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Two-Variable Expansions for Three-Body Decays Involving Particles with Arbitrary Spins

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Helicity amplitudes for the decay process $1 \to 2+3+4$ involving particles with arbitrary spins are expanded in terms of matrix elements of the O(4) group transformations. The resulting expansions display the entire dependence on the kinematic parameters (energies and angles) explicitly and can be interpreted as standard O(3) partial-wave expansions of the amplitudes in a center-of-mass-like frame of reference, supplemented by O(4) expansions of the partial-wave helicity amplitudes. Restrictions on the O(4) amplitudes due to parity conservation are established, the physical meaning of all the quantum numbers figuring in the expansion is clarified, and the expansions are shown to have a reasonable threshold and pseudothreshold behavior. The expansion formula, as well as its inverse, involves amplitudes defined in the physical decay region only. The O(4) expansions of this paper are a modification of O(3, 1) expansions of helicity amplitudes for the scattering of particles with arbitrary spins, suggested in the preceding paper. For spinless particles the corresponding expansions have been derived and applied earlier by Hicks and Winternitz.

I. INTRODUCTION

In the preceding article,[1] further to be referred to as I, we have presented and discussed two-variable expansions of helicity amplitudes for the reaction $1+2 \to 3+4$ involving particles with arbitrary positive masses and spins. The helicity amplitudes were expanded in terms of certain transformation matrices of the homogeneous Lorentz group, written in a basis corresponding to the group reduction $O(3,1) \supset O(3) \supset O(2)$. The amplitudes are considered in the center-of-mass frame of reference; the entire dependence on the c.m. scattering angle (as well as on an azimuthal angle θ) is contained in the usual O(3) group D functions, whereas the dependence on the c.m. energy is contained in O(3, 1) d functions (Wigner boosts). The expansions were shown to have correct threshold and pseudothreshold behavior, total angular momentum is diagonalized, restrictions due to parity conservation can be imposed in a trivial manner, and the expansion can be interpreted as the standard Jacob and Wick[2] partial-wave expansion, in which the O(3) little-group partial-wave helicity amplitudes are further expanded in a manner dictated by the incorporation of the rotation group into the Lorentz group $O(3) \subset O(3,1)$.

The expansions of I are a direct and straightforward generalization of the O(3, 1) two-variable expansions considered previously for the scattering of spinless particles (see, e.g., Refs. 3–7). Let us note that a general formalism, incorporating various single-variable and two-variable expansions, was suggested by Feldman and Matthews[8] and was made use of in I.

One type of application of the two-variable expansions that we have in mind is to perform phenomenological fits to experimental data. Since the dependence on both kinematic parameters (energy and scattering angle, the Mandelstam variables s and t, or some other pair of variables) is explicit, we should be able to perform, e.g., a partial-wave analysis of scattering data, simultaneously over a certain energy region (or over all energies). Alternatively one could perform say an s-channel Regge-pole fit simultaneously over a whole region of squared momentum transfers t. As was discussed previously,[9] such a program for scattering encounters certain difficulties since the O(3, 1) two-variable expansions involve at least one integral (and sometimes two), which must somehow be approximated. We postpone an investigation of this problem to the future and instead turn to the three-body decay process

$$1 \to 2+3+4, \tag{1}$$

where the situation is much simpler. Indeed, for the process (1) the physical region is finite [we have, e.g., $(m_3+m_4)^2 \leq s \leq (m_1-m_2)^2$, $(m_2+m_4)^2 \leq t \leq (m_1-m_3)^2$, and $(m_2+m_3)^2 \leq u \leq (m_1-m_4)^2$] and this can be used to transform the O(3, 1) expansions into O(4) expansions. The group O(4) is compact, all irreducible representations are finite-dimensional and unitary (or equivalent to unitary ones), so they can be labeled by discrete quantum numbers. Hence expansions of decay amplitudes will involve sums only, which can readily be truncated in phenomenological applications. Such O(4) expansions for spinless particles were obtained previously[9] and have been applied to analyze $K\to 3\pi$ and $\eta\to 3\pi$ Dalitz plots.[10]

In this paper we obtain O(4) expansions of helicity amplitudes for the decay (1) in the case when the particles have arbitrary spins. We also discuss the relation of these expansions to other approaches, obtain restrictions due to parity conservation, and investigate threshold and pseudothreshold behavior.

II. KINEMATICS OF THE DECAY AND SOME MATHEMATICAL PRELIMINARIES

A. Kinematics and Particle States

As in the spinless case[9,10] we shall consider the decay $1\to 2+3+4$ in a center-of-mass-like frame of reference, in which the particle momenta are (see Fig. 1 of Ref. 9)

$$\begin{aligned} p_1&=m_1(\cosh a_1,\ \sinh a_1\sin\theta\cos\phi,\ \sinh a_1\sin\theta\sin\phi,\ \sinh a_1\cos\theta),\\ p_2&=m_2(\cosh a_2,\ \sinh a_2\sin\theta\cos\phi,\ \sinh a_2\sin\theta\sin\phi,\ \sinh a_2\cos\theta),\\ p_3&=m_3(\cosh a_3,\ 0,\ 0,\ \sinh a_3),\\ p_4&=m_4(\cosh a_4,\ 0,\ 0,\ -\sinh a_4). \end{aligned} \tag{2}$$

We have

$$\begin{aligned} &p_1=p_2+p_3+p_4,\\ &s=(p_1-p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2 \end{aligned} \tag{3}$$

so that

$$\begin{aligned} &\xi\equiv\cosh a_1=\frac{s+m_1^2-m_2^2}{2m_1\sqrt{s}},\quad \cosh a_2=\frac{-s+m_1^2-m_2^2}{2m_2\sqrt{s}},\\ &\cosh a_3=\frac{s+m_3^2-m_4^2}{2m_3\sqrt{s}},\quad \cosh a_4=\frac{s-m_3^2+m_4^2}{2m_4\sqrt{s}}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} z&=\cos\theta\\ &=\frac{2s(t-m_1^2-m_3^2)+(s+m_1^2-m_2^2)(s+m_3^2-m_4^2)}{\{[-s+(m_1+m_2)^2][-s+(m_1-m_2)^2][s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{1/2}}. \end{aligned} \tag{5}$$

In the physical decay region we have

$$-1\le z\le 1\quad\text{and}\quad 1\le\xi\le\frac{(m_3+m_4)^2+m_1^2-m_2^2}{2m_1(m_3+m_4)}. \tag{6}$$

