mesons have a negative metric which, as discussed in Sec. II, is consistent with the covariant scalar product of the Klein-Gordon equation. Since the masses and form factors are all real, the Hamiltonian satisfies the pseudo-Hermiticity condition

$$\eta H^*\eta=H. \tag{5.8}$$

There are two operators which commute with the Hamiltonian and are therefore constants of the motion,

$$B=\int d\vec{p}\left(V_1^\dagger(\vec{p})V_1(\vec{p})-V_2^\dagger(\vec{p})V_2(\vec{p})+\sum_i N_i^\dagger(\vec{p})N_i(\vec{p})\right), \tag{5.9}$$

$$Q=\int d\vec{p}\,[V_1^\dagger(\vec{p})V_1(\vec{p})-V_2^\dagger(\vec{p})V_2(\vec{p})]+\sum\int d\vec{k}\,\epsilon(w_i)a_i^\dagger(\vec{k})a_i(\vec{k}). \tag{5.10}$$

The lowest few sectors are trivial since the state vectors coincide for the bare and physical vacuum, the one-N fermion states, and the one-meson states, so we consider first the $N\theta$ scattering sector $\{B=1,\ Q=1\}$.

In order that the model be explicitly soluble, we make two simplifying assumptions: that the form factors factorize

$$F_I(\vec{p},\vec{k})=\overline{F}_I(\vec{p})F_I(\vec{k}), \tag{5.11}$$

$$G_I(\vec{p},\vec{k})=\overline{G}_I(\vec{p})G_I(\vec{k}),\quad I=1,2 \tag{5.12}$$

and that the form factors for the V fields are universal, i.e., $\overline{F}_I(\vec{p})=u_1(\vec{p})$, $\overline{G}_I(\vec{p})=u_2(\vec{p})$. With these assumptions, the model is soluble in all partial waves.

The most general state for the $N\theta$ scattering sector can be represented by

$$|\Phi(\vec{p})\rangle=\sum_I C_I u_I(\vec{p})V_I^\dagger(\vec{p})|0\rangle+\sum_i\int d\vec{k}\,\phi^i(\vec{k})N_i^\dagger(\vec{p}-\vec{k})a_i^\dagger(\vec{k})|0\rangle, \tag{5.13}$$

which we denote by

$$|\Phi\rangle=\left|\begin{matrix}\phi^1(\vec{k})\\ C_1\\ \phi^2(\vec{k})\\ C_2\\ \phi^3(\vec{k})\end{matrix}\right\rangle. \tag{5.14}$$

Applying the Hamiltonian, we have

$$(H_0+H_I)|\Phi\rangle=\lambda|\Phi\rangle \tag{5.15}$$

or

$$\begin{aligned}(\lambda-w)\phi^1(\vec{k})&=\frac{C_1F_1(\vec{k})}{(2\omega)^{1/2}},\\ (\lambda-m_1^0)C_1&=\int d\vec{k}\,\epsilon(w)\frac{F_1(\vec{k})}{(2\omega)^{1/2}}\phi^1(\vec{k})+\int d\vec{k}\,\epsilon(w)\frac{F_2(\vec{k})}{(2\omega)^{1/2}}\phi^2(\vec{k}),\\ (\lambda-w)\phi^2(\vec{k})&=C_1\frac{F_2(\vec{k})}{(2\omega)^{1/2}}-C_2\frac{G_2(\vec{k})}{(2\omega)^{1/2}},\\ (\lambda+m_2^0)C_2&=\int d\vec{k}\,\epsilon(w)\frac{G_2(\vec{k})}{(2\omega)^{1/2}}\phi^2(\vec{k})+\int d\vec{k}\,\epsilon(w)\frac{G_1(\vec{k})}{(2\omega)^{1/2}}\phi^3(\vec{k}),\\ (\lambda-w)\phi^3(\vec{k})&=\frac{-C_2G_1(\vec{k})}{(2\omega)^{1/2}}.\end{aligned} \tag{5.16}$$

Now, it is simplest to work in a spherical basis, so we define

$$a_{ilm}(k)\equiv i^l\int d\Omega\,Y_{lm}^*(\theta,\phi)a_i(\vec{k}). \tag{5.17}$$

Thus,

$$[a_{ilm}(k),a_{i'l'm'}^\dagger(k')]=\frac{\epsilon(w)}{k^2}\delta_{ii'}\delta_{ll'}\delta_{mm'}\delta(k-k') \tag{5.18}$$

with the metric operator

$$\eta=\exp\left[-i\pi\left(\int d\vec{p}\,V_2^\dagger(\vec{p})V_2(\vec{p})+\sum_i\int k^2dk\,\theta(-w)\sum_{lm}a_{ilm}^\dagger(k)a_{ilm}(k)\right)\right]. \tag{5.19}$$

The Y_{lm}'s are normalized conventionally,

$$\int d\Omega\,Y_{lm}^*(\theta,\phi)Y_{l'm'}(\theta,\phi)=\delta_{ll'}\delta_{mm'}, \tag{5.20}$$

$$\sum_{lm}Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi')=\delta(\cos\theta-\cos\theta')\delta(\phi-\phi').$$

From Eqs. (5.13)–(5.16) it is apparent that with no loss in generality we may confine our calculation to θ particles in the "lm" partial wave. Doing this, we have

$$\phi^i(\vec{k}) \to \frac{1}{k} Y_{lm}(\theta,\phi)\phi^i_{lm}(w), \quad F_I(\vec{k}) \to Y_{lm}(\theta,\phi) f_I^{lm}(w), \tag{5.21}$$

and likewise for $G_I(\vec{k})$ and $g_I^{lm}(w)$. Then in a convenient matrix notation the eigenvalue equation becomes (cf. the shadow diagram of Fig. 4)

