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Two-Variable Lorentz-Group Expansions of Physical Scattering Amplitudes for Particles with Arbitrary Spins

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Two-variable expansions of relativistic scattering amplitudes that have previously been suggested for the scattering and decays of spinless particles are generalized to the case of two-body scattering of particles with arbitrary spins. The usual helicity amplitudes are expanded in terms of the transformation matrices of the homogeneous Lorentz group in a basis, corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$. The expansion can be interpreted as the usual Jacob and Wick partial-wave expansion, in which the energy dependence of the partial-wave helicity amplitudes is further expanded in terms of the $O(3,1)$ d functions. Restrictions due to parity and time-reversal invariance are discussed. The $O(3,1)$ expansions are shown to have the correct threshold behavior "term by term". Further generalizations of the formalism to include $O(2,1)$ expansions (and thus Regge-pole theory) are discussed as well as applications to particle decays (these will be presented separately).

I. INTRODUCTION

A series of previous articles has been devoted to an elementary-particle reaction theory, based on the use of two-variable expansions of scattering amplitudes (see, e.g., Refs. 1–7 and further references contained there). The expansions under consideration are provided by the representation theory of the homogeneous Lorentz group $O(3,1)$ for reactions of the type

$$1+2\to 3+4 \tag{1}$$

and by the representation theory of the rotation group $O(4)$ for three-body decays

$$1\to 2+3+4. \tag{2}$$

Previously the expansions have been written for spinless particles only. Let us here briefly summarize the situation. The amplitudes for the reactions (1) and (2) can be written as $F(s,t,u)$, where s, t, and u are the usual Mandelstam variables[8] satisfying

$$\begin{aligned} &s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_1-p_4)^2, \\ &s+t+u=m_1^2+m_2^2+m_3^3+m_4^2, \end{aligned} \tag{3}$$

m_i are the particle masses (we assume $m_i>0$, $i=1,\ldots,4$), and the momenta satisfy the usual conservation laws.

The $O(3,1)$ two-variable expansions are obtained in the following manner.

(1) A definite frame of reference is chosen by standardizing two of the four-momenta p_i, e.g., the center-of-mass system by putting

$$\begin{aligned} &p_1+p_2=(\sqrt{s},0,0,0), \\ &p_1=(E_1,0,0,p), \quad p_2=(E_2,0,0,-p), \end{aligned} \tag{4}$$

with

$$\begin{aligned} &E_1=\frac{s+m_1^2-m_2^2}{2\sqrt{s}}, \quad E_2=\frac{s-m_1^2+m_2^2}{2\sqrt{s}}, \\ &p=\left(\frac{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}{4s}\right)^{1/2}. \end{aligned} \tag{5}$$

The amplitude $F(s,t)$ can then be considered to be a function of one of the remaining momenta only, e.g., of the components of p_3.

(2) Definite curvilinear coordinates are chosen on the hyperboloid $p_3^2=m_3^2$, i.e., on the mass shell of one of the particles. (It is more convenient to interpret this hyperboloid as the common velocity space $v^2=v_0^2-\vec{v}^2=1$ of all the particles, putting $v=p/m$.) For instance, we can choose spherical coordinates, putting

$$\begin{aligned} p&=mv \\ &=m(\cosh a, \sinh a\sin\theta\cos\phi, \sinh a\sin\theta\sin\phi, \sinh a\cos\theta). \end{aligned} \tag{6}$$

The scattering amplitude can now be considered to be a function of the spherical coordinates of p_3, and since ϕ simply defines the position of the scattering plane, it must be a cyclic variable. Thus we obtain

$$F(s,t)=F'(p_3)=F''(a,\theta),$$

where $m\cosh a$ and $\cos\theta$ characterize the c.m. energy and scattering angle. As s and t run through the physical scattering region, we have

$$0 \leqslant a < \infty, \quad 0 \leqslant \theta \leqslant \pi \quad (\text{and } 0 \leqslant \phi < 2\pi). \tag{7}$$

Thus, the scattering amplitude is now a function, defined over the entire upper sheet of the hyperboloid $v^2 = 1$. This is a homogeneous manifold for the group O(3, 1) and it is thus very natural to expand $F(s, t)$ in terms of the basis functions of this group (figuring as the group of motions of the space of independent kinematic parameters, rather than as an invariance group of the amplitude).

(3) A definite basis is chosen for the representations of the group O(3, 1), e.g., by choosing a convenient complete set of commuting operators [consisting of the Casimir operators of O(3, 1), some further operators from the enveloping algebra of the algebra of O(3, 1), and possibly some discrete operators] and finding their common eigenfunctions. The amplitude can then be expanded in terms of the obtained basis functions and we obtain the required two-variable expansions (for details see, e.g., Ref. 5, 6).

If we choose the c.m. system as a frame of reference, spherical coordinates on the hyperboloid, and a basis corresponding to the group reduction $O(3,1) \supset O(3) \supset O(2)$, we obtain the expansion

$$F(s,t) = \sum_{l=0}^{\infty} (2l+1) \int_{\delta - i\infty}^{\delta + i\infty} (\sigma+1)^2 d\sigma \frac{\Gamma(\sigma+1)}{\Gamma(\sigma - l + 1)} A_l(\sigma) \frac{1}{(\sinh a)^{1/2}} P_{1/2+\sigma}^{-l-1/2}(\cosh a) P_l(\cos\theta), \tag{8}$$

with

$$\cosh a = \frac{s + m_3^2 - m_4^2}{2 m_3 \sqrt{s}}, \tag{9}$$

$$\cos\theta = \frac{2s(t - m_1^2 - m_3^2) + (s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)}{\{[s - (m_1+m_2)^2][s - (m_1 - m_2)^2][s - (m_3 + m_4)^2][s - (m_3 - m_4)^2]\}^{1/2}}. \tag{10}$$

In (8) $P_\nu^\mu(z)$ are Legendre functions and $A_l(\sigma)$ are the "Lorentz amplitudes" carrying all the "dynamics." We see that the entire dependence on the kinematic parameters a and θ (s and t) is displayed explicitly in known functions. Further, if we put

$$a_l(s) = \int_{\delta - i\infty}^{\delta + i\infty} (\sigma+1)^2 d\sigma \frac{\Gamma(\sigma+1)}{\Gamma(\sigma - l + 1)} \times A_l(\sigma) \frac{1}{(\sinh a)^{1/2}} P_{1/2+\sigma}^{-l-1/2}(\cosh a), \tag{11}$$

then (8) reduces to the usual partial-wave expansion [or O(3) little-group expansion[9]]. Thus, our two-variable expansion (8) can be interpreted as the O(3) little-group expansion, supplemented by an integral representation of the partial-wave amplitude $a_l(s)$.

Expansion (8) is only one example of the O(3, 1) two-variable expansions. Other expansions have been obtained, corresponding to the reduction of O(3, 1) to the subgroups[1-3,5,6] O(2, 1) and E_2 and also corresponding to "non-subgroup" type bases for O(3, 1) representations.[4]

The aim of this approach is to separate the dynamics and the kinematics of reactions as much as possible. The two-variable expansions should thus be suitable for enforcing the consequences of general principles of scattering theory (Lorentz invariance, analyticity and crossing symmetry, unitarity, etc.), for performing phenomenological fits to larger bodies of data than can be treated by single-variable expansions, and for formulating dynamical hypotheses.

In previous articles we have thus obtained a number of different two-variable expansions for the amplitude of reaction (1) (with spinless particles), having the following features.

The two-variable expansions incorporate all the "little-group expansions" currently used in the literature.[9-11] In particular they incorporate the Regge-pole expansion and the Toller expansion for elastic forward scattering. Mandelstam analyticity for $F(s, t)$ is reflected in simple analyticity properties of the Lorentz amplitudes.[3] Explicitly crossing-symmetric expansions have been obtained.[4] The expansions demonstrate the correct kinematical threshold behavior, reasonable asymptotic behavior, etc. For scattering the expansions involve at least one integral, sometimes two integrals. For decays they have been modified[7] into O(4) expansions, so as to involve double sums. These O(4) expansions have been applied to analyze Dalitz-plot distributions in $K \to 3\pi$ and $\eta \to 3\pi$ decays.[7]

Our present aim is to generalize the O(3, 1) expansions to reactions among particles with arbitrary spins. In this article we restrict ourselves to the simplest case, namely the expansion (8) corresponding to the group reduction $O(3,1) \supset O(3) \supset O(2)$. In a future article we shall also generalize the expansions corresponding to the reductions $O(3,1)\supset O(2,1)\supset O(2)$ and $O(3,1)\supset E_2 \supset O(2)$. In this generalization we want to preserve the main features of the spinless case, in particular the direct connection with the O(3) little-group expansion, correct behavior at the physical threshold, and the use of amplitudes defined in physical regions. For decays we again want to obtain O(4) expansions, suitable for a phenomenological treatment of Dalitz plot distributions (and polarizations, etc.) in three-body decays involving particles with spins.

For our present purpose the most convenient amplitudes to expand turn out to be the Jacob and Wick helicity amplitudes.[12] Feldman and Matthews[13] have suggested some general expansions of scattering amplitudes which can be specified to give little-group expansions, O(3, 1) expansions, etc. In our derivation we shall combine the approach sketched above with some of the techniques suggested by Feldman and Matthews, which prove to be very useful for keeping track in a relativistic manner of spin and linear-momentum variables separately.