We shall choose $a\equiv a_1$ and θ to be the independent variables and we see that $0\le a\le a_{\max}$ and $0\le\theta\le\pi$. Following our general approach[1,3-7,9,10] we notice that a and θ are spherical coordinates of the momentum p_1, i.e., the coordinates of a point on the upper sheet of the hyperboloid $p_1^2=m_1^2$. The scattering amplitudes are functions of this point. Relations (6) however show that p_1 does not range over the entire hyperboloid, but only over a "cup" (6) close to the point $p_1=m_1(1, 0, 0, 0)$. We can now construct a parallel mapping[9] of this hyperbolic cup onto a four-dimensional sphere of radius R,

$$R=\sinh a_{\max}=\frac{\{[(m_1+m_2)^2-(m_3+m_4)^2][(m_1-m_2)^2-(m_3+m_4)^2]\}^{1/2}}{2m_1(m_3+m_4)}. \tag{7}$$

A point on this O(4) sphere can be parametrized as

$$p_s=R(\cos\tfrac{1}{2}\alpha,\ \sin\tfrac{1}{2}\alpha\sin\theta\cos\phi,\ \sin\tfrac{1}{2}\alpha\sin\theta\sin\phi,\ \sin\tfrac{1}{2}\alpha\cos\theta), \tag{8}$$

where

$$\cos\alpha = 1 - \frac{[(m_1+m_2)^2 - s][(m_1-m_2)^2 - s]}{2m_1{}^2R^2s},$$
$$\sin\alpha = \frac{\{[(m_1+m_2)^2-s][(m_1-m_2)^2-s][s-(m_3+m_4)^2][(m_1{}^2-m_2{}^2)^2-(m_3+m_4)^2s]\}^{1/2}}{2m_1{}^2(m_3+m_4)R^2s} \tag{9}$$

and $\cos\theta$ is given by (5). We see that the scattering amplitudes can now be considered to be functions of the components of p_s, i.e., functions of a point on an O(4) sphere. The range of variables is

$$0 \leqslant \alpha \leqslant \pi, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi. \tag{10}$$

Below we shall make use of this mapping of the momentum p_1 of (2) onto the momentum p_s of (8) to expand the amplitudes in terms of representations of O(4).

The single-particle states will be considered to be the usual helicity states[2] which we write as

$$|1\rangle = R(\phi,\theta,-\phi)e^{-ia_1K_3}O(m_1)|\eta_1 s_1\lambda_1\rangle,$$
$$|2\rangle = R(\phi,\theta,-\phi)e^{-ia_2K_3}O(m_2)|\eta_2 s_2\lambda_2\rangle, \tag{11}$$
$$|3\rangle = e^{-ia_3K_3}O(m_3)|\eta_3 s_3\lambda_3\rangle,$$
$$|4\rangle = e^{ia_4K_3}O(m_4)|\eta_4 s_4 -\lambda_4\rangle.$$

In formula (11) we have

$$R(\phi,\theta,-\phi) = e^{-iJ_3\phi}e^{-iJ_2\theta}e^{iJ_3\phi},$$

where J_i are the generators of rotations (i.e., the total angular momentum operators). The operator K_i is a generator of pure Lorentz transformations along the ith axis. The state $O(m)|\eta s\lambda\rangle$ thus represents a particle of mass m, spin s, intrinsic parity η, and helicity λ, at rest. The vector $|\eta s\lambda\rangle$ is a basis vector of the homogeneous Lorentz group for a specific finite-dimensional representation, characterized by the pair of integer or half-integer numbers $(j_0, c) = (s, \pm(s+1))$ (in general j_0 is always integer or half-integer but c can be any complex number[11]). These specific representations, upon reduction to O(3), remain irreducible, i.e., contain only one representation of O(3) (with angular momentum s). These have been studied in detail by Joos[12] and Weinberg.[13] The operator $O(m)$ and more generally $O(p)$ was introduced by Feldman and Matthews[8] and was used extensively in I. It is a projection operator that serves to transform a Lorentz group state $|\eta s\lambda\rangle$ into a Poincaré-group state of momentum $m_\mu = (m,0,0,0)$, or more generally momentum $p_\mu = (p_0, \vec{p})$. All relevant properties[8] of the operator O have been discussed in I and we shall not repeat them here.

B. Some Properties of the Representations of O(3,1) and O(4)

All properties of the representations of O(3, 1) relevant for this article have been summarized in I. Let us just note that the transformation matrices of the O(3, 1) group for the finite-dimensional representations $(s, \pm(s+1))$ in a basis corresponding to the reduction $O(3,1) \supset O(3) \supset O(2)$ can be written as

$$\begin{aligned} u_{\lambda_1\lambda_2}(p,\eta_1\eta_2) &= \langle s\lambda_1\eta_1|\,U(p)\,|s\lambda_2\eta_2\rangle \\ &= \langle s\lambda_1\eta_1|\,e^{-i\phi J_3}e^{-i\theta J_2}e^{i\phi J_3}e^{-iaK_3}|s\lambda_2\eta_2\rangle \\ &= D^s_{\lambda_1\lambda_2}(\phi,\theta,-\phi)\tfrac{1}{2}(e^{a\lambda_2}+\eta_1\eta_2e^{-a\lambda_2}). \end{aligned} \tag{12}$$

We also have

$$\begin{aligned} \bar{u}_{\lambda_1\lambda_2}(p,\eta_1\eta_2) &= \langle s\lambda_1\eta_1|\,U^{-1}(p)\,|s\lambda_2\eta_2\rangle \\ &= \tfrac{1}{2}(e^{-a\lambda_1}+\eta_1\eta_2e^{a\lambda_1})D^{s*}_{\lambda_2\lambda_1}(\phi,\theta,-\phi) \end{aligned} \tag{13}$$

and

$$\sum_{\lambda\eta} u_{\lambda_1\lambda}(p,\eta_1\eta)\bar{u}_{\lambda\lambda_2}(p,\eta\eta_2) = \delta_{\lambda_1\lambda_2}\delta_{\eta_1\eta_2},$$
$$\sum_{\lambda\eta} \bar{u}_{\lambda_1\lambda}(p,\eta_1\eta)u_{\lambda\lambda_2}(p_1,\eta\eta_2) = \delta_{\lambda_1\lambda_2}\delta_{\eta_1\eta_2}. \tag{14}$$

We shall also use the notation

$$f_0(a,\lambda,\eta\bar{\eta}) = \tfrac{1}{2}(e^{a\lambda}+\eta\bar{\eta}e^{-a\lambda}). \tag{15}$$

The representations of O(4) have been studied by many authors[14] (see also Ref. 9). Let us here state several results that we shall need below (we have not succeeded in locating all of them in the literature, so some might actually be new).