$$\int k^2 dk \begin{bmatrix} \frac{1}{k}\delta(w-\xi) & \frac{f_1(\xi)k}{(2\omega)^{1/2}} & 0 & 0 & 0 \\ \frac{\epsilon(\xi)f_1(\xi)}{k(2\omega)^{1/2}} & m_1^0 & \frac{\epsilon(\xi)f_2(\xi)}{k(2\omega)^{1/2}} & 0 & 0 \\ 0 & \frac{f_2(\xi)k}{(2\omega)^{1/2}} & \frac{1}{k}\delta(w-\xi) & -\frac{g_2(\xi)k}{(2\omega)^{1/2}} & 0 \\ 0 & 0 & \frac{\epsilon(\xi)g_2(\xi)}{k(2\omega)^{1/2}} & -m_2^0 & \frac{\epsilon(\xi)g_1(\xi)}{k(2\omega)^{1/2}} \\ 0 & 0 & 0 & -\frac{g_1(\xi)k}{(2\omega)^{1/2}} & \frac{1}{k}\delta(w-\xi) \end{bmatrix} \begin{bmatrix} \phi^1_\lambda(\xi) \\ C_1 \\ \phi^2_\lambda(\xi) \\ C_2 \\ \phi^3_\lambda(\xi) \end{bmatrix} = \lambda \begin{bmatrix} \phi^1_\lambda(\xi) \\ C_1 \\ \phi^2_\lambda(\xi) \\ C_2 \\ \phi^3_\lambda(\xi) \end{bmatrix}, \tag{5.22}$$

where we have suppressed the lm superscripts and subscripts. Notice that the Hamiltonian matrix is pseudo-Hermitian with respect to the diagonal metric matrix

$$\eta = \mathrm{diag}\left(\epsilon(w)\delta(w-w')\left|\frac{w}{k}\right|, 1, \epsilon(w)\delta(w-w')\left|\frac{w}{k}\right|, -1, \epsilon(w)\delta(w-w')\left|\frac{w}{k}\right|\right). \tag{5.23}$$

The only single-particle states which differ from the bare-particle states are the V_1 bound state and the V_2 ghost state with discrete eigenvalues determined respectively by

$$\eta(\Lambda)=0, \quad 0<\Lambda<\mu$$
$$\eta(K)=0, \quad -\mu<K<0 \tag{5.24}$$

where

$$\eta(\lambda)=\tau(\lambda)-\frac{\sigma(\lambda)\nu(\lambda)}{\tau(\lambda)} \tag{5.25}$$

is given in terms of several useful spectral functions. These representations, which will be used to express the physical state vectors in a simple rational form, are

$$\sigma^\pm(\lambda) = -\lambda + m_1^0 + \int k^2 dk \,\frac{\epsilon(\xi)[f_1(\xi)^2+f_2(\xi)^2]}{(2\omega)(\lambda-\xi\pm i\epsilon)},$$

$$\nu^\pm(\lambda) = \lambda + m_2^0 + \int k^2 dk \,\frac{\epsilon(\xi)[g_1(\xi)^2+g_2(\xi)^2]}{(2\omega)(\lambda-\xi\pm i\epsilon)}, \tag{5.26}$$

$$\tau^\pm(\lambda) = \int k^2 dk \,\frac{\epsilon(\xi)f_2(\xi)g_2(\xi)}{(2\omega)(\lambda-\xi\pm i\epsilon)}.$$

Explicitly, the single-particle state vectors are given by

$$V^1_\Lambda(w) = C_1 \begin{bmatrix} \frac{f_1(w)k}{(2\omega)(\Lambda-w)} \\ 1 \\ \frac{[f_2(w)+(\sigma/\tau)g_2(w)]k}{(2\omega)^{1/2}(\Lambda-w)} \\ -\sigma/\tau \\ \frac{\sigma g_1(w)k}{\tau(2\omega)^{1/2}(\lambda-w)} \end{bmatrix}, \tag{5.27}$$

$$V^2_K(w) = C_2 \begin{bmatrix} \frac{\nu f_1(w)k}{\tau(2\omega)^{1/2}(K-w)} \\ -\nu/\tau \\ -\frac{[g_2(w)+(\nu/\tau)f_2(w)]k}{(2\omega)^{1/2}(K-w)} \\ 1 \\ \frac{g_1(w)k}{(2\omega)^{1/2}(K-w)} \end{bmatrix}, \tag{5.28}$$

where for proper normalization

$$|C_1|^2 = \frac{\nu(\Lambda)}{\tau(\Lambda)\eta'(\Lambda)}, \quad |C_2|^2 = -\frac{\tau(K)}{\nu(K)\eta'(K)}. \tag{5.29}$$

Their respective eigenvalue equations are

$$\tau^2(\Lambda)-\sigma(\Lambda)\nu(\Lambda)=0, \quad \tau^2(K)-\sigma(K)\nu(K)=0. \tag{5.30}$$

A. Scattering Sector: One Shadow Channel

Again, as in Sec. III, we introduce the notion of shadow states by the choice of boundary conditions. We will first consider the case when there is but one shadow channel.

Suppose, say, channel 3 contains a shadow state. Then for an incident plane wave in channel 1, we have, for $\lambda > 0$,

$$\begin{aligned}
&\phi_\lambda^1(w) \sim \text{plane wave plus outgoing wave},\\
&\phi_\lambda^2(w) \sim \text{outgoing wave},\\
&\phi_\lambda^3(w) \sim \text{principal-value Green's function},\\
&\phi_\lambda^{3'}(w) \sim \text{noninteracting standing wave},
\end{aligned} \tag{5.31}$$

or, explicitly,

$$\begin{aligned}
\phi_\lambda^1(w) &= \delta(w-\lambda) + \frac{C_1 f_1(w)k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)},\\
\phi_\lambda^2(w) &= \frac{[C_1 f_2(w) - C_2 g_2(w)]k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)},\\
\phi_\lambda^3(w) &= -\mathcal{P}\frac{C_2 g_1(w)k}{(2\omega)^{1/2}(\lambda - w)},\\
\phi_\lambda^{3'}(w) &= -\frac{\pi C_2 g_1(\lambda)k}{(2\lambda)^{1/2}}\delta(\lambda - w),
\end{aligned} \tag{5.32}$$

where

$$H' = \begin{pmatrix} H & 0 \\ 0 & (1/k)\delta(w-\xi) \end{pmatrix},$$
$$\eta' = \begin{pmatrix} \eta & 0 \\ 0 & -\epsilon(w)\delta(w-w')|w/k| \end{pmatrix} \tag{5.33}$$

define the shadow ghost channel. To express the physical state vectors simply in a rational form, also define

$$\xi^\pm = \tau^\pm - \frac{\sigma^\pm \mu^\pm}{\tau^\pm}, \tag{5.34}$$

where

$$\mu^\pm(\lambda) = \lambda + m_2^0 + \int k^2 dk\, \frac{\epsilon(\xi) g_2{}^2(\xi)}{(2\omega)(\lambda - \xi \pm i\epsilon)} + \mathcal{P}\int k^2 dk\, \frac{\epsilon(\xi) g_1{}^2(\xi)}{(2\omega)(\lambda - \xi)}. \tag{5.35}$$

