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Light-Cone Behavior of Perturbation Theory

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A technique introduced by Symanzik is used to derive a series of equations obeyed order by order in perturbation theory by the structure functions W_1 and νW_2 entering the cross section for inelastic electron scattering. These equations relate the q^2, ν, and coupling-constant dependence of W_1 and νW_2 in a manner reminiscent of the renormalization-group results of Gell-Mann and Low. The equations are used to compute the leading logarithmic contribution to νW_2 in a theory of fermions coupled to pseudoscalar particles and a theory of fermions coupled to vector particles.

I. INTRODUCTION

The simple scaling behavior[1] of the structure functions W_1 and νW_2 (Ref. 2) observed[3] for q^2 and $m\nu \gtrsim 2$ BeV^2 has caused considerable interest in the large q^2 and ν dependence of the matrix element

$$\frac{1}{8\pi m}\sum_{s=\pm 1/2}\int e^{-iq\cdot x}d^4x\,\langle p,s|J_\mu(x)J_\nu(0)|p,s\rangle=\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\omega)+\frac{1}{m^2}\left(p^\mu-q^\mu\frac{p\cdot q}{q^2}\right)\left(p^\nu-q^\nu\frac{p\cdot q}{q^2}\right)W_2(q^2,\omega), \tag{1}$$

where $|p,s\rangle$ is a single nucleon state with four-momentum p and z component of spin s, $J_\mu(x)$ is the usual electromagnetic current.[4] In this paper we investigate the behavior of W_1 and νW_2 for large q^2 and fixed $\omega=2m\nu/q^2$ as computed to arbitrary order in the perturbation expansion of a renormalizable field theory.

As is well known,[5] the large q^2 and ν behavior of the matrix element (1) can be determined from the singularity of the product $J_\mu(x)J_\nu(0)$ on the light cone, $x^2=0$. We begin with Wilson's operator expansion[6,7] for the short-distance limit of the product $J_\mu(\frac{1}{2}(x+y))J_\nu(\frac{1}{2}(-x+y))$:

$$\begin{aligned}J_\mu\!\left(\frac{x+y}{2}\right)J_\nu\!\left(\frac{-x+y}{2}\right)=&\left(\delta^{\mu\nu}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}-\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\frac{1}{x^2+i\epsilon x_0}\left\{\sum_{n=0}^{N}\sum_{i=0}^{u_n}F_n^{(i)}(x^2+i\epsilon x_0)O^{(i)}_{\mu_1\cdots\mu_n}(y)x_{\mu_1}\cdots x_{\mu_n}+R_N'^{(i)}(x,y)\right\}\\&+\left(\delta^{\mu\nu}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}+\delta_{\alpha\mu}\delta_{\beta\nu}\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial x_\rho}-\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\mu}\delta_{\beta\nu}-\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\nu}\delta_{\beta\mu}\right)\\&\times\left\{\sum_{n=0}^{N}\sum_{i=0}^{u_{n+2}}E_n^{(i)}(x^2+i\epsilon x_0)O^{(i)}_{\alpha,\beta,\mu_1,\ldots,\mu_n}(y)x_{\mu_1}\cdots x_{\mu_n}+R_N^{(i)}(x,y)\right\},\end{aligned} \tag{2}$$

where $O^{(i)}_{\alpha_1\cdots\alpha_n}(y)$ are finite local operators, traceless and symmetric with respect to each pair of Lorentz indices.[8] $F_n^{(i)}(x^2)$ and $E_n^{(i)}(x^2)$ are c-number functions given by a perturbation expansion of the form

$$F_n^{(i)}(x^2)=\sum_{l=0}^{\infty}\sum_{r=0}^{l+1}\mathfrak{F}_n^{(i)}(l,r)g^{2l}\ln^r(x^2),$$
$$E_n^{(i)}(x^2)=\sum_{l=0}^{\infty}\sum_{r=0}^{l+1}\mathcal{E}_n^{(i)}(l,r)g^{2l}\ln^r(x^2), \tag{3}$$

where g is the coupling constant. The quantities $R_N^{(i)}(\lambda x,y)$ and $R_N'^{(i)}(\lambda x,y)$ are composed of terms which either approach zero as λ^{N+1} for λ approaching zero or vanish when $x^2=0$.[9,10] The structure functions W_1 and νW_2 can be directly determined from the coefficients $F_n^{(i)}(x^2)$, $E_n^{(i)}(x^2)$, $0\le n<\infty$, by substituting the expansion (2) into Eq. (1) and carrying out the indicated Fourier transformation.