We shall label the representations of O(4) by a pair of numbers (ν, n), which are simultaneously integer or half-integer and satisfy $n \geqslant |\nu|$. The group O(4) is locally isomorphic to $O(3)\otimes O(3)$, so its representations can alternatively be labeled by two "angular momenta" j and $\tilde{j}$, satisfying

$$j = \frac{n+\nu}{2}, \quad \tilde{j} = \frac{n-\nu}{2}. \tag{16}$$

We shall consider the representations of O(4) in an $O(4)\supset O(3)\supset O(2)$ basis in which the basis vectors $|\nu nLM\rangle$ satisfy

$$\begin{aligned}
(\vec{L}^2+\vec{A}^2)|\nu nLM\rangle &= [\nu^2+(n+1)^2-1]|\nu nLM\rangle,\\
\vec{L}\cdot\vec{A}|\nu nLM\rangle &= (n+1)\nu|\nu nLM\rangle,\\
\vec{L}^2|\nu nLM\rangle &= L(L+1)|\nu nLM\rangle,\\
L_3|\nu nLM\rangle &= M|\nu nLM\rangle
\end{aligned} \tag{17}$$

as well as some phase conventions specified previously.[9] Above L_1, L_2, and L_3 are the generators of rotations in the 23, 31, and 12 planes. A_1, A_2, and A_3 generate rotations in the 41, 42, and 43 planes. Furthermore $\vec{L}^2+\vec{A}^2$ and $\vec{L}\cdot\vec{A}$ are the Casimir operators of O(4).

Making use of the local isomorphism $O(4)\sim O(3)\otimes O(3)$ it is easy to calculate the O(4) finite transformation matrices in the basis (17). Indeed, we have

$$\begin{aligned}
D^{\nu n}_{L_1M_1L_2M_2}(\phi,\theta,\psi,\alpha,\eta,\chi) &= D^{\nu n}_{L_1M_1L_2M_2}(g)\\
&= \langle \nu nL_1M_1|e^{-iL_3\phi}e^{-iL_2\theta}e^{-iL_3\psi}e^{-iA_3\alpha}e^{-iL_2\eta}e^{-iL_3\chi}|\nu nL_2M_2\rangle\\
&= \sum_\lambda D^{L_1}_{M_1\lambda}(\phi,\theta,\psi)d^{\nu n}_{L_1L_2\lambda}(\alpha)D^{L_2}_{\lambda M_2}(0,\eta,\chi)\,.
\end{aligned} \tag{18}$$

In (18) $D^{L_1}_{M_1\lambda}(\phi,\theta,\psi)$ and $D^{L_2}_{\lambda M_2}(0,\eta,\chi)$ are O(3) D functions, and the O(4) d functions can be calculated to be

$$d^{\nu n}_{L_1L_2\lambda}(\alpha)=[(2L_1+1)(2L_2+1)]^{1/2}e^{-i\lambda\alpha}\sum_m \begin{pmatrix}\frac{n+\nu}{2} & \frac{n-\nu}{2} & L_1\\ \lambda-m & m & -\lambda\end{pmatrix}\begin{pmatrix}\frac{n+\nu}{2} & \frac{n-\nu}{2} & L_2\\ \lambda-m & m & -\lambda\end{pmatrix}e^{2im\alpha}, \tag{19}$$

where the brackets denote O(3) $3j$ symbols.[15]

The following useful properties of the O(4) d functions can be obtained by inspecting formula (19) directly.

(i) Symmetry properties:

$$\begin{aligned}
&d^{\nu n}_{L_2L_1\lambda}(\alpha)=d^{\nu n}_{L_1L_2\lambda}(\alpha),\\
&d^{-\nu n}_{L_1L_2\lambda}(\alpha)=(-1)^{2n-L_1-L_2}d^{\nu n*}_{L_1L_2\lambda}(\alpha)=d^{\nu n}_{L_1L_2-\lambda}(\alpha),\\
&d^{\nu n}_{L_1L_2\lambda}(-\alpha)=d^{\nu n*}_{L_1L_2\lambda}(\alpha),
\end{aligned} \tag{20}$$

(ii) Normalization: The d functions satisfy

$$\sum_{\lambda=-\min\{\lambda_1\lambda_2\}}^{\min\{\lambda_1\lambda_2\}}\int_0^\pi \sin^2\alpha\, d\alpha\, d^{\nu n}_{L_1L_2\lambda}(\alpha)d^{\nu' n'}_{L_1L_2\lambda}(\alpha)=\frac{\pi}{2}\frac{(2L_1+1)(2L_2+1)}{(n+1)^2-\nu^2}\delta_{\nu\nu'}\delta_{nn'}, \tag{21}$$

so that

$$\int_0^\pi \sin^2\alpha\, d\alpha\int_0^\pi \sin^2\theta\, d\theta\int_0^{2\pi}d\phi\int_0^{2\pi}d\psi\int_0^\pi \sin\eta\, d\eta\int_0^{2\pi}d\chi\, D^{\nu n}_{L_1M_1L_2M_2}(g)D^{\nu'n'}_{L_1'M_1'L_2'M_2'}(g) = \frac{(2\pi)^4}{(n+1)^2-\nu^2}\delta_{\nu\nu'}\delta_{nn'}\delta_{L_1L_1'}\delta_{L_2L_2'}\delta_{M_1M_1'}\delta_{M_2M_2'}\,. \tag{22}$$

(iii) The d functions can be written as homogeneous polynomials in $\cos\alpha$ and $\sin\alpha$. Consider $d^{\nu n}_{L_1L_2\lambda}(\alpha)$ with $\nu\geq 0$ and $\lambda\geq 0$ [all other values can be obtained using the symmetry relations (20)]. The order of $d^{\nu n}_{L_1L_2\lambda}(\alpha)$ as a polynomial in $\cos\alpha$ and $\sin\alpha$ is

$$N=n-|\lambda-\nu|\,, \tag{23}$$

i.e., we have

$$d^{\nu n}_{L_1L_2\lambda}(\alpha)=\sum_{k=0}^N A_k(\cos\alpha)^{N-k}(\sin\alpha)^k,\quad \nu\geq 0,\ \lambda\geq 0, \tag{24}$$

where N is given by (23) and A_k are certain finite coefficients (some of them may vanish).