Then we substitute (5.32) into (5.22), obtaining the properly normalized state vector

$$\Phi_\lambda^1(w) = \begin{bmatrix} \delta(w-\lambda) + \dfrac{F^+\mu^+ f_1(w)k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ F^+\mu^+ \\ \dfrac{F^+[\mu^+ f_2(w) + \tau^+ g_2(w)]k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ -F^+\tau^+ \\ \mathcal{P}\dfrac{F^+\tau^+ g_1(w)k}{(2\omega)^{1/2}(\lambda - \omega)} \\ \dfrac{\pi F^+\tau^+ g_1 k}{(2\lambda)^{1/2}}\delta(w-\lambda) \end{bmatrix}, \tag{5.36}$$

where

$$F^+ = \frac{\epsilon f_1 \lambda}{\tau^+ \xi^+ (2\lambda)^{1/2}}. \tag{5.37}$$

Similarly for channel 2 which is also physical

$$\Phi_\lambda^2(w) = \begin{bmatrix} \dfrac{H^+ f_1(w)k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ H^+ \\ \delta(w-\lambda) + \dfrac{[H^+ f_2(w) + G^+ g_2(w)]k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ -G^+ \\ \mathcal{P}\dfrac{G^+ g_1(w)k}{(2\omega)^{1/2}(\lambda - w)} \\ \dfrac{\pi G^+ g_1 k}{(2\lambda)^{1/2}}\delta(w-\lambda) \end{bmatrix}, \tag{5.38}$$

where

$$\begin{aligned}
G^+ &= \frac{\epsilon(\tau^+ f_2 + \sigma^+ g_2)\lambda}{\tau^+ \xi^+ (2\lambda)^{1/2}},\\
H^+ &= \frac{\epsilon(\mu^+ f_2 + \tau^+ g_2)\lambda}{\tau^+ \xi^+ (2\lambda)^{1/2}}.
\end{aligned} \tag{5.39}$$

These state vectors are orthogonal[12] with the scalar product given by

$$\begin{aligned}
\langle \Phi_\lambda^I | \Phi_{\lambda'}^{I'} \rangle &= \int dw\, \phi^I(w)^\dagger \eta(w) \phi^{I'}(w)\\
&= C_1^* C_{1'} - C_2^* C_2\\
&\quad + \sum_i \int_{-\infty}^{+\infty} \left|\frac{w}{k}\right| d\omega\, \epsilon(w) \phi_\lambda^i(\omega)^* \phi_{\lambda'}^i(\omega)\\
&\quad - \int_{-\infty}^{\infty} \left|\frac{w}{k}\right| dw\, \epsilon(w) \phi_\lambda^{3'}(w)^* \phi_{\lambda'}^{3'}(w)
\end{aligned} \tag{5.40}$$

and are also orthogonal to the bound state, V_Λ^1, and ghost, V_K^2.

For these physical scattering states, then, the S matrix for the positive-frequency solutions can be found by (see discussion in Secs. II and IV)

$$S_{II'} = \langle \text{out state}, I \,|\, \text{in state}, I' \rangle = \int_{-\infty}^{\infty} dw\, \phi^I(w)\eta(w)\phi^{I'}(w)\,, \tag{5.41}$$

and for

$$S = 1 + i2T$$

we obtain

$$T = -\frac{\pi k}{2\tau^+\xi^+} \times \begin{pmatrix} \mu^+ f_1^{\,2} & f_1(\mu^+ f_2 + \tau^+ g_2) \\ f_1(\mu^+ f_2 + \tau^+ g_2) & \mu^+ f_2^{\,2} + 2\tau^+ f_2 g_2 + \sigma^+ g_2^{\,2} \end{pmatrix}, \tag{5.42}$$

where $k = (\lambda^2 - \mu^2)^{1/2}$. It is easy to verify using (5.26) and (5.35) that T is unitary, that is,

$$TT^+ = \tfrac{1}{2} i (T^+ - T)\,. \tag{5.43}$$

Renormalization can again be carried out. For complex λ, ν is the piecewise-analytic extension of μ from the domain $\lambda \in \{[-\infty, -\mu], [\mu, \infty]\}$. To simplify the algebra now, and to make contact with the substitution law later, we assume the symmetric relationship

$$f_{1,2}(\xi)^2 = g_{1,2}(\pm\xi)^2\,, \quad m_1^0 = m_2^0 \tag{5.44}$$

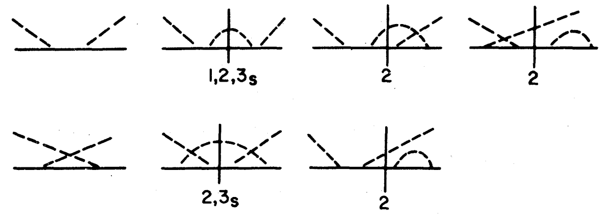

FIG. 5. Field-theoretic equivalents, to the first and second order, for elastic scattering in channel 2.

holds. Then it follows from Eq. (5.30) that $m_V = \Lambda = -K$. The spectral representations can be reexpressed as

$$\left(m_1^0 = M - \int k^2 dk \frac{\epsilon(f_1^{\,2} + f_2^{\,2})}{(2\omega)(m_V - \xi)}\right)$$

$$\sigma = -\lambda + M + (m_V - \lambda)\int k^2 dk \frac{\epsilon(f_1^{\,2} + f_2^{\,2})}{(2\omega)(\xi - \lambda)(\xi - m_V)}\,,$$

$$\nu = \lambda + M + (m_V - \lambda)\int k^2 dk \frac{\epsilon(g_1^{\,2} + g_2^{\,2})}{(2\omega)(\xi - \lambda)(\xi - m_V)}\,, \tag{5.45}$$

$$\tau = \pm \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - \lambda)} \quad \text{as } f_2 = \pm g_2\,,$$

where

$$M \equiv \left[m_V^{\,2} + \left(\int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - m_V)} \right)^2 \right]^{1/2}. \tag{5.46}$$