II. FREE-PARTICLE STATES AND REPRESENTATIONS OF THE LORENTZ GROUP

A. Free-Particle States

We shall describe single-particle states in the standard manner, introduced by Wigner,[14] namely their state vectors will transform under unitary irreducible representations of the Poincaré group. Throughout this article we make use of the Jacob and Wick helicity formalism,[12] i.e., quantize the spin of each particle along the direction of its motion.

In order to establish notation, let us summarize a few well-known facts concerning the Poincaré group. We use a metric tensor $g_{\mu\nu}$, such that $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$ (and $g_{\mu\nu}=0$ for $\mu\neq\nu$). Greek letters run from 0 to 3, Latin ones from 1 to 3. The generators of the Poincaré group in the usual notation satisfy the commutation relations

$$\begin{aligned} &[P_\mu, P_\nu]=0,\\ &[P_\lambda, J_{\mu\nu}]=i(g_{\lambda\mu}P_\nu-g_{\nu\lambda}P_\mu), \qquad (12)\\ &[J_{\mu\nu}, J_{\rho\sigma}]=i(g_{\mu\sigma}J_{\nu\rho}+g_{\nu\rho}J_{\mu\sigma}-g_{\mu\rho}J_{\nu\sigma}-g_{\nu\sigma}J_{\mu\rho}).\end{aligned}$$

As usual, we define $\vec{J}$ and $\vec{K}$ such that

$$J_i=\tfrac{1}{2}\epsilon_{0ijk}J^{jk}\,,\quad K_i=J_{0i}\,. \qquad (13)$$

The relativistic (Pauli-Lubanski) spin operator is

$$W_\mu=-\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}J^{\nu\lambda}P^\rho \qquad (14)$$

and the two Casimir operators of the Poincaré group are

$$P^2=P_\mu P^\mu=m^2\,,\quad W^2=W_\mu W^\mu=m^2 s(s+1). \qquad (15)$$

Single-particle states will be labeled by the mass m ($m^2>0$), spin s (integer or half-integer), momentum p_μ, and spin projection (helicity) λ and possibly also intrinsic parity η:

$$|ms\eta p\lambda\rangle\equiv|p\lambda\rangle\,. \qquad (16)$$

Making use of the Wigner boost operator $U(p)$ we can express the general single-particle states in terms of states corresponding to particles at rest:

$$|p\lambda\rangle=U(p)|m_\mu\lambda\rangle\,, \qquad (17)$$

where we have $m_\mu=(m,0,0,0)$ and we choose[12]

$$U(p)=e^{-iJ_3\phi}e^{-iJ_2\theta}e^{+iJ_3\phi}e^{-iK_3 a}\,. \qquad (18)$$

We then choose

$$p=m(\cosh a,\ \sinh a\sin\theta\cos\phi,\ \sinh a\sin\theta\sin\phi,\ \sinh a\cos\theta), \qquad (19)$$

with

$$0\leqslant a<\infty,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\phi<2\pi\,. \qquad (20)$$

The states $|mp\eta p\lambda\rangle$ are then simply the Jacob and Wick helicity states, i.e., $s(s+1)$ and λ are eigenvalues of

$$\frac{-W^2}{p^2}=U(p)J^2U^{-1}(p)$$

and (21)

$$\frac{W_0}{|\vec{p}|}=\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}=U(p)J_3U^{-1}(p)\,.$$

For the purpose of obtaining O(3, 1) expansions it is very useful to consider explicitly transformations of particle states under the homogeneous Lorentz group. In order to relate the particle states $|ms\eta p\lambda\rangle$ to basis vectors of irreducible representations of the Lorentz group, we make use of a projection operator $O(p)$, introduced by Feldman and Matthews.[13] Some relevant properties of this operator are

$$\begin{aligned} &[J_{\mu\nu}, O(p)]=i\left(p_\mu\frac{\partial}{p_\nu}-p_\nu\frac{\partial}{\partial p_\mu}\right)O(p),\\ &O(p)O(q)=O(q)O(p)=O(p)\delta^4(p-q), \qquad (22)\\ &U(\Lambda)O(p)U^{-1}(\Lambda)=O(\Lambda p)\end{aligned}$$

(Λ is a Lorentz transformation), and

$$\int O(p)d^4p = 1. \tag{23}$$

We can put

$$P_\mu = \int p_\mu O(p)d^4p, \tag{24}$$

where p_μ is an eigenvalue of P_μ and then $J_{\mu\nu}$ and P_μ satisfy the commutation relations (12). We make no attempt at mathematical rigor but do note that (24) is a standard formula – the spectral resolution of the momentum operator.[15]

We now use the operator $O(p)$ to relate the basis functions (16) of irreducible representations of the Poincaré group to basis functions of irreducible representations of the Lorentz group. For the Lorentz group we make use of the notations and results of Naimark.[16] The irreducible representations of the Lorentz group are labeled by a pair of numbers (j_0, c) where j_0 is integer or half-integer and c is in general an arbitrary complex number. Their relation to the two Casimir operators of the homogeneous Lorentz group is

$$\begin{aligned}(\vec{J}^2 - \vec{K}^2)|j_0cs\lambda\rangle &= (j_0^2 + c^2 - 1)|j_0cs\lambda\rangle, \\ \vec{J}\cdot\vec{K}|j_0cs\lambda\rangle &= -ij_0c|j_0cs\lambda\rangle.\end{aligned} \tag{25}$$

Here $|j_0cs\lambda\rangle$ are basis vectors for the representation (j_0, c) and s, λ label these basis vectors.

There exists a large degree of arbitrariness as to the choice of a basis. In this article we are only interested in the simplest possibility, namely a basis corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$. The basis functions then also satisfy

$$\begin{aligned}\vec{J}^2|j_0cs\lambda\rangle &= s(s+1)|j_0cs\lambda\rangle, \\ J_3|j_0cs\lambda\rangle &= \lambda|j_0cs\lambda\rangle,\end{aligned} \tag{26}$$

i.e., s and λ correspond to the usual angular momentum and its projection onto the third axis.

Two types of representations of $O(3,1)$ will be of special importance for us:

(a) unitary irreducible representations of the principal series, for which c is pure imaginary,

$$j_0 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \qquad -i\infty < c < i\infty \tag{27}$$

(these are, naturally, infinite-dimensional):

(b) finite-dimensional representations (nonunitary)

$$j_0 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \quad c^2 = (j_0 + n)^2, \tag{28}$$

where n is a positive integer. A representation $(j_0, c = \pm(j_0 + n))$ contains representations of the rotation group $O(3)$ with $s = j_0, j_0 + 1, \ldots, j_0 + n - 1$. We shall make use in particular of the representation $(j_0, c = -j_0 - 1)$, which has been studied extensively by Joos[9] and Weinberg[17] and contains only one representation of $O(3)$ (with $s = j_0$). Note that the representations (j_0c) and $(-j_0, -c)$ are equivalent – this allows us to take $j_0 \geq 0$.

Let us consider the representation $(j_0, c) = (s, -s-1)$ and a basis vector $|s, -s-1, s\lambda\rangle \equiv |s\lambda\rangle$. Let us act on this function with the operator $O(p)$ of (22)–(24) taking $p_\mu = m_\mu = (m, 0, 0, 0)$. The function $O(m)|j_0cs\lambda\rangle$ satisfies

$$\begin{aligned}P_\mu O(m)|j_0cs\lambda\rangle &= m_\mu O(m)|j_0cs\lambda\rangle, \\ -\frac{W^2}{m^2}O(m)|j_0cs\lambda\rangle &= \vec{J}^2 O(m)|j_0cs\lambda\rangle \\ &= s(s+1)O(m)|j_0cs\lambda\rangle, \\ \frac{W_0(m)}{m}O(m)|j_0cs\lambda\rangle &= \lambda O(m)|j_0cs\lambda\rangle.\end{aligned}$$

Thus, we can identify $O(m)|j_0cs\lambda\rangle$ with the Wigner one-particle state at rest. For the $(s, -s-1)$ representation we have

$$|msm_\mu\lambda\rangle = O(m)|s\lambda\rangle. \tag{29}$$

A general single-particle state can then be written as

$$\begin{aligned}|msp\lambda\rangle &= U(p)O(m)|s\lambda\rangle \\ &= O(p)U(p)|s\lambda\rangle,\end{aligned} \tag{30}$$

where the boost $U(p)$ is given by (18). Note that the states $|s\lambda\rangle$ are eigenfunctions of $\vec{J}^2 - \vec{K}^2$, $\vec{J}\cdot\vec{K}$, $\vec{J}^2$, and J_3, whereas $|msp\lambda\rangle$ are eigenfunctions of P^2, $-W^2/m^2$, P_μ, and $\vec{J}\cdot\vec{p}/|\vec{p}| = W_0(p)/|\vec{p}|$.