(iv) The behavior of the d functions for $\sin\alpha \to 0$ is given by the formula

$$d^{\nu n}_{L_1 L_2 \lambda}(\alpha) = e^{-i\lambda\alpha}(\sin\alpha)^{|L_1 - L_2|}\, p^{\nu n}_{L_1 L_2 \lambda}(\alpha)\,, \tag{25}$$

where $p^{\nu n}_{L_1 L_2}(\alpha)$ is a polynomial in $\cos\alpha$ and $\sin\alpha$, which is finite (nonzero) for $\sin\alpha \to 0$. We see that $d^{\nu n}_{L_1 L_2 \lambda}(\alpha)$ vanishes for $\alpha \to 0$ or $\alpha \to \pi$, unless $L_1 = L_2$.

(v) The basis functions for O(4) representations with $\nu = 0$ are specially well known. Using our phase conventions, they are[9]

$$\phi_{nLM}(\alpha, \theta, \phi) = e^{-i(\pi/2)L}\frac{2^{L-1/2}L!}{\pi}\left((2L+1)\frac{(L-M)!}{(L+M)!}\,\frac{(N+1)(N-L)!}{(N+L+1)!}\right)^{1/2}(\sin\alpha)^L C^{L+1}_{n-L}(\cos\alpha)P^M_L(\cos\theta)e^{iM\phi}. \tag{26}$$

Their relation to the O(4) D functions is

$$D^{0n}_{00LM}(\chi, \eta, \psi, \alpha, \theta, \pi - \phi) = \frac{\sqrt{2}\,\pi}{n+1}\,\phi_{nLM}(\alpha, \theta, \phi) \tag{27}$$

and

$$d^{0n}_{0L0}(\alpha) = e^{-i(\pi/2)L}2^L L!\left(\frac{2L+1}{n+1}\,\frac{(n-L)!}{(n+L+1)!}\right)^{1/2}(\sin\alpha)^L C^{L+1}_{n-L}(\cos\alpha) \tag{28}$$

[$C^\lambda_n(x)$ and $P^M_L(x)$ are Gegenbauer and Legendre polynomials, respectively].

III. EXPANSIONS OF DECAY AMPLITUDES

Let us now consider the matrix elements of the transition matrix for the process

$$1 \to 2+3+4$$

taken between the "center-of-mass" helicity states (11).

Using some elementary properties[1,8] of the operators $O(p)$ we have

$$\begin{aligned}\langle 234|\,T\,|\,1\rangle = \langle s_2\lambda_2\eta_2, s_3\lambda_3\eta_3, s_4 - \lambda_4\eta_4|\,e^{ia_2K_3^{(2)}}R^{(2)}(\phi, -\theta, -\phi)e^{ia_3K_3^{(3)}}\\ \times e^{-ia_4K_3^{(4)}}O(p_2)O(p_3)O(p_4)TO(p_1)R^{(1)}(\phi, \theta, -\phi)e^{-ia_1K_3^{(1)}}\,|s_1\eta_1\lambda_1\rangle\end{aligned} \tag{29}$$

(the superscripts in brackets denote the particle on which the operators act). Using linear momentum conservation, we can put

$$\begin{aligned}O(p_2)O(p_3)O(p_4)TO(p_1) &= O(p_3)O(p_4)O(p_2+p_3+p_4)TO(p_1)\\ &= \delta(p_1 - p_2 - p_3 - p_4)O(p_3)O(p_4)TO(p_1)\,.\end{aligned} \tag{30}$$

Similarly as in I (and Refs. 3–7) we use two of the momenta, this time p_3 and p_4, to specify the frame of reference. We put

$$\begin{aligned}n &= \frac{1}{\sqrt{s}}\,(p_3 + p_4)\,,\\ \gamma &= \{s[s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{-1/2}[(s - m_3^{\,2} + m_4^{\,2})p_3 - (s + m_3^{\,2} - m_4^{\,2})p_4]\,.\end{aligned} \tag{31}$$

In our frame of reference (2) we have

$$n = (1, 0, 0, 0)\,, \quad \gamma = (0, 0, 0, 1)\,, \tag{32}$$

i.e., they do not depend on s and t (or on α, θ, and ϕ).

Further, we have

$$O(p_3)O(p_4) = g(s)O(n)O(\gamma)\,, \tag{33}$$

where g is the Jacobian for the transformation $p_3p_4 \to n, \gamma$:

$$g(s) = \left(\frac{4}{[s-(m_3+m_4)^2][s-(m_3-m_4)^2]}\right)^2\,. \tag{34}$$

We return to formula (29), insert complete sets of states for particles 2, 3, and 4 on both sides of the corresponding Lorentz and rotation group operators, and make use of (30)–(34) to obtain

$$\langle 234|T|1\rangle=\delta(p_1-p_2-p_3-p_4)g(s)\sum_{\bar\lambda_2\bar\eta_2\bar\eta_3\bar\eta_4}\bar u_{\lambda_2\bar\lambda_2}(p_2,\eta_2\bar\eta_2)f_0(a_3,-\lambda_3,\eta_3\bar\eta_3)f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)$$
$$\times\langle s_2\bar\lambda_2\bar\eta_2,s_3\lambda_3\bar\eta_3,s_4-\lambda_4\bar\eta_4|O(n)O(\gamma)TU(p_1)O(m_1)|s_1\eta_1\lambda_1\rangle. \quad (35)$$

Notice that $\bar u_{\lambda\bar\lambda}(p,\eta\bar\eta)$ and $f_0(a,\lambda,\eta\bar\eta)$, given by (13) and (15), respectively, are O(3, 1) transformation matrices for the finite-dimensional representation $(s,\pm(s+1))$. The only remaining unknown dependence on the variables s and t is in the boost $U(p_1)$. In order to extract this dependence explicitly, as in I for scattering, we insert a complete set of one-particle states on both sides of $U(p_1)$. We shall make use of the finiteness of the physical scattering region to obtain O(4) expansions, as discussed in the previous section. The boost

$$U(p_1)=e^{-iJ_3\phi}e^{-iJ_2\theta}e^{iJ_3\phi}e^{-iK_3a}, \quad (36)$$

where the parameters a, θ, and ϕ determine the c.m. momentum p_1 [see (2)], by definition represents an element of O(3, 1)

$$g=(\phi,\theta,-\phi,a,0,0). \quad (37)$$

If we use the mapping $p_1\to p_s$ [see (8)] where p_s lies on an O(4) sphere, then we obtain a new mapping $g\to\tilde g$, where

$$\tilde g=(\phi,\theta,-\phi,\alpha,0,0) \quad (38)$$

is an element of O(4) [the range of α, θ, and ϕ is given by (10)].