The important denominator spectral function is

$$\begin{aligned} D &\equiv \tau^2 - \nu\sigma \\ &= (\lambda - m_V)\Bigg[\lambda + m_V + \left(2M + \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - \lambda)} + \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - m_V)} \right) \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - \lambda)(\xi - m_V)} \\ &\qquad + 2M \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - \lambda)(\xi - m_V)} + (m_V - \lambda)\left(\int k^2 dk \frac{\epsilon(f_1^{\,2} + f_2^{\,2})}{(2\omega)(\xi - \lambda)(\xi - m_V)} \right)^2 \Bigg], \end{aligned} \tag{5.47}$$

which is symmetric under $m_V \leftrightarrow -m_V$ and also under $\lambda \leftrightarrow -\lambda$. Thus, from Eq. (5.42) we have the corresponding elastic amplitudes

$$\begin{aligned} T_{11}(\lambda) &= -\frac{\pi f_1^{\,2} k}{2D}\left(\lambda + M + (m_V - \lambda)\int k^2 dk \frac{\epsilon(f_1^{\,2} + f_2^{\,2})}{(2\omega)(\xi - \lambda)(\xi - m_V)} \right), \\ T_{22}(\lambda) &= -\frac{\pi f_2^{\,2} k}{D}\left(M + \int k^2 dk \frac{\epsilon f_2^{\,2}}{(2\omega)(\xi - m_V)} + (m_V - \lambda)\int k^2 dk \frac{\epsilon(f_1^{\,2} + f_2^{\,2})}{(2\omega)(\xi - \lambda)(\xi - m_V)} \right). \end{aligned} \tag{5.48}$$

Requiring

$$T_{11} \underset{\lambda \to m_V}{\sim} \frac{\pi f_1^{r\,2} k}{2(m_V - \lambda)}$$

and

$$T_{22} \underset{\lambda \to m_V}{\sim} \frac{\pi f_2^{r\,2} k}{2(m_V - \lambda)}\,, \tag{5.49}$$

the renormalized coupling constants are given by

$$f_1^{r\,2}=\frac{f_1^{\,2}(M+m_V)}{2\left[\Lambda+\left(M+\int k^2dk\,\frac{\epsilon f_2^{\,2}}{\xi-m_V}\right)\int k^2dk\,\frac{\epsilon f_2^{\,2}}{(2\omega)(\xi-m_V)^2}+M\int k^2dk\,\frac{\epsilon f_1^{\,2}}{(2\omega)(\xi-m_V)^2}\right]},$$

$$f_2^{r\,2}=\frac{f_2^{\,2}[M+\int k^2dk\epsilon f_2^{\,2}/(\xi-m_V)]}{\Lambda+\left(M+\int k^2dk\,\frac{\epsilon f_2^{\,2}}{\xi-m_V}\right)\int k^2dk\,\frac{\epsilon f_2^{\,2}}{(2\omega)(\xi-m_V)^2}+M\int k^2dk\,\frac{\epsilon f_1^{\,2}}{(2\omega)(\xi-m_V)^2}}. \tag{5.50}$$

It then follows that

$$T_{22}\underset{\lambda\to-m_V}{\sim}\frac{\pi f_2^{r\,2}k}{2(m_V+\lambda)} \tag{5.51}$$

and that the inelastic Born term is

$$T_{12}\underset{\lambda\to m_V}{\sim}\frac{\pi f_1^r f_2^r}{2(m_V-\lambda)}. \tag{5.52}$$

These same expressions are obtained in the weak coupling limit in terms of the bare coupling constants, and in that limit the second-order expansion terms involve both the physical channels and the shadow channels. In Fig. 5 the field-theoretic equivalents to the first- and second-order diagrams for elastic scattering in channel 2 are given.

Again it is interesting to notice that a *purely real* state vector corresponding to the shadow channel can be constructed which is orthogonal to each of the physical $N\theta$ sector states. The normalized state vector is

$$\Phi_\lambda^S(w)=\frac{1}{\sqrt{N_S}}\begin{bmatrix}\mathrm{Re}\left(\dfrac{N^-f_1(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)}\right)\\ \mathrm{Re}(N^-)\\ \dfrac{g_2}{g_1}\left(1-\dfrac{\eta^+\eta^-}{\xi^+\xi^-}\right)\delta(\lambda-w)+\mathrm{Re}\left(\dfrac{N^-f_2(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)}\right)+\mathrm{Re}\left(\dfrac{\sigma^-I^-g_2(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)}\right)\\ -\mathrm{Re}(\sigma^-I^-)\\ \delta(\lambda-w)+\mathrm{Re}\left(\dfrac{\sigma^-I^-g_1(w)k}{(2\omega)^{1/2}(\lambda-w+i\epsilon)}\right)\end{bmatrix}, \tag{5.53}$$

where

$$I^+=\frac{\epsilon g_1\lambda}{\tau^+\xi^+(2\lambda)^{1/2}},$$

$$N^+=\frac{\epsilon g_1\lambda}{\xi^+\xi^-(2\lambda)^{1/2}}, \tag{5.54}$$

and

$$N_S=[1+\pi^2g_+^{\,2}(\tau^+\tau^--F^+F^-+G^+G^-)\tfrac{1}{2}k^2]\left(\frac{\eta^+\eta^-}{\xi^+\xi^-}\right). \tag{5.55}$$

The wave functions can be verified to be complete where now both the V_K^1 bound state and V_K^2 ghost state are included in the generalized Møller matrix along with the ghost shadow (negative-metric) state. Like the shadow state, it is orthonormal and *purely real* and is given by

$$\Phi_\lambda^G(w) = \left(\frac{1}{N_G}\right)^{1/2} \begin{bmatrix} \mathrm{Re}\left(\dfrac{i\tau^+ I^+ f_1(w) k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)}\right) \\ \mathrm{Re}(i\tau^+ I^+) \\ \mathrm{Re}\left(\dfrac{iI^+[\tau^+ f_2(w) + \sigma^+ g_2(w)]\, k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)}\right) \\ -\mathrm{Re}(i\sigma^+ I^+) \\ \mathrm{Re}(i\sigma^+ I^+)\,\mathcal{P}\left(\dfrac{g_1(w) k}{(2\omega)^{1/2}(\lambda - w)}\right) \\ -\frac{1}{2}\epsilon N_G k \delta(\lambda - w) \end{bmatrix}, \tag{5.56}$$

where

$$N_G = 1 + \pi^2 g_1{}^2 (\tau^+ F^+ \tau^- F^- + G^+ G^-) \tfrac{1}{2} k^2 . \tag{5.57}$$