The normalization of states is

$$\langle j_0cs\lambda|j_0cs'\lambda'\rangle = \delta_{ss'}\delta_{\lambda\lambda'},$$

$$\langle p\lambda|p'\lambda'\rangle = (2\pi)^3(2p_0)\delta(p - p')\delta_{\lambda\lambda'}$$

and the completeness relation is

$$\sum_\lambda \int \frac{dp}{2p_0(2\pi)^3}|p\lambda\rangle\langle p\lambda| = 1.$$

In order to describe states with a definite intrinsic parity we must consider simultaneously two representations of the Lorentz group, namely (j_0, c) and $(j_0, -c)$. Using the conventions of Naimark,[16] we have

$$|\eta s\lambda\rangle = \frac{1}{\sqrt{2}}[|j_0cs\lambda\rangle + \eta(-1)^{[s]}|j_0 - cs\lambda\rangle], \tag{31}$$

with $[s] = s$ for s integer and $[s] = s - \frac{1}{2}$ for s half-odd-integer.

A single-particle state of definite mass m, spin s, momentum p_μ, helicity λ, and parity η can be written as

$$|p\lambda\eta\rangle = U(p)O(m)\frac{1}{\sqrt{2}}[|j_0 cs\lambda\rangle + \eta(-1)^{[s]}|j_0 - cs\lambda\rangle]. \tag{32}$$

B. Transformation Matrices

When writing expansions we shall need the O(3, 1) transformation matrices, written in an O(3, 1) $\supset$O(3)$\supset$O(2) basis. More precisely, we shall use the Naimark canonical basis,[16] corresponding to the above reduction, with such phase conventions that the matrix elements of equivalent representations satisfy

$$D^{j_0 c}_{j_1 m_1 j_2 m_2}(g) = D^{-j_0 -c}_{j_1 m_1 j_2 m_2}(g).$$

The matrix elements for unitary representations of the principal series have been calculated by numerous authors (see, e.g., Refs. 18–20). Since conventions (and misprints) differ from paper to paper we find it expedient to present the relevant formulas here. A canonical basis[16] in the Hilbert space $L^2(z)$ of functions satisfying

$$\int |f(z)|^2 dz \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 dx\,dy < \infty$$

can be written as

$$\begin{aligned}|j_0\rho j\mu\rangle &= f^{j_0\rho}_{j\mu}(z)\\ &= \kappa^{j_0\rho}_j(-1)^{j-\mu}\left(\frac{(2j+1)}{\pi}\,\frac{(j-j_0)!(j+j_0)!}{(j-\mu)!(j+\mu)!}\right)^{1/2}(1+|z|^2)^{-1-j+i\rho/2}\\ &\times \sum_{d=\max(0,-\mu-j_0)}^{\min(j-\mu,\,j-j_0)} \frac{(j-\mu)!(j+\mu)!}{d!(j-\mu-d)!(j-j_0-d)!(\mu+j_0+d)!}(-1)^{-d}\bar{z}^{\,j-j_0-d} z^{j-\mu-d},\end{aligned} \tag{33}$$

with $\rho = 2i(\operatorname{sgn} j_0)c$ and

$$\kappa^{j_0\rho}_j = \prod_{\nu=|j_0|}^{j} \frac{-2\nu+i\rho}{(4\nu^2+\rho^2)^{1/2}} = e^{i\pi(j-|j_0|+1)(j+|j_0|)/2}\left(\frac{\Gamma(j-i\rho+1)\Gamma(2|j_0|+i\rho)}{\Gamma(j+i\rho+1)\Gamma(2|j_0|-i\rho)}\right)^{1/2}. \tag{34}$$

We define the matrix element (or the D function) of the Lorentz group to be

$$\begin{aligned}D^{j_0\rho}_{j_1 m_1 j_2 m_2}(g) &= D^{j_0\rho}_{j_1 m_1 j_2 m_2}(\phi,\theta,\psi,a,\alpha,\beta)\\ &= \langle j_0\rho j_1 m_1|e^{-i\phi J_3}e^{-i\rho J_2}e^{-i\psi J_3}e^{-iaK_3}e^{-i\alpha J_2}e^{-i\beta J_3}|j_0\rho j_2 m_2\rangle\\ &= \sum_\lambda D^{j_1}_{m_1\lambda}(\phi,\theta,\psi)\,d^{j_0\rho}_{j_1 j_2\lambda}(a)\,D^{j_2}_{\lambda m_2}(0,\alpha,\beta).\end{aligned} \tag{35}$$

In (35), e.g., $D^j_{m\lambda}(\phi,\theta,\psi)$ is a Wigner D function, i.e., a matrix element of the rotation group O(3), and $d^{j_0\rho}_{j_1 j_2\lambda}(a)$ is a "reduced" O(3, 1) matrix element, which can be written in the basis (33) as

$$\begin{aligned}d^{j_0\rho}_{j_1 j_2\lambda}(a) &= \langle j_0\rho j_1\lambda|e^{-iaK_3}|j_0\rho j_2\lambda\rangle\\ &= \kappa^{j_0\rho}_{j_1}\kappa^{j_0\rho\,*}_{j_2}(-1)^{j_1+j_2-2\lambda}\left((2j_1+1)(2j_2+1)\frac{(j_1-j_0)!(j_1+j_0)!(j_2-j_0)!(j_2+j_0)!}{(j_1-\lambda)!(j_1+\lambda)!(j_2-\lambda)!(j_2+\lambda)!}\right)^{1/2}\\ &\times \sum_{d_1 d_2}\frac{(j_1-\lambda)!(j_1+\lambda)!(j_2-\lambda)!(j_2+\lambda)!}{d_1!d_2!(j_1-\lambda-d_1)!(j_2-\lambda-d_2)!(j_1-j_0-d_1)!(j_2-j_0-d_2)!(\lambda+j_0+d_1)!(\lambda+j_0+d_2)!}(-1)^{d_1+d_2}\\ &\times \exp[-2a(\tfrac{1}{4}i\rho+\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}j_0+d_2)]\int_0^\infty dx\,x^{j_1+j_2-j_0-\lambda-d_1-d_2}(1+x)^{-1-j_1+i\rho/2}(e^{-2a}+x)^{-1-j_2-i\rho/2}.\end{aligned}$$

The integral can be calculated in various manners to give, e.g.,

$$\begin{aligned}I &= \frac{(j_1+j_2-j_0-\lambda-d_1-d_2)!(j_0+\lambda+d_1+d_2)!}{(j_1+j_2+1)!}\exp[-2a(j_1-j_0-\lambda-d_1-d_2-\tfrac{1}{2}i\rho)]\\ &\times {}_2F_1(1+j_1-\tfrac{1}{2}i\rho,\ j_1+j_2-j_0-\lambda-d_1-d_2+1,\ j_1+j_2+2,\ 1-e^{-2a})\end{aligned} \tag{37}$$

or

$$I=\frac{1}{2^{j_1+j_2}}\sum_{r,s}\frac{(j_0+\lambda+d_1+d_2)!\,(j_1+j_2-j_0-\lambda-d_1-d_2)!}{r!\,s!\,(j_0+\lambda+d_1+d_2-r)!\,(j_1+j_2-j_0-\lambda-d_1-d_2-s)!}\,\frac{(-1)^{-j_0-\lambda-d_1-d_2+r-s}}{(r+s-j_1+\frac{1}{2}i\rho)}$$

$$\times e^{a(j_2+1+r-s+i\rho/2)}(\sinh a)^{-j_1-j_2-1}\sinh(r+s-2j_1+i\rho)a\,. \qquad (38)$$

Some of the useful properties of the d functions are

$$d^{j_0\rho}_{j_1j_2,\lambda}(0)=\delta_{j_1j_2}, \qquad (39)$$

$$d^{j_0\rho}_{j_1j_2\lambda}(a)=[d^{j_0\rho}_{j_2j_1\lambda}(-a)]^*,\qquad d^{j_0\rho}_{j_1j_2\lambda}(a)=d^{-j_0-\rho}_{j_1j_2\lambda}(a),\qquad d^{j_0\rho}_{j_1j_2\lambda}(a)=d^{-j_0\rho}_{j_1j_2-\lambda}(a)\,. \qquad (40)$$

We also have

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\psi\int_0^{\infty}\sinh^2 a\,da\int_0^{\pi}\sin\alpha\,d\alpha\int_0^{2\pi}d\beta\,D^{j_0\rho *}_{j_1m_1j_2m_2}(g)D^{j_0'\rho'}_{j_1'm_1'j_2'm_2'}(g)$$

$$=k\delta_{j_0j_0'}\,\frac{\delta(\rho-\rho')}{j_0^{\,2}+\frac{1}{4}\rho^2}\,\delta_{j_1j_1'}\delta_{j_2j_2'}\delta_{m_1m_1'}\delta_{m_2m_2'}\,, \qquad (41)$$

where k is a numeric constant. In the following we shall usually label the O(3, 1) transformations by c instead of ρ.