The boost $U(p_1)$ can now be considered to be a function defined over an O(4) group manifold and we can expand it in terms of the matrix elements of O(4). In order to obtain this expansion we must insert a suitable complete set of intermediate states, labeled by O(4) quantum numbers (as well as mass, spin, parity, etc.). For scattering the appropriate procedure used in I was to insert Poincaré group states in a basis, corresponding to the reduction P ⊃ O(3, 1) ⊃ O(3) ⊃ O(2). For the decay amplitude (35) we extend the group P to the complex Poincaré group[16] P* and consider the group reduction

$$P^*\supset O(4)\supset O(3)\supset O(2). \quad (39)$$

The states can then be denoted $|ms\eta\nu nLM\rangle$ [see (17)]. Upon inserting these states for particle 1 on both sides of $U(p_1)$ in (35) we obtain the following factors:

$$\text{(i)}\ \langle\bar\eta_2 s_2\bar\lambda_2,\bar\eta_3 s_3\lambda_3,\bar\eta_4 s_4-\lambda_4|O(n)O(\gamma)T|m_1s_1\eta_1\nu nLM\rangle,$$
$$\text{(ii)}\ \langle m_1s_1\eta_1\nu ns_1\lambda_1|O(m_1)|\eta_1s_1\lambda_1\rangle=[O(m_1)]^{\eta_1s_1}_{\nu n}, \quad (40)$$
$$\text{(iii)}\ \langle\nu nLM|U(p_1)|\nu ns_1\lambda_1\rangle=D^L_{M\lambda_1}(\phi,\theta,-\phi)d^{\nu n}_{Ls_1\lambda_1}(\alpha).$$

Thus, the third factor produces the O(4) D functions [see (18) and (19)]. The other two factors do not contain any dependence on s and t. Notice that the Wigner-Eckart theorem implies that the second factor does not depend on λ_1. We can combine the first two factors into an expansion coefficient, put

$$\langle 234|T|1\rangle=\delta(p_1-p_2-p_3-p_4)f_{\lambda_i}(a,\theta,\phi), \quad (41)$$

and obtain

$$f_{\lambda_i}(a,\theta,\phi)=g(s)\sum_{\bar\lambda_2\bar\eta_2\bar\eta_3\bar\eta_4}\bar u_{\lambda_2\bar\lambda_2}(p_2,\eta_2\bar\eta_2)f_0(a_3,-\lambda_3,\eta_3\bar\eta_3)f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)$$
$$\times\sum_{\nu=-s_1}^{s_1}\sum_{n=s_1}^{\infty}\sum_{L=|\nu|}^{n}\sum_{M=-\min\{L,s_1\}}^{\min\{L,s_1\}}A^{\nu n}_{LM}(\bar\lambda_2\lambda_3\lambda_4,\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)d^{\nu n}_{Ls_1\lambda_1}(\alpha)D^L_{M\lambda_1}(\phi,\theta,-\phi). \quad (42)$$

Expansion (42) can be further simplified by combining the entire dependence on θ and ϕ into one O(3) D function and by making use of the explicit dependence of the left-hand side on the azimuthal angle ϕ. Indeed, the helicity amplitudes $f_{\lambda_i}(a,\theta,\phi)$ contain ϕ in the factor $e^{i\phi(\lambda_1-\lambda_2-\lambda_3+\lambda_4)}$ only.[2] On the right-hand side we have the same behavior if we put $M=\bar\lambda_2+\lambda_3-\lambda_4$. In the following we shall make this substitution and drop the subscript M. Combining the two O(3) D functions using standard angular momentum theory we finally obtain the O(4) expansion of the helicity amplitudes for three-body decays:

$$f_{\lambda_i}(a,\theta,\phi)=g(s)\sum_{\bar\lambda_2\bar\eta_2\bar\eta_3\bar\eta_4}\sum_{\nu=-s_1}^{s_1}\sum_{n=s_1}^{\infty}\sum_{L=|\nu|}^{n}\sum_{J=|s_2-L|}^{s_2+L}(2J+1)(-1)^{-\lambda_1+\bar\lambda_2+\lambda_3-\lambda_4}f_0(a_2,-\lambda_2,\eta_2\bar\eta_2)f_0(a_3,-\lambda_3,\eta_3\bar\eta_3)$$

$$\times f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)\begin{pmatrix} L & s_2 & J\\ -\lambda_1 & \lambda_2 & \lambda_1-\lambda_2\end{pmatrix}\begin{pmatrix} L & s_2 & J\\ -\bar\lambda_2-\lambda_3+\lambda_4 & \bar\lambda_2 & \lambda_3-\lambda_4\end{pmatrix}$$

$$\times A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4,\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)d^{\nu n *}_{Ls_1\lambda_1}(-\alpha)D^J_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}(\phi,-\theta,-\phi)\,. \tag{43}$$

The expansion formula (43) can readily be inverted and since we are dealing with harmonic analysis on a compact group, no convergence difficulties arise [contrary to the O(3, 1) expansions of scattering amplitudes, considered in I]. Indeed, making use of the orthogonality properties (14) and (21), as well as the properties of the O(3) D functions and $3j$ symbols, we obtain

$$A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4,\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)$$

$$=\frac{(n+1)^2-\nu^2}{2\pi^2(2s_1+1)}\int_0^{\pi}\sin^2\alpha\,d\alpha\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\sum_{\lambda_1\lambda_2\eta_2\eta_3\eta_4}\sum_J(2J+1)f_0(a_2,-\lambda_2,\eta_2\bar\eta_2)f_0(a_3,\lambda_3,\eta_3\bar\eta_3)f_0(a_4,\lambda_4,\eta_4\bar\eta_4)$$

$$\times(-1)^{\lambda_1-\bar\lambda_2-\lambda_3+\lambda_4}\begin{pmatrix} L & s_2 & J\\ -\bar\lambda_2-\lambda_3+\lambda_4 & \bar\lambda_2 & \lambda_3-\lambda_4\end{pmatrix}\begin{pmatrix} L & s_2 & J\\ -\lambda_1 & \lambda_2 & \lambda_1-\lambda_2\end{pmatrix}$$

$$\times f_{\lambda_i}(a,\theta,\phi)d^{\nu n}_{Ls_1\lambda_1}(-\alpha)D^J_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}(\alpha,-\theta,-\phi)\,. \tag{44}$$

Several comments are in order:

(1) The reason why the angle $(-\theta)$ figures instead of θ as in the Jacob and Wick expansion is that particles 3 and 4 are along the z axis in our case, i.e., we have

$$\theta=-\theta_{\text{scatt}}\,,$$

where θ is the angle used in this article and θ_{scatt} is the usual c.m. scattering angle.