B. Scattering Sector: Two Shadow Channels

We consider the case when channels 1 and 2 contain shadow states and channel 3 contains a physical state. When a physical plane wave is incident, the boundary conditions are

$$\begin{aligned} &\chi_\lambda^{1,2}(w) \sim \text{principal-value Green's function,} \\ &\chi_\lambda^{3}(w) \sim \text{plane wave plus } +i\epsilon \text{ prescription,} \\ &\chi_\lambda^{1',2'}(w) \sim \text{noninteracting standing wave.} \end{aligned} \tag{5.58}$$

Define

$$\rho^\pm = \lambda + m_2^0 + \int k^2 dk \frac{\epsilon(\xi) g_1{}^2(\xi)}{(2\omega)(\lambda - \xi \pm i\epsilon)} + \mathcal{P}\int k^2 dk \frac{\epsilon(\xi) g_2{}^2(\xi)}{(2\omega)(\lambda - \xi)} \tag{5.59}$$

and

$$\zeta^\pm = \tau^P - \frac{\sigma^P \rho^\pm}{\tau^P}, \tag{5.60}$$

where the superscript P denotes the principal value, i.e., $\tau^P = \frac{1}{2}(\tau^+ + \tau^-)$. Substitute (5.58) into (5.22) to obtain again the properly normalized state vector

$$X_\lambda^3(w) = \begin{bmatrix} -\mathcal{P}\dfrac{M^+ \tau^P f_1(w) k}{(2\omega)^{1/2}(\lambda - w)} \\ -M^+ \tau^P \\ -\mathcal{P}\dfrac{M^+[\tau^P f_2(w) + \sigma^P g_2(w)]\, k}{(2\omega)^{1/2}(\lambda - w)} \\ M^+ \sigma^P \\ \delta(\lambda - w) + \dfrac{M^+ \sigma^P g_1(w) k}{(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ -\dfrac{\pi M^+ \tau^P f_1 k}{(2\lambda)^{1/2}} \delta(\lambda - w) \\ -\dfrac{\pi M^+ (\tau^P f_2 + \sigma^+ g_2) k}{(2\lambda)^{1/2}} \delta(\lambda - w) \end{bmatrix} \tag{5.61}$$

where

$$M^{+}=\frac{\epsilon g_{+}\lambda}{(2\lambda)^{1/2}\tau^{P}\zeta^{+}}, \tag{5.62}$$

which is orthogonal to the V-particle states. The corresponding T-matrix element is

$$T=-\frac{\pi\sigma^{P}g_{1}^{2}k}{2\tau^{P}\zeta^{+}}, \tag{5.63}$$

and it is unitary.

In terms of the renormalized quantities

$$T(\lambda)\underset{\lambda\to -m_V}{\simeq}\frac{\pi f_1^{r\,2}k}{2(m_V+\lambda)}, \tag{5.64}$$

where

$$T(\lambda)=\frac{\pi f_1^{r\,2}(M+m_V)k}{4(m_V+\lambda)\left[m_V+\left(M+\int k^2dk\,\frac{\epsilon f_2^{\,2}}{(2\omega)(\xi+m_V)}\right)\int k^2dk\,\frac{\epsilon f_2^{\,2}}{(2\omega)(\xi-m_V)^2}+M\int k^2dk\,\frac{\epsilon f_1^{\,2}}{(2\omega)(\xi-m_V)^2}\right]}. \tag{5.65}$$

VI. SHADOW STATES AND THE SUBSTITUTION LAW

In this section we wish to discuss the role of the substitution law in the presence of shadow states. Historically, in radiation theory special cases of this law were first recognized,[13] e.g., the relation between pair production and bremsstrahlung, and then quite some time later in the framework of quantum electrodynamics it was appreciated that this law is a general consequence of the structure of S-matrix elements as obtained by iterative solution.[7] Not so well known is that the law also follows in simple soluble models having meson fields of both positive and negative energy, say, as in the conventional charged scalar theory of $\pi^- p$ and $\pi^+ p$ elastic scattering. (We briefly review it here as presented in Ref. 6.) In Fig. 6 this theory is described by two of our diagrams which are related by the substitution law; in fact, they are crossing-symmetric. For the first diagram, the $\pi^- p$ scattering amplitude is given by

$$T_{\pi^- p}(\lambda)=\frac{\pi k f_-(\lambda)^2}{2\alpha^+(\lambda)}, \tag{6.1}$$

where

$$\alpha^{\pm}(\lambda)=-\lambda+m^0+\int k^2dk\,\frac{\epsilon(\xi)f_-(\xi)^2}{(2\omega)(\lambda-\xi\pm i\epsilon)}, \tag{6.2}$$

and for the second diagram, the $\pi^+ p$ amplitude is

$$T_{\pi^+ p}(\lambda)=\frac{\pi k f_+(\lambda)^2}{2\beta^+(\lambda)}, \tag{6.3}$$

where

$$\beta^{\pm}(\lambda)=\lambda+m^0+\int k^2dk\,\frac{\epsilon(\xi)f_+(\xi)^2}{(2\omega)(\lambda-\xi\pm i\epsilon)}, \tag{6.4}$$

and thus by inspection it is seen that

$$T_{\pi^- p}(-\lambda-i\epsilon)=T_{\pi^+ p}(\lambda+i\epsilon) \tag{6.5}$$

does hold for $f_-(\xi)^2=f_+(-\xi)^2$.

The remarkable thing is that when shadow states are introduced in the simplest manner into the charged scalar theory, this substitution-law relationship is maintained. It is natural to assume that both charged processes are dynamically coupled to shadow states, with possible transitions being

$$\begin{aligned}&\pi^- p \rightleftarrows n \rightleftarrows \theta_S N_S(\text{shadow}),\\ &\pi^+ p \rightleftarrows n_G \rightleftarrows \theta_S' N_S'(\text{shadow}),\end{aligned} \tag{6.6}$$

say, as in the two-channel model considered in Sec. III. (n denotes the physical neutron "bound state" and n_G the similar "ghost" or negative-metric state.) From the scattering amplitude which was obtained for the two-channel model, Eq. (3.31), it is seen that the $\pi^- p$ amplitude (6.1) is unchanged when the shadow process is coupled in except for the replacement of $\alpha^+(\lambda)$ with

$$\nu^+(\lambda)=-\lambda+m^0+\int k^2dk\,\frac{\epsilon(\xi)f_-(\xi)^2}{(2\omega)(\lambda-\xi+i\epsilon)}+\mathcal{P}\int k^2dk\,\frac{\epsilon(\xi)f_-'(\xi)^2}{(2\omega)(\lambda-\xi)}. \tag{6.7}$$

FIG. 6. Crossing-symmetric charged scalar theory.