The transformation matrices for the finite-dimensional representations $(s,-(s+1))$ and $(s,(s+1))$ can be written[17] in the same basis as

$$u_{\lambda_1\lambda_2}(p,\eta_1\eta_2)=\langle s\lambda_1\eta_1|U(p)|s\lambda_2\eta_2\rangle$$

$$=\langle s\lambda_1\eta_1|e^{-i\phi J_3}e^{-i\theta J_3}e^{i\phi J_3}e^{-iaK_3}|s\lambda_2\eta_2\rangle$$

$$=D^s_{\lambda_1\lambda_2}(\phi,\theta,-\phi)\tfrac{1}{2}(e^{a\lambda_2}+\eta_1\eta_2e^{-a\lambda_2})\,. \qquad (42)$$

We also have

$$\bar{u}_{\lambda_1\lambda_2}(p,\eta_1\eta_2)=\langle s\lambda_1\eta_1|U^{-1}(p)|s\lambda_2\eta_2\rangle$$

$$=\tfrac{1}{2}(e^{-a\lambda_1}+\eta_1\eta_2e^{a\lambda_1})D^s_{\lambda_2\lambda_1}(\phi,\theta,-\phi) \qquad (43)$$

and

$$\sum_{\lambda\eta}u_{\lambda_1\lambda}(p,\eta_1\eta)\bar{u}_{\lambda\lambda_2}(p,\eta\eta_2)=\delta_{\lambda_1\lambda_2}\delta_{\eta_1\eta_2}\,,$$

$$\sum_{\lambda\eta}\bar{u}_{\lambda_1\lambda}(p,\eta_1\eta)u_{\lambda\lambda_2}(p,\eta\eta_2)=\delta_{\lambda_1\lambda_2}\delta_{\eta_1\eta_2}\,. \qquad (44)$$

III. LORENTZ-GROUP EXPANSIONS OF SCATTERING AMPLITUDES

Let us now consider a scattering amplitude

$$\langle 34|T|12\rangle=\langle p_3s_3\lambda_3\eta_3\,,p_4s_4\lambda_4\eta_4|T|\,p_1s_1\lambda_1\eta_1\,,p_2s_2\lambda_2\eta_2\rangle\,, \qquad (45)$$

i.e., a matrix element of the T matrix (related to the S matrix by $S=1+iT$) taken between two-particle initial and final states (which are direct products of the single-particle states considered above).

As suggested in the Introduction, we consider the reaction $1+2\rightarrow 3+4$ in the center-of-mass system, characterized by Eqs. (4) and (5).

The particle momenta can be written as

$$p_1=m_1(\cosh a_1,\,0,\,0,\,\sinh a_1),$$

$$p_2=m_2(\cosh a_2,\,0,\,0,\,-\sinh a_2),$$

$$p_3=m_3(\cosh a_3,\,\sinh a_3\sin\theta\cos\phi,\,\sinh a_3\sin\theta\sin\phi,\,\sinh a_3\cos\theta),$$

$$p_4=m_4(\cosh a_4,\,-\sinh a_4\sin\theta\cos\phi,\,-\sinh a_4\sin\theta\sin\phi,\,-\sinh a_4\cos\theta), \qquad (46)$$

with

$$m_1\sinh a_1 = m_2\sinh a_2, \quad m_3\sinh a_3 = m_4\sinh a_4,$$
$$m_1\cosh a_1 + m_2\cosh a_2 = m_3\cosh a_3 + m_4\cosh a_4. \tag{47}$$

The variable $\cosh a \equiv \cosh a_3$ and $\cos\theta$ are given in terms of s and t by (9) and (10).

Using (30) and the conventions of Jacob and Wick[12] we can write the states as

$$\begin{aligned}
|p_1 s_1\lambda_1\eta_1\rangle &= e^{-ia_1K_3^{(1)}}O(m_1)|s_1\lambda_1\eta_1\rangle,\\
|p_2 s_2\lambda_2\eta_2\rangle &= e^{ia_2K_3^{(2)}}O(m_2)|s_2-\lambda_2\eta_2\rangle,\\
|p_3 s_3\lambda_3\eta_3\rangle &= R^{(3)}(\phi,\theta,-\phi)e^{-ia_3K_3^{(3)}}O(m_3)|s_3\lambda_3\eta_3\rangle,\\
|p_4 s_4\lambda_4\eta_4\rangle &= R^{(4)}(\phi,\theta,-\phi)e^{ia_4K_3^{(4)}}O(m_4)|s_4-\lambda_4\eta_4\rangle,
\end{aligned} \tag{48}$$

with

$$R^{(i)}(\phi\theta-\phi) = e^{-iJ_3^{(i)}\phi}e^{-iJ_2^{(i)}\theta}e^{iJ_3^{(i)}\phi}. \tag{49}$$

($\vec{J}^{(i)}$ and $\vec{K}^{(i)}$ are the generators for the ith particle.) Note that we are using the Weinberg-Joos O(3, 1) states (31), with $j_0 = s$ and $c = -(s+1)$.

Using formulas (48) [in the general form (30)], we can write the scattering amplitude as

$$\langle 34|T|12\rangle = \langle s_3\lambda_3\eta_3, s_4-\lambda_4\eta_4|U^{-1}(p_4)U^{-1}(p_3)O(p_3)O(p_4)TO(p_1)O(p_2)U(p_1)U(p_2)|s_1\lambda_1\eta_1, s_2-\lambda_2\eta_2\rangle. \tag{50}$$

Using (42) and (43) we can extract some of the transformation matrices $U(p)$ to obtain

$$\begin{aligned}
\langle 34|T|12\rangle = \sum_{\bar\lambda_i,\bar\eta_i} &\bar{u}_{-\lambda_4\bar\lambda_4}(p_4,\eta_4\bar\eta_4)u_{\bar\lambda_1\lambda_1}(p_1,\bar\eta_1\eta_1)u_{\bar\lambda_2-\lambda_2}(p_2\bar\eta_2\eta_2)\\
&\times\langle s_3\lambda_3\eta_3, s_4\bar\lambda_4\bar\eta_4|U^{-1}(p_3)O(p_3)O(p_4)TO(p_1)O(p_2)|s_1\bar\lambda_1\bar\eta_1, s_2\bar\lambda_2\bar\eta_2\rangle
\end{aligned} \tag{51}$$

(the summation is over $\bar\lambda_1$, $\bar\lambda_2$, $\bar\lambda_4$, $\bar\eta_1$, $\bar\eta_2$, and $\bar\eta_4$).

Let us now make use of energy-momentum conservation, of the fact that we are in the c.m. system, and of the properties of the projection operators $O(p)$. Since the total momentum P_1+P_2 commutes with the T matrix, we have

$$\begin{aligned}
U^{-1}(p_3)O(p_3)O(p_4)TO(p_1)O(p_2) &= U^{-1}(p_3)O(p_3)O(p_3+p_4)TO(p_1+p_2)O(p_2)\\
&= U^{-1}(p_3)O(p_3)TO(p_3+p_4)O(p_1+p_2)O(p_2)\\
&= \delta(p_1+p_2-p_3-p_4)O(m_3)U^{-1}(p_3)TO(p_1)O(p_2).
\end{aligned} \tag{52}$$

Our aim is to make the entire dependence of the scattering amplitude on the kinematic parameters s and t explicit. Thus, we must extract this dependence from the operators $U^{-1}(p_3)$, $O(p_1)$, and $O(p_2)$. Since we are using p_1 and p_2 to specify the frame of reference [see (46)] we introduce a tetrad of orthogonal unit vectors[13]: n, α, β, and γ.

$$\begin{aligned}
n &= \frac{1}{\sqrt{s}}(p_1+p_2),\\
\gamma &= \{s[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{-1/2}[(s-m_1^2+m_2^2)p_1-(s+m_1^2-m_2^2)p_2],\\
\alpha &= N_\alpha(p_3-(p_3n)n+(p_3\gamma)\gamma),\\
\beta_\mu &= \epsilon_{\mu\nu\lambda\rho}n_\nu\gamma_\lambda\alpha_\rho,
\end{aligned} \tag{53}$$

satisfying $n^2 = -\alpha^2 = -\beta^2 = -\gamma^2 = 1$. It is easy to check that if p_i are given by (46), then the vectors (53) reduce to

$$n=(1,0,0,0), \quad \alpha=(0,1,0,0), \quad \beta=(0,0,1,0), \quad \gamma=(0,0,0,1) \tag{54}$$

and they obviously do not depend on s and t. We now notice that (23) implies that

$$O(p_1)O(p_2) = g(s)O(n)O(\gamma),$$

where g is the inverse Jacobian for the transformation $p_1, p_2 \to n, \gamma$.

We have

$$g(s)=\left(\frac{4}{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}\right)^2 \quad . \tag{55}$$

The expansion of the scattering amplitude can now be written as

$$\langle 34|T|12\rangle=\delta(p_1+p_2-p_3-p_4)g(s)\sum_{\bar\lambda_i\bar\eta_i}\bar u_{-\lambda_4\bar\lambda_4}(p_4,\eta_4\bar\eta_4)u_{\bar\lambda_1\lambda_1}(p_1,\bar\eta_1\eta_1)u_{\bar\lambda_2-\lambda_2}(p_2,\bar\eta_2\eta_2)$$
$$\times\langle s_3\lambda_3\eta_3,\, s_4\bar\lambda_4\bar\eta_4|O(m_3)U^{-1}(p_3)TO(n)O(\gamma)|s_1\bar\lambda_1\bar\eta_1,\, s_2\bar\lambda_2\bar\lambda_2\rangle\ . \tag{56}$$

The dependence on s and t is now completely in the matrix elements $u_{\lambda_i\bar\lambda_i}$ and in the boost operator $U^{-1}(p_3)$. [The factor $g(s)$ (55) does of course also depend on s. This dependence can either be kept explicitly as in (56) and further in this article or be dropped by redefining the expanded amplitude. Alternatively (and maybe preferably) we could combine the factor $g(s)$ with the boost operator $U^{-1}(p_3)$, consider them both to be functions over the Lorentz group (see below), and expand their product in terms of the transformation matrices of the Lorentz group.] In order to make the entire dependence explicit let us insert a complete set of single-particle states for particle 3 on both sides of the boost operator. Contrary to the labeling of single-particle states by linear momenta (as in Sec. II) we use a basis corresponding to the following reduction of the Poincaré group P (including space reflections):

$$\mathrm{P}\supset \mathrm{O}(3,1)\supset \mathrm{O}(3)\supset \mathrm{O}(2).$$

The states can thus be denoted $|ms\eta j_0cj\mu\rangle$, or for brevity $|j_0cj\mu\rangle$. Inserting the complete set of functions gives rise to the following three factors:

(i) $\langle s_3\lambda_3\eta_3|O(m_3)|m_3s_3\eta j_0cj\mu\rangle=\delta_{\eta\eta_3}\delta_{js_3}\delta_{\lambda_3\mu}\langle s_3\eta_3|O(m_3)|j_0cs_3\rangle$.