(2) Note that we could formally simplify the expansion (43) by defining a new expansion coefficient

$$B_{LJ}^{\nu n}(\lambda_3\lambda_4,\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)=\sum_{\bar\lambda_2}\begin{pmatrix} L & s_2 & J\\ -\bar\lambda_2-\lambda_3+\lambda_4 & \bar\lambda_2 & \lambda_3-\lambda_4\end{pmatrix}(-1)^{\bar\lambda_2+\lambda_3-\lambda_4}A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4,\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)\,.$$

We shall however keep the coefficient A as in (43).

(3) It may seem somewhat inconsistent that expansion (43) contains the functions $f_0(a\lambda\eta\bar\eta)$ which are actually d functions for certain finite-dimensional representations of the Lorentz group O(3, 1) as well as $d^{\nu n}_{Ls\lambda}(\alpha)$, which are d functions of O(4). We see no contradiction here; on the contrary it is a manifestation of the dual role that the Lorentz group plays in the derivation and interpretation of our two-variable (or multivariable) expansions in general. Thus, the appearance of the f_0 functions is due to the fact that the initial and final free-particle states transform under irreducible unitary representations of the Poincaré group [which can of course be reduced to its O(3, 1) subgroup] and is thus purely a manifestation of Lorentz invariance.[17] The unitary O(3, 1) d functions for scattering and O(4) d functions for decays, on the other hand, appear because of our particular choice of a complete set of intermediate states. The group, in this case characterizes the space of independent kinematic parameters (a, θ, and ϕ or α, θ, and ϕ), i.e., the manifold over which the amplitudes are defined (for a previous discussion of the dual role of the Lorentz group in the derivations of two-variable expansions for the scattering of spinless particles see the first of Refs. 5).

IV. RELATION TO O(3) LITTLE-GROUP EXPANSION, PHYSICAL MEANING OF QUANTUM NUMBERS, AND CONSEQUENCES OF PARITY CONSERVATION

In I we have shown that the O(3, 1) two-variable expansions for scattering can be interpreted as the Jacob and Wick O(3) little-group expansions of helicity amplitudes, supplemented by an O(3, 1) expansion of the partial-wave helicity amplitudes. Let us show that expansion (43) can be interpreted in precisely the same manner. To do this we must invert some of the summations in (43). This is somewhat tedious, but yields a simple result, namely,

$$\sum_{\nu=-s_1}^{s_1}\sum_{n=s_1}^{\infty}\sum_{L=|\nu|}^{n}\sum_{j=|s_2-L|}^{s_2+L} = \sum_{j=\{0,1/2\}}^{\infty}\sum_{L=|J-s_2|}^{J+s_2}\sum_{n=\max(s_1,L)}^{\infty}\sum_{\nu=-\min(L,s_1)}^{\min(L,s_1)}, \tag{45}$$

where the summation over j starts at 0 or $\frac{1}{2}$, depending on whether j is integer or half-odd-integer. The only other restriction on the above sums is obvious, namely,

$$J \geq \max\{|\lambda|, |\mu|\}, \quad \lambda = \lambda_1 - \lambda_2, \quad \mu = \lambda_3 - \lambda_4. \tag{46}$$

The two-variable expansion (43) (the third variable ϕ is irrelevant) can now indeed be written as the usual partial-wave expansion

$$f_{\lambda_i}(a, \theta, \phi) = \sum_{J=\max(|\lambda|,|\mu|)}^{\infty} (2J+1)\langle\lambda_2\lambda_3\lambda_4|T^J(a)|\lambda_1\rangle D^{J*}_{\lambda\mu}(\phi, -\theta, -\phi), \tag{47}$$

and the partial-wave helicity amplitude is expanded as

$$\begin{aligned}\langle\lambda_2\lambda_3\lambda_4|T^J(a)|\lambda_1\rangle = g(s) \sum_{L=|J-s_2|}^{J+s_2}\sum_{n=\max(s_1,L)}^{\infty}\sum_{\nu=-\min(L,s_1)}^{\min(L,s_1)}\sum_{\bar\lambda_2\bar\eta_2\bar\eta_3\bar\eta_4} & (-1)^{-\lambda_1+\bar\lambda_2+\lambda_3-\lambda_4} \\ & \times f_0(a_2, -\lambda_2, \eta_2\bar\eta_2) f_0(a_3, -\lambda_3, \eta_3\bar\eta_3) f_0(a_4, -\lambda_4, \eta_4\bar\eta_4) \\ & \times \begin{pmatrix} L & s_2 & J \\ -\lambda_1 & \lambda_2 & \lambda_1-\lambda_2\end{pmatrix}\begin{pmatrix} L & s_2 & J \\ -\bar\lambda_2-\lambda_3+\lambda_4 & \bar\lambda_2 & \lambda_3-\lambda_4\end{pmatrix} \\ & \times A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4, \eta_1\bar\eta_2\bar\eta_3\bar\eta_4) d^{\nu n *}_{Ls_1\lambda_1}(-\alpha). \end{aligned} \tag{48}$$

The meaning of the quantum numbers now becomes obvious. Thus J is the angular momentum of particles 3 and 4 (not however in general the total angular momentum of the final particles or the spin of the initial particle). In turn L is the vector sum of J and s_2, and can thus be identified with the total final-state angular momentum in a frame in which particle 2 is at rest, $|\nu|$ is the lower bound of L, n the upper bound (for J fixed).