For the $\pi^+ p$ process, n_G is a negative-metric "bound state" of mass $-m^0$ so when shadow states in the two-channel model also mediate this process, the amplitude is

$$T_{\pi^+ p}(\lambda) = \frac{\pi k f_+(\lambda)^2}{2\eta^+(\lambda)}, \tag{6.8}$$

where

$$\eta^+(\lambda) = \lambda + m^0 + \int k^2 dk \frac{\epsilon(\xi) f_+(\xi)^2}{(2\omega)(\lambda - \xi + i\epsilon)} + \mathcal{P}\int k^2 dk \frac{\epsilon(\xi) f_+'(\xi)^2}{(2\omega)(\lambda - \xi)}. \tag{6.9}$$

Notice that η has the same analytic structure as ν and σ as shown in Fig. 2, and is the piecewise-analytic extension of them along the "left-hand cut." Hence,

$$T_{\pi^- p}(-\lambda - i\epsilon) = T_{\pi^+ p}(\lambda + i\epsilon) \tag{6.10}$$

for

$$f_-(\xi)^2 = f_+(-\xi)^2 \text{ and } f_-'(\xi)^2 = f_+'(-\xi)^2.$$

The transformation involved here is one of substitution and *not* analytic continuation since here the thresholds for the physical and shadow channels coincide, and thus the threshold point is a point of nonanalyticity. It should also be noted that while the scattering amplitude is continuous as a function of energy along the real axis since for $\lambda \in [-\mu, \mu]$, $\nu(\lambda)$ has the piecewise-analytic extension $\sigma(\lambda)$ as discussed in Sec. III.

However, it is only in the multichannel situation that we see the full power of the substitution law for intertwining apparently distinct shadow theories. For example, in Sec. V we considered separately two three-channel models; the first, (a), has one shadow channel with the transitions

$$N_1\theta_1 \rightleftarrows V_1 \rightleftarrows N_2\theta_2 \rightleftarrows V_2 \rightleftarrows N_3\theta_3(\text{shadow}), \tag{6.11}$$

and the second, (b), has two shadow channels with the transitions

$$N_1\theta_1(\text{shadow}) \rightleftarrows V_1 \rightleftarrows N_2\theta_2(\text{shadow}) \rightleftarrows V_2 \rightleftarrows N_3\theta_3. \tag{6.12}$$

But we now see that the $N_1\theta_1$ elastic scattering amplitude of case (a),

$$T_{11}(\lambda + i\epsilon) = -\frac{\pi k f_1(\lambda)^2 \mu^+}{2\tau^P \xi^+}, \tag{6.13}$$

is the piecewise-analytic extension of the $N_3\theta_3$ elastic amplitude of case (b),

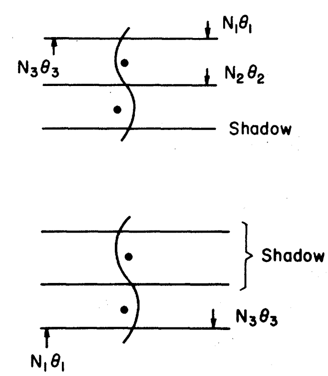

FIG. 7. Three-channel shadow diagrams as related by the substitution law.

$$T_{33}(\lambda + i\epsilon) = -\frac{\pi k g_1(\lambda)^2 \sigma^P}{2\tau^P \xi^+}, \tag{6.14}$$

in the symmetric situation in which $f_{1,2}(\xi)^2 = g_{1,2}(\pm\xi)^2$, i.e., they are related by the substitution law. (See Fig. 7.) Notice that the scattering amplitude is again continuous as a function of energy along the real axis.

VII. MULTICHANNEL GENERALIZATION

We now wish to consider the many-channel situation in which there are an arbitrary number of shadow channels. In order to do this, we construct a model possessing a mathematical structure quite similar to that of the previous ones but which permits a simpler notation. Field-theoretically, this model is the four-point interaction counterpart of those which have already been studied with three-point interactions, so for the ith channel there is again an N_i fermion field and a θ_i meson field of both positive and negative energy; however, now there are no explicit V fields associated with the bound states and/or ghost particles. Thus the possible transitions are

$$N_i\theta_i \rightleftarrows N_j\theta_j \tag{7.1}$$

with conserved quantum numbers

$$B = \sum_i \int d\vec{p}\, N_i^\dagger(\vec{p}) N_i(\vec{p}), \tag{7.2}$$

$$Q = \sum_i \int d\vec{k}\, \epsilon(w_i) a_i^\dagger(\vec{k}) a_i(\vec{k}). \tag{7.3}$$

The Hamiltonian is

$$H = H_0 + H_I, \tag{7.4}$$

where (setting $E_i^N = 0$)

$$H_0=\sum_i \int d\vec{k}\, w_i(k) a_i^\dagger(\vec{k}) a_i(\vec{k}), \tag{7.5}$$

$$H_I=\tfrac{1}{2}\sum_{i,j}\int\int\int d\vec{p}\,d\vec{k}\,d\vec{k}'\,\frac{G_{ij}(\vec{k},\vec{k}')}{2[\omega_i(k)\omega_j(k')]^{1/2}} \times[N_i^\dagger(\vec{p}-\vec{k})a_i^\dagger(\vec{k})N_j(\vec{p}-\vec{k}')a_j(\vec{k}')+N_i(\vec{p}-\vec{k})a_i(\vec{k})N_j^\dagger(\vec{p}-\vec{k}')a_j^\dagger(\vec{k}')] \tag{7.6}$$

with the same commutation relations as before. The metric operator is given by

$$\eta=\exp\left(i\pi\sum_i\int d\vec{k}\,\theta(-w_i)a_i^\dagger(\vec{k})a_i(\vec{k})\right), \tag{7.7}$$

so, since the form factor G_{ij} is purely real,

$$\eta H^*\eta=H. \tag{7.8}$$

Applying the Hamiltonian to the physical $N\theta$ scattering states

$$|\Phi(\vec{p})\rangle=\sum_i\int d\vec{k}\,\phi^i(\vec{k})N_i^\dagger(\vec{p}-\vec{k})\theta_i^\dagger(\vec{k})|0\rangle, \tag{7.9}$$

we obtain

$$[\lambda-w_i(k)]\,\phi^i(\mathrm{k})=\frac{1}{[2\omega_i(k)]^{1/2}}\sum_i\int d\mathrm{k}'\,\frac{\epsilon(w_j')G_{ij}(\vec{k},\vec{k}')}{[2\omega_j(k')]^{1/2}}\,\phi^j(\vec{k}'). \tag{7.10}$$