(It follows from the Wigner-Eckart theorem that there is no dependence on λ_3.)

(ii) $$\langle m_3s_3\eta j_0cj\mu|U^{-1}(p_3)|m_3s_3\eta' j_0'c'j'\mu'\rangle=\delta_{j_0j_0'}\delta_{cc'}\langle m_3s_3\eta j_0cj\mu|U^{-1}(p_3)|\eta' j_0cj'\mu'\rangle$$
$$=\delta_{j_0j_0'}\delta_{cc'}a_{\eta\eta'}\langle j_0cj\mu|e^{iK_3a}e^{-iJ_3\phi}e^{iJ_2\theta}e^{iJ_3\phi}|j_0cj'\mu'\rangle$$
$$=\delta_{j_0j_0'}\delta_{cc'}a_{\eta\eta'}d^{j_0c*}_{j'j\mu}(a)D^{j'*}_{\mu'\mu}(\phi,\theta,-\phi).$$

Here $a_{\eta\eta'}$ are numbers, depending on the parities only.

(iii) $\langle m_3s_3\eta' j_0'c'j'\mu',\, s_4\bar\lambda_4\bar\eta_4|TO(n)O(\gamma)|s_1\bar\lambda_1\bar\eta_1,\, s_2\bar\lambda_2\bar\eta_2\rangle$.

Let us now introduce the notation

$$A^{j_0c}_{j\mu}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\,\bar\lambda_1\bar\lambda_2\bar\lambda_4)=\sum_\eta a_{\eta_3\eta}\langle s_3\eta_3|O(m_3)|j_0cs_3\rangle\langle m_3s_3\eta j_0cj\mu,\, s_4\bar\lambda_4\bar\eta_4|TO(n)O(\gamma)|s_1\bar\lambda_1\bar\eta_1,\, s_2\bar\lambda_2\bar\eta_2\rangle\ . \tag{57}$$

Introducing the helicity amplitudes $f_{\lambda_i}(a,\theta,\phi)$ by putting

$$\langle 34|T|12\rangle=\delta(p_1+p_2-p_3-p_4)f_{\lambda_i}(a,\theta,\phi), \tag{58}$$

we can rewrite the expansion (56) as

$$f_{\lambda_i}(a,\theta,\phi)=g(s)\sum_{\bar\lambda_1\bar\lambda_2\bar\lambda_4}\sum_{\bar\eta_1\bar\eta_2\bar\eta_4}\sum_{j_0cj\mu}\bar u_{-\lambda_4\bar\lambda_4}(p_4,\eta_4\bar\eta_4)\bar u_{\bar\lambda_1\lambda_1}(p_1,\eta_1\bar\eta_1)u_{\bar\lambda_2-\lambda_2}(p_2,\eta_2\bar\eta_2)$$
$$\times A^{j_0c}_{j\mu}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\,\bar\lambda_1\bar\lambda_2\bar\lambda_4)d^{j_0c*}_{js_3\lambda_3}(a)D^{j*}_{\mu\lambda_3}(\phi,\theta,-\phi). \tag{59}$$

The sum over j_0 is such that the representation (j_0,c) contains the angular momentum s_3: $-s_3\leq j_0\leq s_3$. We can however restrict ourselves to $j_0\geq 0$, in view of the equivalence of the representations (j_0,c) and $(-j_0,-c)$. We then have $j=j_0,\ j_0+1,\ j_0+2,\ \ldots$. The range of summation of the projections λ_i and μ is obvious. The range of the variable c depends on the behavior of the amplitude $f_{\lambda_i}(a,\theta,\phi)$ and can run through both continuous and discrete values (this sum is to be interpreted as a sum and an integral). In particular, if the amplitudes satisfy a square-integrability condition

$$\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{|f_{\lambda_i}(a,\theta,\phi)|^2}{|g(s)|^2}\sinh^2 a\,da\,\sin\theta\,d\theta\,d\phi<\infty, \tag{60}$$

then it follows from Plancherel's theorem[16] that the expansion will be over the principal series of unitary representations only, i.e.,

$$\sum_c \to \int_{-i\infty}^{+i\infty} dc\,. \tag{61}$$

In general the condition (60) is unphysical, since it requires that the amplitudes fall off to zero asymptotically. This can be remedied by considering more general integration paths in the complex c plane; however we will not go into these problems here and will restrict ourselves to the principal series.

Let us further simplify expansion (59). Using (42) and (43) we have

$$\begin{aligned} &u_{\bar\lambda_1\lambda_1}(p_1,\eta_1\bar\eta_1)=\delta_{\lambda_1\bar\lambda_1}\tfrac{1}{2}(e^{-a_1\lambda_1}+\eta_1\bar\eta_1 e^{a_1\lambda_1})=f_0(a_1,\lambda_1,\eta_1\bar\eta_1)\delta_{\lambda_1\bar\lambda_1},\\ &u_{\bar\lambda_2,-\lambda_2}(p_2,\eta_2\bar\eta_2)=f_0(a_2,\lambda_2,\eta_2\bar\eta_2)\delta_{\lambda_2,-\bar\lambda_2},\\ &\bar u_{-\lambda_4\lambda_4}(p_2,\eta_4\bar\eta_4)=\tfrac{1}{2}(e^{a_4\lambda_4}+\eta_4\bar\eta_4 e^{-a_4\lambda_4})D^{s_4}_{\bar\lambda_4-\lambda_4}(\phi,\theta,-\phi)=f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)D^{s_4}_{\bar\lambda_4-\lambda_4}(\phi,\theta,-\phi). \end{aligned} \tag{62}$$

We can combine the two O(3) D functions together to obtain

$$\begin{aligned} f_{\lambda_i}(a,\theta,\phi)=g(s)\sum_{\bar\lambda_4}\sum_{\bar\eta_1\bar\eta_2\bar\eta_4}\sum_{j_0 j\mu}\sum_J(2J+1)\int_{-i\infty}^{+i\infty}\frac{j_0^{\,2}-c^2}{k}dc\, f_0(a_1,\lambda_1,\eta_1\bar\eta_1)f_0(a_2,\lambda_2,\eta_2\bar\eta_2)\\ \times f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)(-1)^{-\lambda_3+\lambda_4+\mu+\bar\lambda_4}\begin{pmatrix} j & s_4 & J\\ \lambda_3 & -\lambda_4 & -\lambda_3+\lambda_4\end{pmatrix}\begin{pmatrix} j & s_4 & J\\ \mu & \bar\lambda & -\mu-\bar\lambda_4\end{pmatrix}\\ \times A^{j_0c}_{j\mu}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)d^{j_0c*}_{js_3\lambda_3}(a)D^{J*}_{\mu+\bar\lambda_4,\lambda_3-\lambda_4}(\phi,\theta,-\phi). \end{aligned} \tag{63}$$

The brackets represent the ordinary $3j$ symbols of O(3). The dependence of the helicity amplitudes on the angle ϕ is given[12] by the Wigner D function $D^{J*}_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}(\phi,\theta,-\phi)$. Thus the left-hand side of (63) contains ϕ only in the factor $e^{i(\lambda_1-\lambda_2-\lambda_3+\lambda_4)\phi}$. On the right-hand side we have $e^{i(\mu+\bar\lambda_4-\lambda_3+\lambda_4)\phi}$. Hence we obtain $\lambda_1-\lambda_2=\mu+\bar\lambda_4$, i.e., we can drop the summation over μ, put $\mu=\lambda_1-\lambda_2-\bar\lambda_4$, and drop the label μ on the expansion coefficient. Finally, we obtain the following expansion formula for the helicity amplitudes:

$$\begin{aligned} f_{\lambda_i}(a,\theta,\phi)=g(s)\sum_{\bar\eta_1\bar\eta_2\bar\eta_4}\sum_{\bar\lambda_4}\sum_{j_0=\{0,\frac{1}{2}\}}^{s_3}\sum_{j=j_0}^{\infty}\sum_{J=|j-s_4|}^{j+s_4}(2J+1)\int_{-i\infty}^{+i\infty}\frac{j_0^{\,2}-c^2}{k}dc\, f_0(a_1,\lambda_1,\eta_1\bar\eta_1)\\ \times f_0(a_2,\lambda_2,\eta_2\bar\eta_2)f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)(-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}\\ \times\begin{pmatrix} j & s_4 & J\\ \lambda_3 & -\lambda_4 & \lambda_3-\lambda_4\end{pmatrix}\begin{pmatrix} j & s_4 & J\\ \lambda_1-\lambda_2-\bar\lambda_4 & \bar\lambda_4 & -\lambda_1+\lambda_2\end{pmatrix}\\ \times A^{j_0c}_j(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)d^{j_0c*}_{js_3\lambda_3}(a)D^{J*}_{\lambda_1-\lambda_2\lambda_3-\lambda_4}(\phi,\theta,-\phi). \end{aligned} \tag{64}$$

Thus we have obtained an expansion of the helicity amplitude $f_{\lambda_i}(a,\theta,\phi)$ in which the entire dependence on s and t (a and θ) is contained explicitly in the O(3, 1) and O(3) transformation matrices. We have introduced the Plancherel measure $j_0^{\,2}-c^2$ [see (41)] into the formula in order to simplify the inversion formula (see next section). The sum over j_0 is from 0 or $\frac{1}{2}$ depending on whether s_3 is integer or half-odd-integer.