If parity is conserved in the decay $1 \to 2+3+4$ then not all of the helicity amplitudes and hence not all the Lorentz amplitudes [or O(4) amplitudes] $A_L^{\nu n}$, are independent. Accepting the same parity conventions as for scattering[2,18] we have

$$\langle -\lambda_3 - \lambda_4|T^J(a)| -\lambda_1 - \lambda_2\rangle = \eta\langle\lambda_3\lambda_4|T^J(a)|\lambda_1\lambda_2\rangle, \quad \eta = \frac{\eta_3\eta_4}{\eta_1\eta_2}(-1)^{s_3+s_4-s_1-s_2}. \tag{49}$$

Using (48) to expand both sides of (49), making use of the relation $d^{-\nu n}_{Ls_1-\lambda}(\alpha) = d^{-\nu n}_{Ls_1\lambda}(\alpha)$ and

$$f_0(a, \lambda, \eta\bar\eta) = \eta\bar\eta f_0(a, -\lambda, \eta\bar\eta),$$

we find that if parity is conserved then the Lorentz amplitudes satisfy

$$A_L^{-\nu n}(-\bar\lambda_2 - \lambda_3 - \lambda_4, \eta_1\bar\eta_2\bar\eta_3\bar\eta_4) = \frac{\bar\eta_3\bar\eta_4}{\eta_1\bar\eta_2}(-1)^{s_3+s_4-s_1-s_2} A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4, \eta_1\bar\eta_2\bar\eta_3\bar\eta_4). \tag{50}$$

We shall not consider restrictions due to time-reversal invariance, which are somewhat complicated and not particularly important for decay amplitudes.

V. THRESHOLD AND PSEUDOTHRESHOLD BEHAVIOR OF DECAY AMPLITUDES

The most important feature of the O(4) expansion (43) that we have derived is that the entire dependence on the kinematic variables s and t is contained explicitly in the O(3, 1) functions f_0, in $d^{\nu n}_{Ls_1\lambda_1}(\alpha)$, and in the O(3) D function [and in the factor $g(s)$]. It is of obvious interest to consider the behavior of the individual terms of the expansion in various accessible limits. For decays the limits of interest are the boundary of the physical region and the various thresholds and pseudothresholds.

The boundary of the physical region is simply given by the condition

$$\cos\theta = \pm 1 . \tag{51}$$

The dependence on θ is entirely contained in the O(3) function $d^J_{\lambda\mu}(\theta)$, $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$ and is thus given by the usual partial-wave expansion (47) of the helicity amplitudes. This expansion has been extensively studied in the literature[18] and we shall not discuss it here.

On the other hand, we can also consider the behavior at the physical threshold and pseudothreshold

$$s = (m_3 + m_4)^2 \text{ and } s = (m_1 - m_2)^2 \tag{52}$$

(which is where $\cos\theta$ changes sign).

The "kinematic" behavior of the partial-wave helicity amplitudes (48) at the corresponding threshold and pseudothreshold is known to be[18-20]

$$\langle \lambda_2\lambda_3\lambda_4 | T^J(a) | \lambda_1 \rangle = [s - (m_3 + m_4)^2]^{l_n/2} [s - (m_1 - m_2)^2]^{l_p/2} \times \langle \lambda_2\lambda_3\lambda_4 | T^J(a) | \lambda_1 \rangle' , \tag{53}$$

where l_n and l_p are the minimal possible values of the angular momentum of particles 3 and 4 or the minimal value of the difference between the momenta of particles 1 and 2, respectively. The primed quantity in (53) is regular and nonzero at the points of interest.

Let us now check the term-by-term behavior of the expansion (48).

First consider the pseudothreshold and put

$$s = (m_1 - m_2)^2 (1 - \epsilon) , \quad \epsilon > 0 . \tag{54}$$

From (4) and (9) we see that

$$\cosh a_2 \underset{\epsilon\to 0}{\sim} 1 + \frac{\epsilon}{2}\frac{m_1}{m_2} , \qquad \sin\alpha \underset{\epsilon\to 0}{\sim} \frac{2}{R}\left(\epsilon \frac{m_2}{m_1}\right)^{1/2} . \tag{55}$$

From (15) we obtain

$$f_0(a_2, -\lambda_2, \eta_2\bar{\eta}_2) = \begin{cases} 1 + \lambda_2{}^2 \dfrac{\epsilon}{2}\dfrac{m_1}{m_2} , & \text{for } \eta_2\bar{\eta}_2 = 1 \\ -\lambda_2\left(\epsilon\dfrac{m_1}{m_2}\right)^{1/2} , & \text{for } \eta_2\bar{\eta}_2 = -1 . \end{cases} \tag{56}$$

Using (55), (56), and (25) we can in the limit $\epsilon \to 0$ rewrite (43) as

$$f_{\lambda_i}(a, \theta, \phi) \underset{\epsilon\to 0}{\sim} g \sum_{\bar{\eta}_3\bar{\eta}_4} \sum_{\bar{\lambda}_2} \sum_{\nu=-s_1}^{s_1} \sum_{n=s_1}^{\infty} \sum_{J=|s_1-s_2|}^{s_1+s_2} (2J+1) f_0(a_3, -\lambda_3, \eta_3\bar{\eta}_3) f_0(a_4, -\lambda_4, \eta_4\bar{\eta}_4)(-1)^{\lambda_1 - \bar{\lambda}_2 - \lambda_3 + \lambda_4}$$
$$\times \begin{pmatrix} s_1 & s_2 & J \\ -\lambda_1 & \lambda_2 & \lambda_1 - \lambda_2 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & J \\ -\bar{\lambda}_2 - \lambda_3 + \lambda_4 & \bar{\lambda}_2 & \lambda_3 - \lambda_4 \end{pmatrix}$$
$$\times A^{\nu n}_{s_1}(\bar{\lambda}_2\lambda_3\lambda_4, \eta_1\eta_2\bar{\eta}_3\bar{\eta}_4) \; D^{J*}_{\lambda_1-\lambda_2, \lambda_3-\lambda_4}(\phi, -\theta, -\phi) . \tag{57}$$

Formula (57) shows that only terms with

$$|s_1 - s_2| \le J \le s_1 + s_2 \tag{58}$$

survive in the expansion at $s \to (m_1 - m_2)^2$. It follows from (53) that only terms with $l_p = 0$ should survive. In the c.m. frame, determined by (2), we can split the angular momentum into its spin s and orbital l part. We have

$$\vec{s}_1 + \vec{l}_1 = \vec{s}_2 + \vec{l}_2 + \vec{s}_3 + \vec{l}_3 + \vec{s}_4 + \vec{l}_4 = \vec{s}_2 + \vec{l}_2 + \vec{J} = \vec{L} + \vec{l}_2 . \tag{59}$$

Thus, $\vec{J} = \vec{s}_1 - \vec{s}_2 + \vec{l}_1 - \vec{l}_2$ and the limits (58) indicate that $\vec{l}_1 - \vec{l}_2 = 0$, i.e., $l_p = 0$, as required by (53). Using (25), (55), and (43) we see that all other partial waves vanish as prescribed by (53).