As before, we assume factorization of the form factor, $G_{ij}(\vec{k},\vec{k}')=G_i(\vec{k})G_j(\vec{k}')$, and then to reduce (7.10), expand the factors in a spherical basis

$$G_i(\mathrm{k})=\sum_{lm}Y_{lm}(\theta,\phi)g_j^{lm}(w). \tag{7.11}$$

Taking $\phi_{lm}^i(w)$'s norm such that

$$\phi_{lm}^i(w)=k\int d\Omega\;Y_{lm}^*(\theta,\phi)\phi^i(\vec{k}), \tag{7.12}$$

we restrict our attention to the lmth partial wave so (6.10) becomes

$$[\lambda-w_i(k)]\,\phi^i(w)=\frac{kg_1(w)}{(2\omega_i)^{1/2}}\sum_j\int k'^2dk'\,\frac{\epsilon(\xi_j)g_j(\xi)\phi^j(\xi)}{(2\omega_j')k'}. \tag{7.13}$$

We now order the channels by their associated boundary conditions

$$i=\begin{cases}a,\ b,\ \ldots & \text{outgoing-wave boundary condition}\\ m,\ \ldots & \text{shadow-wave boundary condition}\end{cases} \tag{7.14}$$

and let (i not summed)

$$x_i(\lambda)\equiv\int k^2dk\,\frac{\epsilon(\xi)g_i(\xi)\phi_i(\lambda,\xi)}{(2\omega_i)^{1/2}k}. \tag{7.15}$$

Then for an incident plane wave in a particular one of the physical channels, let us call this channel A, the solution of Eq. (7.13) is of the form

$$\phi_i^{(A)}(\lambda,w)=\delta_{iA}\delta(\lambda-w)+\delta_{ia}\frac{g_a(w)\sum_j x_j(\lambda)k}{(2\omega_a)^{1/2}(\lambda-w+i\epsilon)}+\delta_{im}\mathcal{P}\frac{g_m(w)\sum_j x_j(\lambda)k}{(2\omega_m)^{1/2}(\lambda-w)}, \tag{7.16}$$

where for $i=a, b, \ldots$ the third term does not appear and for $i=m, \ldots$ the second term does not appear. Substituting into (7.15) and solving the resulting system of equations, we get

$$x_i(\lambda)=\frac{\epsilon(\lambda)g_A(\lambda)\,|\,d_{Ai}^+(\lambda)\,|\,\lambda}{(2\lambda)^{1/2}\,|D^+(\lambda)|}, \tag{7.17}$$

where

$$|D^+| \equiv \det(\delta_{ij} - \alpha^+_{aj} - \beta^P_{mj}) = 1 - \sum_a \alpha^+_{aj} - \sum_m \beta^P_{mj} \tag{7.18}$$

with

$$\alpha^{\pm}_{aj} = \int k^2 dk \frac{\epsilon(\xi) g_a(\xi)^2}{(2\omega_a)(\lambda - \xi \pm i\epsilon)}, \qquad \beta^P_{mj} = \mathcal{P} \int k^2 dk \frac{\epsilon(\xi) g_m(\xi)^2}{(2\omega_m)(\lambda - \xi)}, \quad \forall j \tag{7.19}$$

and where $d^+_{Aj}(\lambda)$ is the cofactor of element D^+_{Aj} in D^+. Now the shadow ghost channels are introduced and the fact that $\sum_j |d^+_{Aj}(\lambda)| = 1$ is used to write the orthonormal set of physical state vectors,

$$\Phi^A_\lambda(w) = \begin{pmatrix} \delta_{aA}\delta(w-\lambda) + \dfrac{\epsilon g_A \lambda g_a(w) k}{|D^+(\lambda)|(2\lambda)^{1/2}(2\omega)^{1/2}(\lambda - w + i\epsilon)} \\ \mathcal{P}\dfrac{\epsilon g_a \lambda g_m(w) k}{|D^+(\lambda)|(2\lambda)^{1/2}(2\omega)^{1/2}(\lambda - w)} \\ \tfrac{1}{2}\epsilon\pi g_A g_m k\, \delta(w-\lambda) \end{pmatrix}. \tag{7.20}$$

By Eq. (5.18) it is easy to verify that these are indeed orthonormal and that the corresponding physical T matrix is given by

$$T^{(A,B)} = -\frac{\pi g_A g_B k}{2|D^+|}. \tag{7.21}$$

It is unitary.

VIII. DISCUSSION

In this paper we have dealt with several models of soluble quantum field theories involving shadow states. They are to serve as prototypes for realistic models of particle-physics phenomena. In these models the scattering amplitude is relativistically invariant and explicitly unitary. The novel feature is the appearance of shadow channels which influence the scattering but which do not interfere with the unitarity of the scattering among the physical channels alone. This obliging nature of shadow states, insofar as they contribute dynamical effects essentially as if they are physical states and not meddling in the probability interpretation of scattering processes, makes them very valuable tools in the construction of a finite relativistic quantum field theory. It has been recognized for some time now that the identification of physical particle states in a quantum field theory is part of the dynamical problem. The models discussed in this paper again call attention to this basic fact.

The choice of a standing-wave boundary condition for the shadow states was shown to be completely consistent with the physical description of the scattering process in terms of wave packets.

We have remarked in the introduction that the substitution law[7] is valid in this theory since scattering at negative energy is a well-defined process. To this extent the situation coincides with standard quantum field theory, and, hence, the scheme is quite different from the one-meson approximation in, say, the Tamm-Dancoff formalism. Of course the one-meson approximation mutilates standard quantum field theory and makes it soluble in closed form; this is reflected in the fact that the states are limited to one-meson states.

The most notable feature of the results is that the scattering amplitude so obtained is an analytic function in almost all neighborhoods of the real energies at which scattering takes place. But it is not the boundary value of a single analytic function. Rather, the function is only piecewise-analytic. Along the real axis the function is continuous as a function of the energy, but in different regions the scattering amplitude takes on the boundary values of different analytic functions. In particular, the onset of the shadow channel is at a junction between two analytic functions. From a systematic study of a variety of theories[3-5,14] we have learned that this is a fundamental property of the scattering amplitude in any theory, mutilated or otherwise, in which we have shadow states.