The coefficients $A^{j_0c}_j(\eta_i,\lambda_i)$ are generalizations of the "Lorentz amplitudes," introduced previously in the spinless case.[1-7] These Lorentz amplitudes carry the entire dynamics of each individual scattering process. It is important to minimize the "kinematical constraints" upon the Lorentz amplitudes, i.e., the correct kinematical behavior of the total amplitudes at thresholds, at the boundary of the physical region, at $s=0$ or $t=0$, etc. should as far as possible be ensured by the behavior of the transformation matrices in (64) (see below).

IV. INVERSION OF THE EXPANSION FORMULA

In order to obtain an expression for the expansion coefficient $A_j^{j_0c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)$, let us first rewrite formula (59) as

$$g^{-1}(s)f_{\lambda_i}(a,\theta,\phi)=\sum_{\bar\eta_1\bar\eta_2\bar\eta_4}\sum_{\bar\lambda_4}\sum_{j_0=\{0,\frac12\}}^{s_3}\sum_{j=j_0}^{\infty}\int_{-i\infty}^{+i\infty}\frac{j_0^2-c^2}{k}dc$$
$$\times f_0(a_1,\lambda_1,\eta_1\bar\eta_1)f_0(a_2,\lambda_2,\eta_2\bar\eta_2)f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)D^{s_4*}_{\bar\lambda_4,-\lambda_4}(\phi,\theta,-\phi)$$
$$\times A_j^{j_0c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)d^{j_0c*}_{js_3\lambda_3}(a)D^{j*}_{\lambda_1-\lambda_2-\bar\lambda_4,\lambda_3}(\phi,\theta,-\phi)\,. \tag{65}$$

Using formulas (44) and (41) we can readily obtain

$$A_j^{j_0c}=\frac{8\pi^2}{2s_3+1}\int_0^\infty\sinh^2a\,da\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\phi\sum_{\lambda_3\lambda_4}\sum_{\eta_1\eta_2\eta_4}f_0(a_1,-\lambda_1,\eta_1\bar\eta_1)f_0(a_2,-\lambda_2,\eta_2\bar\eta_2)f_0(a_4,\lambda_4,\eta_4\bar\eta_4)$$
$$\times D^{s_4}_{\bar\lambda_4,-\lambda_4}(\phi,\theta,-\phi)d^{j_0c}_{js_3\lambda_3}(a)D^{j}_{\lambda_1-\lambda_2-\bar\lambda_4,\lambda_3}(\phi,\theta,-\phi)$$
$$\times g^{-1}(s)f_{\lambda_i}(a,\theta,\phi). \tag{66}$$

Finally, we can write

$$A_j^{j_0c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)$$
$$=\frac{8\pi^2}{2s_3+1}\int_0^\infty\sinh^2a\,da\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\phi\sum_{\lambda_3\lambda_4}\sum_{\eta_1\eta_2\eta_4}\sum_{J}f_0(a_1,-\lambda_1,\eta_1\bar\eta_1)f_0(a_2,-\lambda_2,\eta_2\bar\eta_2)f_0(a_4,\lambda_4,\eta_4\bar\eta_4)$$
$$\times(-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}(2J+1)\begin{pmatrix}s_4 & j & J\\ \bar\lambda_4 & \lambda_1-\lambda_2-\bar\lambda_4 & -\lambda_1+\lambda_2\end{pmatrix}\begin{pmatrix}s_4 & j & J\\ -\lambda_4 & \lambda_3 & -\lambda_3+\lambda_4\end{pmatrix}$$
$$\times d^{j_0c}_{js_3\lambda}(a)D^{J}_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}(\phi,\theta,-\phi)g^{-1}(s)f_{\lambda_i}(a,\theta,\phi). \tag{67}$$

Formula (67) is only valid for unitary representations of the principal series, i.e., for c = pure imaginary. In this article we shall not discuss any generalizations to nonunitary representations, still less the corresponding generalizations of (67).

V. RELATION TO THE JACOB AND WICK EXPANSION AND PHYSICAL MEANING OF THE QUANTUM NUMBERS

The Jacob and Wick expansion,[12] which is simply the O(3) little-group expansion of a helicity amplitude, is

$$f_{\lambda_i}(a,\theta,\phi)=\sum_{J=\max\{|\lambda|,|\mu|\}}^{\infty}(2J+1)\langle\lambda_3\lambda_4|S^J(a)|\lambda_1\lambda_2\rangle D^{J*}_{\lambda\mu}(\phi,\theta,-\phi), \tag{68}$$

where J is the total angular momentum, $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$, and $\langle\lambda_3\lambda_4|S^J(a)|\lambda_1\lambda_2\rangle$ are the partial-wave helicity amplitudes.

In order to find the relation between our O(3, 1) expansion (64) and the expansion (68) we must reorder the summation in formula (64). This is a somewhat tedious procedure but the result is very simple, namely,

$$\sum_{j_0=\{0,\frac12\}}^{s_3}\sum_{j=j_0}^{\infty}\sum_{J=|j-s_4|}^{j+s_4}=\sum_{J=\{0,\frac12\}}^{\infty}\sum_{j_0=\{0,\frac12\}}^{\min(J+s_4,s_3)}\sum_{j=\max\{|j_0|,|J-s_4|\}}^{J+s_4}$$

Finally, we obtain a formula, coinciding with (68), if we put the partial-wave amplitude equal to

$$\langle\lambda_3\lambda_4|S^J(a)|\lambda_1\lambda_2\rangle$$
$$= g(s)\sum_{\bar\eta_1\bar\eta_2\bar\eta_4\bar\lambda_4}\;\sum_{j_0=\{0,\frac{1}{2}\}}^{\min(J+s_4,s_3)}\;\sum_{j=\max\{j_0,(J-s_4)\}}^{J+s_4}(2J+1)\int_{-i\infty}^{+i\infty}dc\,\frac{j_0^2-c^2}{k}\,f_0(a_1,\lambda_1,\eta_1\bar\eta_1)f_0(a_2,\lambda_2,\eta_2\bar\eta_2)f_0(a_4,-\lambda_4,\eta_4\bar\eta_4)$$
$$\times(-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}\begin{pmatrix} j & s_4 & J\\ \lambda_3 & -\lambda_4 & -\lambda_3+\lambda_4\end{pmatrix}\begin{pmatrix} j & s_4 & J\\ \lambda_1-\lambda_2-\bar\lambda_4 & \bar\lambda_4 & -\lambda_1+\lambda_2\end{pmatrix}$$
$$\times A_j^{j_0c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)d^{j_0c*}_{js_3\lambda_3}(a)\,. \tag{69}$$

Thus, the O(3, 1) expansion (64) can be interpreted in exactly the same manner as in the spinless case – it is the O(3) little-group expansion (68), supplemented by an integral expansion of the O(3) partial wave amplitude, given by (69).

The meaning of some of the quantum numbers in (64) and (69) now becomes clear. The symbol J simply denotes the total angular momentum of the initial (and final) state in the c.m. system, whereas j, being the vector sum of the spin s_4 and J, is the angular momentum of particle 3 in the frame in which 4 is at rest. The Lorentz group parameter j_0 is the minimum possible value of j (for given J); η_i and λ_i are the intrinsic parities and helicities of the particles. Finally, the meaning of the continuous parameter c is less straightforward and is related to the high-energy (or short-distance) behavior of the amplitudes. We hope to return to this question in the future.

VI. RESTRICTIONS ON LORENTZ AMPLITUDES DUE TO PARITY CONSERVATION AND TIME-REVERSAL INVARIANCE

If we assume that parity is conserved in the reaction $1+2\to3+4$ or that the process is elastic and that the scattering is invariant under time reversal, then not all the helicity amplitudes and hence not all the Lorentz amplitudes will be independent.