Similarly we can consider the (normal) threshold by putting

$$s = (m_3 + m_4)^2 (1 + \epsilon) , \quad \epsilon > 0 . \tag{60}$$

Proceeding in an identical manner as above, we find that again only partial waves satisfying (58) survive for $\epsilon \to 0$ in (60). Since we also have $\vec{J} = \vec{s}_3 + \vec{s}_4 + \vec{l}_3 + \vec{l}_4$, Eq. (53) would require $l_n = 0$, i.e., $\vec{l}_3 + \vec{l}_4 = 0$, i.e.,

$$|s_3 - s_4| \le J \le s_3 + s_4 . \tag{61}$$

This is only satisfied if $s_1 = s_3$, $s_2 = s_4$ (or $s_1 = s_4$, $s_2 = s_3$); thus the behavior of the O(4) expansion at the threshold is in general not completely correct. It should however be stressed that both the requirement (53) and condition (58) mean that an infinite number of partial-wave helicity amplitudes vanish and only a finite number survive. If the two conditions do not coincide then we obtain at most a finite number of constraints upon the Lorentz amplitudes $A^{\nu n}_L$ and not an infinite number as might have been expected *a priori*.

VI. CONCLUSIONS

The main result of this paper is the expansion formula (43), together with its inverse (44). It represents an O(4) group expansion of the c.m. helicity amplitudes and for any fixed value of the energy variable s it coincides with a standard O(3) partial-wave expansion. The appearance of the group O(4), as opposed to the Lorentz group O(3, 1), is due to the finiteness of the physical decay region, which we have used to map part of the O(3, 1) group manifold onto the entire O(4) manifold. Expansion (43) is thus a generalization of the O(4) two-variables expansions, suggested and applied previously for decays involving spinless particles only.[9,10] Indeed, in the spinless case $s_i=\lambda_i=0$ ($i=1,\ldots,4$) so obviously $\nu=0$ and $J=L$. Using (27) it is a simple matter to check that expansion (43) reduces to the one considered earlier.[9] On the other hand this expansion formula is to be viewed as a modification of the two-variable expansions of helicity amplitudes for two-body scattering, suggested in I. In I the scattering amplitudes were expanded in terms of O(3, 1) D functions, calculated in a basis corresponding to the reduction $O(3,1)\supset O(3)\supset O(2)$.

Our main interest is in the application of two-variable (or more generally multivariable) expansions to elementary particle scattering. However, since the O(4) expansions are simpler in that they involve summations only, we are considering their phenomenological applications first. The spinless O(4) expansions have already been applied to analyze $K\to 3\pi$ and $\eta\to 3\pi$ Dalitz plot distributions.[10] Obviously expansion (43) has a wide field of applicability, namely to analyze, in terms of a few O(4) amplitudes $A_L^{\nu n}$, experimental data (Dalitz-plot distributions, polarizations, etc.) in strong, weak, and electromagnetic decays of the type $\omega\to 3\pi$, $\Sigma\to Nl\nu$, $K\to\pi l\nu$, etc. In particular we have applied this O(4) expansion to analyze the Dalitz plot[21] for $\bar{p}n\to 3\pi$ annihilations at rest (the initial $\bar{p}n$ system is treated as a single particle).[22]

When applying (43) in such a phenomenological manner we obviously have to cut off the sum over n at some finite value n_0. When fitting the data it is important to establish: How many parameters do you need? How good is the fit (as characterized, e.g., by the χ^2 value)? How stable is the fit with respect to the choice of the cutoff parameter? How sensitive are the coefficients with respect to interesting dynamical features (violations of symmetries, etc.)? How unique are the solutions? Are the coefficients in (43) to some degree statistically independent, etc.?

Since our approach is quite general, we have obviously not incorporated any specific dynamics. The emphasis on Lorentz invariance, the "natural" mapping onto an O(4) manifold, the occurrence of the angular momenta J and L, etc. in (43), the relation to the O(3) partial-wave analysis, and the correct pseudothreshold and "reasonable" threshold behavior suggest that much of the kinematics has been incorporated, which is a good starting point for both phenomenology and dynamics.

In the future we plan to investigate more dynamical problems using the presented approach. Three-body decays serve, among other things, as a source of information on two-body resonances and also on the phase shifts, characterizing the final-state interactions. We hope to be able to investigate the sensitivity of the O(4) coefficients to resonances on one hand and to relate them to various two-body interactions on the other. We also intend to investigate various models in which the amplitude can be calculated and to find the coefficients $A_L^{\nu n}$. For scattering we plan to generalize the results of I so as to obtain expansions corresponding to other bases than the simplest canonical one, in particular the bases generated by the group reductions $O(3,1)\supset O(2,1)\supset O(2)$ and $O(3,1)\supset E_2\supset O_2$. (For decays these "noncompact bases" have no analogies.)

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Multiplicity Distribution in the Multiperipheral Model and a One-Dimensional Gas*

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The multiplicity distribution at high energy in the multiperipheral model for the ϕ^3 theory is shown to be identical to the grand canonical ensemble distribution of a particular one-dimensional gas with only repulsive forces, which can be decomposed into two-body, three-body, and other multibody forces. The specific form of these forces and the corresponding virial expansion of the gas system are discussed. An alternative systematic expansion method is developed which is different from the virial series but appears to be of a greater practical value for this particular class of physical problems.

I. INTRODUCTION

In this paper, we shall discuss the exact gas-analog problem in statistical mechanics that corresponds to the multiplicity distribution at high energy in the multiperipheral model of Amati, Fubini, and Stanghellini[1] for the ϕ^3 theory (hereafter referred to either as the ϕ^3-multiperipheral model or simply as the multiperipheral model). The interaction Lagrangian is assumed to be

$$(3!)^{-1}mg\phi^3, \tag{1}$$

where ϕ is a scalar field, m denotes its mass, and g is the dimensionless coupling constant. In the ϕ^3-multiperipheral model, the two-body elastic scattering is given simply by the sum of all t-channel ladder diagrams; the corresponding absorptive parts then give the multiplicity distribution. Such a sum of ladder diagrams is of interest since, as is well known, it represents on the one hand the sum of all "leading" diagrams in a perturbation expansion of the ϕ^3 theory at high energy, and on the other hand, it gives the simplest prototype of field-theoretic models that exhibit Regge behavior for elastic scattering,[2] and a $\ln s$ dependence for multiplicity.[1] There exists already quite a sizable literature[3-6] which discusses the similarity between the meson distribution in a multiperipheral-type model and the ensemble distribution of a gas system in statistical mechanics. However, as yet, the precise formulation and the explicit interaction of the gas-analog system have not been given. The purpose of this note is to provide this needed information in order to com-