In the present theory the physical and shadow thresholds coincide; consequently, the threshold is a point of nonanalyticity. The transformations involved in the substitution law are therefore obtained *not* by analytic continuation but by *substitution*. The concept of a master analytic function and the implementation of crossing symmetry by analytic continuation have been the basic postulates of modern S-matrix theory and hence we cannot lightly embrace piecewise-analytic functions: We must systematically examine various consequences of such a step; and also examine the question of experimental evidence for or against such a step.[16] We shall do this in subsequent papers of this series.

Success in applications to particle physics would give added support to such a systematic study. In the next paper we consider the application to low-energy pion-nucleon scattering. This is only a first step since we must at least reproduce the desirable general features of standard S-matrix theory before accepting such a serious step as legitimate.

In this context it is relevant to point out that the concepts of shadow states and indefinite metric have been successfully applied to quantum electrodynamics,[3,15] the one field theory where we have the possibility of quantitative comparisons.

To the extent we have investigated we have not encountered any logical inconsistencies in the formalism developed here, and we have every hope that this is a step in the right direction.

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[10]Both the sum and the difference of the retarded and advanced Green's functions are time-symmetric standing-wave structures. The sum, $G^{p}=\frac{1}{2}(G^{+}+G^{-})$, is the solution of the inhomogeneous equation for a δ-function source and hence is a Green's function. The difference, $G^{\delta}=\frac{1}{2}(G^{+}-G^{-})$, is a solution of the homogenous equation. The associated noninteracting standing-wave component also must be introduced in formal scattering theory with shadow states to properly orthonormalize the physical state vectors. See C. A. Nelson, University of Texas at Austin Report No. CPT-160, 1972 (unpublished).

[10a]This gives the renormalized perturbation expansion for the scattering amplitude in the presence of standing-wave states. It is equivalent to the unrenormalized expansion given in Ref. 4. Such perturbation expansions for scattering amplitudes, and other physical objects, in the presence of shadow states can be resummed in terms of quantities, often more easily calculated, in the associated theory based on purely in and out boundary conditions; see C. A. Nelson, Lett. Nuovo Cimento (to be published).

[11]In momentum space,

$$\phi^1(\vec{p},\vec{k})=(\tfrac{1}{2}\pi)^{3/2}\int d\vec{r}\,e^{-i\vec{k}\cdot\vec{r}}\phi^1(\vec{p},\vec{r}),$$

$$\phi^1(\vec{p},\vec{k})=(\mu p)^{1/2}\,\delta(\vec{k}-\vec{p})+\left(\frac{p}{\mu}\right)^{1/2}\frac{1}{4\pi}\,\frac{2\mu C_\lambda f_1(w)}{(2w)^{1/2}\,(p^2-k^2+i\epsilon)}\;,$$

to be compared with Eq. (3.17).

[12]In verification of the orthonormality it is helpful to make use of the identities

$$\tau^+F^+\tau^-F^-+G^+G^-=\frac{i\lambda}{4\pi}\left(\frac{\sigma^+}{\tau^+\xi^+}-\frac{\sigma^-}{\tau^-\xi^-}\right),$$

$$\mu^+F^+\mu^-F^-+H^+H^-=\frac{i\lambda}{4\pi}\left(\frac{\mu^+}{\tau^+\xi^+}-\frac{\mu^-}{\tau^-\xi^-}\right),$$

$$\tau^+F^+\mu^-F^-+G^+H^-=\frac{i\lambda}{4\pi}\left(\frac{1}{\xi^+}-\frac{1}{\xi^-}\right).$$

Note also that σ, τ, and μ are related by

$$\tau^+F^+\tau^-F^- = \frac{i\lambda}{4\pi\xi^+\xi^-}\left(\sigma^+ - \sigma^- - \frac{(\tau^+-\tau^-)^2}{\mu^+-\mu^-}\right).$$

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Quantum Field Theories with Shadow States. II. Low-Energy Pion-Nucleon Scattering

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By adapting the methods of the preceding paper, we use the concepts of shadow states and indefinite metric to construct a simple static theory of low-energy pion-nucleon scattering. This theory of s- and p-wave scattering so constructed is both finite and exactly soluble. Scattering at negative energy also is a well-defined process. The calculated scattering amplitude is found to satisfy the substitution law and to be covariant, unitary, and analytic in almost all neighborhoods of physical scattering energies. When given the masses and coupling constants as input, this theory predicts the scattering phase shifts in agreement with experiment. The present limits on the s-wave pion-nucleon scattering total cross sections are compatible with the induced cusps from the opening up of the pseudothreshold of the shadow states.

I. INTRODUCTION

The unfinished quantum field theory of low-energy pion-nucleon scattering has had a particularly long history. After all, it was over three decades ago that the meson theory of nuclear forces with Yukawa couplings was created by analogy with electrodynamics[1] and it was as early as 1942 when, on the basis of strong-coupling theory, the suggestion was first made that an isospin-$\frac{3}{2}$ resonance might exist.[2] In spite of many different theoretical attempts since these earliest beginnings, the basic challenge has remained: Construction of a *convergent, divergence-free* theory which when solved exactly predicts the observed experimental parameters of low-energy pion-nucleon scattering approximately. A true quantum field theory for this physical phenomena has not been constructed. As discussed in the preceding paper,[3] the concepts of indefinite metric[4] and shadow states[5] are valuable tools in the construction of a finite relativistic quantum field theory, and therefore we wish to employ them here in our consideration of low-energy pion-nucleon scattering. Before introducing these ideas it is useful to review briefly some relevant aspects of theoretical approaches to this problem in the past.

In the early fifties[6] in the quantum-field-theory approach to this problem a fundamental question concerned the proper field-theoretic interaction to be used. The pseudoscalar interaction was generally preferred over the pseudovector interaction because the pseudoscalar interaction can be made renormalizable by adding a meson-meson interaction to it.[7] A problem still remained though, as to how to carry out calculations when the coupling constant is large. Another aspect in the choice between these couplings was that in lowest-order perturbation theory both types of interaction were found to yield identical results[8] provided the coupling constants satisfied the relation $G/2M = f/\mu$; but then both were wrong in predicting that s-wave scattering should dominate pion-nucleon scattering at low energies.[9] Yet about the same time it was recognized from the analysis of nucleon-nucleon interactions that some additional s-wave interaction was needed, specifically, a term

$$\int d\vec{r}\, \bar{\psi}\psi\phi^2 \qquad (1.1)$$