Let us first investigate the consequences of parity conservation. The partial-wave helicity amplitudes are known to satisfy[8,12]

$$\langle-\lambda_3-\lambda_4|S^J|-\lambda_1-\lambda_2\rangle=\eta\langle\lambda_3\lambda_4|S^J|\lambda_1\lambda_2\rangle\ , \tag{70}$$

with

$$\eta=\frac{\eta_3\eta_4}{\eta_1\eta_2}(-1)^{s_3+s_4-s_1-s_2}\ .$$

We expand both sides of (70) using (69), make use of the relations

$$f_0(a,-\lambda,\eta\bar\eta)=\eta\bar\eta f_0(a,\lambda,\eta\bar\eta)$$

and

$$d^{j_0c}_{js-\lambda}(a)=d^{-j_0c}_{js\lambda}(a)=d^{j_0-c}_{js\lambda}(a)\ ,$$

and find that parity conservation implies that the Lorentz amplitudes satisfy

$$A_j^{j_0-c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,-\lambda_1-\lambda_2-\bar\lambda_4)=\bar\eta_1\bar\eta_2\eta_3\bar\eta_4(-1)^{s_1+s_2-s_3-s_4}A_j^{j_0c}(\bar\eta_1\bar\eta_2\eta_3\bar\eta_4,\lambda_1\lambda_2\bar\lambda_4)\,. \tag{71}$$

The consequences of time-reversal invariance are somewhat more complicated. Consider elastic scattering, when the masses, spins, and parities satisfy $m_1=m_3$, $m_2=m_4$, $s_1=s_3$, $s_2=s_4$, and $\eta_1=\eta_3$, $\eta_2=\eta_4$. The energies satisfy

$$\cosh a_1=\cosh a_3=\frac{s+m_1^2-m_2^2}{2m_1\sqrt{s}},\qquad \cosh a_2=\cosh a_4=\frac{s-m_1^2+m_2^2}{2m_2\sqrt{s}}$$

For the partial-wave amplitudes time-reversal invariance implies[12]

$$\langle\lambda_3\lambda_4|S^J|\lambda_1\lambda_2\rangle=\langle\lambda_1\lambda_2|S^J|\lambda_3\lambda_4\rangle\ . \tag{72}$$

We can now expand the left- and right-hand sides of (72), using the expansion formula (69). We thus obtain relations between Lorentz amplitudes, following from time-reversal invariance. The result can be written in various forms but we have not succeeded in obtaining any simple relations. One way of expressing the consequences of T invariance is

$$\begin{pmatrix} j & s_2 & J \\ \lambda_3-\lambda_4-\bar{\lambda}_2 & \bar{\lambda}_2 & -\lambda_3+\lambda_4 \end{pmatrix} A_{\bar{j}}^{\bar{j}_0\bar{c}}(\bar{\eta}_1\bar{\eta}_2\eta_3\bar{\eta}_4,\, \lambda_3\lambda_4\bar{\lambda}_2)$$
$$= \frac{32\pi^3}{(2\bar{j}+1)(2s_1+1)} \sum_{\hat{\eta}_1\hat{\eta}_2\eta_3\hat{\eta}_4} \sum_{\lambda_1\lambda_2\hat{\lambda}_4} \sum_{j_0 j} \int_{-i\infty}^{+i\infty} dc\, \frac{j_0^2-c^2}{k} A_j^{j_0c}(\hat{\eta}_1\hat{\eta}_2\eta_3\hat{\eta}_4,\, \lambda_1\lambda_2\hat{\lambda}_4)$$
$$\times\Bigg[\sum_{\lambda\eta_2\eta_4} \begin{pmatrix} \bar{j} & s_2 & J \\ \lambda_1 & \lambda_2 & -\lambda \end{pmatrix} \begin{pmatrix} j & s_4 & J \\ \lambda_3 & -\lambda_4 & -\lambda_3+\lambda_4 \end{pmatrix} \begin{pmatrix} j & s_4 & J \\ \lambda_1-\lambda_2-\hat{\lambda}_4 & \hat{\lambda}_4 & -\lambda \end{pmatrix}$$
$$\times \int_0^\infty \sinh^2 a\, da\, f_0(a_1, \lambda_1, \eta_1\hat{\eta}_1) f_0(a_2, \lambda_2, \eta_2\hat{\eta}_2) f_0(a_4, -\lambda_4, \eta_4\hat{\eta}_4)$$
$$\times f_0(a_3, -\lambda_3, \eta_3\bar{\eta}_1) f_0(a_4, -\lambda_4, \eta_4\bar{\eta}_2)\, f_0(a_2, \lambda_2, \eta_2\bar{\eta}_4)\, d_{js_3\lambda_3}^{j_0c\,*}(a)\, d_{\bar{j}s_1\lambda_1}^{\bar{j}_0\bar{c}}(a)\Bigg]. \tag{73}$$

Thus, T invariance implies a set of linear relations between Lorentz amplitudes, but we shall not discuss them any further here.

VII. THRESHOLD BEHAVIOR OF HELICITY AMPLITUDES

The most important feature of our O(3, 1) expansion (64) is that it is a two-variable expansion (the third variable ϕ is inessential), i.e., that the dependence on the energy and scattering angle is explicit. This makes the expansion suitable for considering the behavior of amplitudes in various kinematical limits. In particular it is possible to investigate problems like kinematical singularities and constraints at various points, asymptotic behavior for $s\to\infty$ and/or $t\to\infty$ in various directions, etc. The expansion presented in this paper corresponding to the group reduction $O(3,1)\supset O(3)\supset O(2)$ is specially suitable for investigating low-energy behavior, i.e., threshold behavior. We plan to return to the question of kinematical singularities and constraints in detail in a future publication. Of special interest is the behavior of amplitudes at $t=0$, when the question of Regge daughters, conspiracies, evasion, etc. arises in Regge-pole theory.[8,21-23] For the case of spin-zero particles the question of kinematic constraints at $t=0$ was treated from the point of view of O(3, 1) two-variable expansions in the second of Ref. 2. The Lorentz group expansion corresponding to the reduction $O(3,1)\supset O(2,1)\supset O(2)$ is the one that turns out to be relevant.

As was shown above in Sec. V the O(3, 1) expansion (64) coincides with the Jacob and Wick O(3) expansion, as far as the dependence on the angles θ and ϕ is concerned. Since the boundary of the physical region is given by the equation

$$|\cos\theta| = 1,$$

the behavior of the helicity amplitudes, given by expansion (64) at the boundary of the physical region, coincides with the usual behavior, given by the O(3) D functions.[8]

Let us now look at the threshold behavior. The kinematic behavior of the partial-wave helicity amplitudes at the thresholds and pseudothresholds is known to be[22]

$$\langle\lambda_3\lambda_4|S^J(a)|\lambda_1\lambda_2\rangle \simeq [s-(m_1+m_2)^2]^{l_n/2}[s-(m_1-m_2)^2]^{l_p/2}[s-(m_3+m_4)^2]^{l'_n/2}[s-(m_3-m_4)^2]^{l'_p/2}\langle\lambda_3\lambda_4|S_J(a)|\lambda_1\lambda_2\rangle', \tag{74}$$

where $\langle\lambda_3\lambda_4|S_J(a)|\lambda_1\lambda_2\rangle'$ is regular at the points of interest and l_n, l_p, l'_n, and l'_p are the minimal possible values of the orbital angular momentum at the threshold and pseudothreshold of the initial and final states, respectively.

We expect the expansion (69) of the partial-wave helicity amplitudes to converge for s and t in the physical scattering region.

We have

$$\cosh a_1 = \frac{s+m_1^2-m_2^2}{2m_1\sqrt{s}}, \qquad \cosh a_2 = \frac{s-m_1^2+m_2^2}{2m_2\sqrt{s}},$$
$$\cosh a \equiv \cosh a_3 = \frac{s+m_3^3-m_4^2}{2m_3\sqrt{s}}, \quad \cosh a_4 = \frac{s-m_3^2+m_4^2}{2m_4\sqrt{s}}. \tag{75}$$

The only threshold (or pseudothreshold) that touches the physical region is $s=\max\{(m_1+m_2)^2, (m_3+m_4)^2\}$.

We are expanding in terms of the momentum of one particle, namely particle 3, but we could just as well have chosen any other particle. Let us assume that we have chosen such a particle, that $\cosh a = 1$ at the physical threshold. In our case this corresponds to assuming that $(m_3+m_4)^2 \geq (m_1+m_2)^2$.

Consider the limit $s=(m_3+m_4)^2(1+\epsilon)$, $\epsilon>0$, $\epsilon\to 0$. We have

$$\cosh a_3 \underset{\epsilon\to 0}{\sim} 1+\epsilon\frac{m_4}{2m_3}\,, \quad \cosh a_4 \underset{\epsilon\to 0}{\sim} 1+\epsilon\frac{m_3}{2m_4}\,,$$

and

$$f_0(a_4, -\lambda_4, \eta_4\bar\eta_4) = \tfrac{1}{2}(e^{a_4\lambda_4}+\eta_4\bar\eta_4 e^{-a_4\lambda_4}) \underset{\epsilon\to 0}{\sim} \begin{cases} 1+\lambda_4{}^2\tfrac{1}{2}\epsilon m_3/m_4 & \text{for } \eta_4\bar\eta_4=1 \\ \lambda_4\sqrt{\epsilon}\,(m_3/m_4)^{1/2} & \text{for } \eta_4\bar\eta_4=-1. \end{cases}$$

Furthermore, Eq. (39) tells us that

$$d^{j_0 c*}_{js_3\lambda_3}(a) \underset{a\to 0}{\sim} \delta_{js_3}.$$

Substituting the above limits into expansion (69) we obtain

$$\langle\lambda_3\lambda_4|S^J(a)|\lambda_1\lambda_2\rangle \underset{\epsilon\to 0}{\sim} g((m_3+m_4)^2)\sum_{\bar\eta_1\bar\eta_2\bar\lambda_4}\sum_{j_0=\{0,\frac{1}{2}\}}^{\min(J+s_4,s_3)}(2J+1)\int_{-i\infty}^{+i\infty}dc\,\frac{j_0{}^2-c^2}{k}\,f_0(a_1,\lambda_1,\eta_1\bar\eta_1)$$
$$\times f_0(a_2,\lambda_2,\eta_2\bar\eta_2)(-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}\begin{pmatrix} s_3 & s_4 & J \\ \lambda_3 & -\lambda_4 & -\lambda_3+\lambda_4\end{pmatrix}$$
$$\times\begin{pmatrix} s_3 & s_4 & J \\ \lambda_1-\lambda_2-\bar\lambda_4 & \bar\lambda_4-\lambda_1+\lambda_2\end{pmatrix}A^{j_0c}_{s_3}(\bar\eta_1\bar\eta_2\eta_3\eta_4, \lambda_1\lambda_2\bar\lambda_4)\,. \tag{76}$$

From the $3j$ symbols of the O(3) group we see that the only nonzero partial-wave helicity amplitudes are those for which $|s_3-s_4|\leq J\leq s_3+s_4$. However, since J is the total angular momentum, we have $\vec{J}=\vec{s}_3+\vec{s}_4+\vec{l}_n'$, where l_n' is the orbital angular momentum in the final state. We see that only those amplitudes for which $l_n'=0$ survive, which is in complete agreement with the general requirement (74).

In order to show that all other partial-wave helicity amplitudes vanish in the proper manner we must investigate the behavior of $d^{j_0c}_{j_1j_2\lambda}(a)$ for $a\to 0$ in greater detail. Let us first of all notice that we have

$$\sinh a = \frac{\{[s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{1/2}}{2m_3\sqrt{s}}\,. \tag{77}$$

The proper kinematic behavior at the final-state threshold and pseudothreshold would thus be guaranteed, if we could show that

$$d^{j_0c}_{j_1j_2\lambda}(a) \underset{a\to 0}{\sim} (\sinh a)^{|j_1-j_2|}\tilde d^{j_0c}_{j_1j_2\lambda}(a)\,, \tag{78}$$

where $\tilde d^{j_0c}_{j_1j_2\lambda}(a)$ is finite (and nonzero) for $a\to 0$. While we have not yet been able to cast the O(3, 1) d function into a form in which the behavior (78) is manifestly true, we do have some indication that such a formula exists. Thus, the d functions of the compact group O(4) can be obtained by analytic continuation[24] in the variable c. For the transformation matrices of O(4) it is indeed possible to show that property (78) holds.[25] Furthermore, it is quite easy to derive a formula of the type (78) in the special case when $j_0=j_1=\lambda=0$. Indeed, we can write the basis functions for representations with $j_0=0$ as[1,2,6]

$$\phi^{0\sigma}_{lm}(a,\theta,\phi) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}\,\frac{1}{(\sinh a)^{1/2}}\,P^{-l-1/2}_{1/2+\sigma}(\cosh a)Y_{lm}(\theta,\phi)\,, \quad \sigma=c-1 \tag{79}$$

which can easily be shown to satisfy

$$\phi^{0\sigma}_{lm}(a,\theta,\phi) = \frac{1}{\sqrt{\pi}}\,\delta_{l0}\delta_{m0}\,. \tag{80}$$

We also have

$$\phi^{0\sigma}_{lm}(a,\theta,\phi)\equiv\phi^{0\sigma}_{lm}(g^{-1}\tilde{x})$$
$$=T_g\phi^{0\sigma}_{lm}(\tilde{x})$$
$$=\sum_{l'm'}D^{0\sigma}_{l'm'lm}(g)\phi^{0\sigma}_{l'm'}(0,\theta,\phi)\,, \tag{81}$$

where $\tilde{x}=(1,0,0,0)$ is the vertex of the upper sheet of the hyperboloid $x^2=x_0^2-\vec{x}^2=1$ and

$$g=e^{-iJ_3\chi}e^{-iJ_2\eta}e^{-iJ_3\psi}e^{-iK_3a}e^{-iJ_2\theta}e^{-iJ_3\phi}\,.$$

From Eqs. (79)–(81) we obtain

$$d^{0\sigma}_{0l0}(\cosh a)=\frac{(2l+1)^{1/2}}{2}\,\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-l)}\,\frac{1}{(\sinh a)^{1/2}}\,P^{-l-1/2}_{1/2+\sigma}(\cosh a)\,. \tag{82}$$

For $1\leq\cosh a<3$ this can finally be rewritten as[26]

$$d^{0\sigma}_{0l0}(\cosh a)=\frac{(2l+1)^{1/2}}{2}\,\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-l)\Gamma(l+\frac{3}{2})}\,\frac{(\sinh a)^l}{(\cosh a+1)^{l+1/2}}\;{}_2F_1(-\sigma-\tfrac{1}{2},\sigma+\tfrac{3}{2},l+\tfrac{3}{2},\tfrac{1}{2}(1-\cosh a))\,. \tag{83}$$

Formula (83) demonstrates, at least in this special case, precisely the correct threshold behavior (78). Assuming that formula (78) is indeed valid in general (we plan to return to this problem), we find that each term individually on the right-hand side of (69) has the correct behavior at the final-state threshold (and pseudothreshold). We consider this to be important since in actual applications the sums in (69) will eventually be truncated and the integral approximated. This would be a complicated procedure if the kinematic behavior at threshold had to be ensured by constraints among many terms (in general infinitely many). The correct behavior at the initial channel threshold is not contained automatically; however, we have assumed that $(m_1+m_2)^2\leq(m_3+m_4)^2$, i.e., $s=(m_1+m_2)^2$ is outside the physical region (except for elastic scattering). Obviously, if $(m_1+m_2)^2>(m_3+m_4)^2$ we must expand in terms of the components of, e.g., the momentum p_1 (or p_2), rather than p_3.

VIII. CONCLUSIONS

The main result of this paper is formula (64) representing an O(3, 1) two-variable expansion for a general helicity amplitude $f_{\lambda_1\lambda_2\lambda_3\lambda_4}(a,\theta,\phi)$ (we use the term two-variable expansion, since the third variable ϕ is irrelevant). The formula was derived assuming the square-integrability condition (60), which is a restriction on the possible asymptotic behavior of the helicity amplitudes. More general asymptotic behavior (power-bounded amplitudes) can be treated by generalizing the integration path in the complex c plane.

We wish to stress that the useful features of the O(3, 1) expansions for spinless particles[1-7] have been preserved. In particular, the expansion can be interpreted as an O(3) little-group expansion supplemented by a representation for the O(3) partial-wave helicity amplitude [see (69)] and the partial-wave amplitudes have the correct threshold and pseudothreshold behavior (either in the initial or in the final-state channel, whichever lies on the boundary of the physical region). The expansion and its inverse formula (67) involve amplitudes defined in the physical region only and are now written for arbitrary (positive) masses and spins. Total angular momentum J is diagonalized and all the quantum numbers figuring in the expansion, with the exception of the continuous Lorentz group parameter c, have a simple physical meaning.

Let us note that the expansion formula (64) reduces to the "S-system" expansions of Refs. 1–6 in the spinless case. Indeed, if $s_i=\lambda_i=0$ the only d function that survives is $d^{0c}_{j00}(a)$ and formula (82) ensures that we obtain an expansion in terms of the correct basis functions of O(3, 1) in an O(3, 1) $\supset$ O(3) $\supset$ O(2) basis. Other two-variable expansions exist in the literature. Those of Balachandran *et al.*[27] are written for arbitrary masses and spins, in general for amplitudes inside the Mandelstam triangle, i.e., in a nonphysical region, and they do not have any obvious group-theoretical interpretation. They do, on the other hand, have very useful properties with respect to the crossing transformation. An O(3, 1) expansion of helicity amplitudes was also suggested by Verdiev[28]; however, in his expansion only part of the s dependence is explicit, whereas an unknown part of the dependence is contained in the expansion coefficients. An expansion of helicity amplitudes for nucleon-nucleon and nucleon-antinucleon scattering in terms of the O(3, 1) basis functions for representations with $j_0=0$ (i.e., appropriate for spinless particles) was suggested by Kuznetsov,[29] who also discusses some consequences of crossing symmetry for the O(3, 1) expansion [for the original treatment of the problem of crossing symmetry in the context of O(3, 1) two-variable expansions for spinless particles see Ref. 3].

In a subsequent article[25] we consider expansions of amplitudes for the three-body decays $1 \to 2+3+4$ involving particles with arbitrary spins. Similarly as in the spinless case[7] we make use of the fact that for decays the physical region of the Mandelstam plane is finite. This region can be mapped onto an O(4) sphere and the expansion (64) can be replaced by an O(4) expansion. The integral over c is replaced by a sum over a discrete variable n. The useful features of the O(3, 1) expansions are preserved (correct threshold behavior, diagonalization of total angular momentum, relation to little-group expansions, etc.) and in addition the fact that the expansions involve only sums and no integrals makes it much easier to apply them phenomenologically. The O(4) expansions have been applied to analyze the Dalitz plot distributions of $\bar{p}n \to 3\pi$ annihilation events at rest[30] and the results will be presented separately.[31]

Further extensions of the formalism presented in this paper are in preparation, in particular two-variable expansions corresponding to the group reductions $O(3,1) \supset O(2,1) \supset O(2)$ and $O(3,1) \supset E_2 \supset O(2)$ (for arbitrary spins), an incorporation of mass-zero particles, and also further applications.

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