Using a technique introduced by Symanzik,[11] we derive a set of coupled, first-order, partial differential equations satisfied by the functions $E_n^{(i)}(x^2)$, $1 \le i \le u_n$, and by the functions $F_n^{(i)}(x^2)$, $1 \le i \le u_n$. The derivation is based on the Callan-Symanzik[12] equations obeyed by Green's functions containing the product $J_\mu(x)J_\nu(0)$. The equations obtained are of the sort predicted in other situations by renormalization group[13,14] arguments and connect the x^2 and coupling-constant dependence of $F_n^{(i)}(x^2)$, $E_n^{(i)}(x^2)$. The equations do not completely determine the functions $E_n^{(i)}(x^2)$, $F_n^{(i)}(x^2)$, but are instead constraints which must be obeyed to arbitrary order in perturbation theory. When combined with explicit calculations in lowest-order perturbation theory, the equations directly determine the coefficients $\mathfrak{F}_n^{(i)}(l, l+1)$, $\mathcal{E}_n^{(i)}(l, l+1)$ of the leading logarithm in x^2 appearing in every order of perturbation theory.

These equations obeyed by the coefficients $E_n^{(i)}(x^2)$ and $F_n^{(i)}(x^2)$ in the Wilson expansion are derived for two specific field theories in Sec. II. We begin the section by reviewing the connection between the light-cone behavior of the product $J_\mu(x)J_\nu(0)$, specified by the expansion (2), and the large q^2 and ν limit of the structure functions W_1 and W_2. Then, in Sec. II B, a theory of neutral pseudoscalar particles interacting with charged spin-$\frac{1}{2}$ particles is considered and the equations for the coefficients $E_n^{(i)}(x^2)$ and $F_n^{(i)}(x^2)$ derived. Next, in Sec. II C, the corresponding equations valid for a theory of neutral vector particles interacting with charged spin-$\frac{1}{2}$ particles are obtained. In both cases there are two distinct operators $O^{(i)}_{\alpha_1\cdots\alpha_n}$, $i=1, 2$, which appear for each n, and the resulting equations are two coupled, first-order, partial differential equations. In Sec. III these equations are combined with lowest-order perturbation-theory calculations to obtain $E_n^{(i)}(x^2)$ in the leading logarithmic approximation for each of these theories. The results are identical to those previously obtained from a detailed analysis of Feynman amplitudes to all orders in perturbation theory by Gribov and Lipatov.[15] In Sec. IV we discuss the general solution to these equations. First, two sets of approximate equations are considered which are obeyed by amplitudes in the pseudoscalar theory containing no self-energy or vertex corrections. One set is valid for all such amplitudes, while the other applies only to those amplitudes which do not contain a two-pseudoscalar intermediate state. Both sets of equations imply a simple power dependence for $E_n^{(i)}(x^2)$

$$E_n^{(i)}(x^2) \sim v_n^{i,1}(x^2)^{\nu_n^{(1)}} + v_n^{i,2}(x^2)^{\nu_n^{(2)}} \tag{4}$$

for small x^2, where the power $\nu_n^{(j)}$ is a nontrivial function of n and the $v_n^{i,j}$ are constants. Finally, the general solution to our equations is obtained for the vector theory, determining the two functions of two variables $E_n^{(i)}(x^2, g)$, $i=1, 2$, in terms of seven functions of a single variable. The possibility that there exists a root g_∞ of the Gell-Mann–Low eigenvalue condition[13] is investigated and shown to determine somewhat more explicitly the small-x^2 behavior of this solution.

II. DERIVATION OF EQUATIONS FOR $E_n^{(i)}(x^2), F_n^{(i)}(x^2)$

In this section we derive a set of first-order partial differential equations obeyed by the functions $E_n^{(i)}(x^2)$, $1 \le i \le u_n$, and by the functions $F_n^{(i)}(x^2)$, $1 \le i \le u_n$, to arbitrary order in perturbation theory. The makeup of the operators $O^{(i)}_{\alpha_1\cdots\alpha_n}$ appearing in the expansion (2) and the precise form of the equations to be derived depend, of course, on the particular field theory considered. We will deal explicitly with two distinct theories. The first contains a charged spinor field $\psi(x)$ coupled bilinearly to a neutral pseudoscalar field $\phi(x)$ through the interaction Lagrangian $\mathfrak{L}_I(x) = ig\bar{\psi}(x) \times\gamma_5\psi(x)\phi(x)$. In the second theory, the charged spinor field couples to a vector field V_μ, and $\mathfrak{L}_I(x) = ig\bar{\psi}(x)\gamma_\mu\psi(x)V_\mu(x)$.

A. Relation Between W_1, νW_2 and $E_n^{(i)}(x^2)$, $F_n^{(i)}(x^2)$

Before deriving these equations for $E_n^{(i)}$ and $F_n^{(i)}$, it is useful to recall the connection between W_1, νW_2 and the coefficients $E_n^{(i)}$ and $F_n^{(i)}$ in the Wilson expansion (2). Consider the invariant amplitudes T_L and T_2 entering the spin-averaged forward Compton scattering amplitude

$$\frac{i}{8}\sum_{s=\pm 1/2}\int e^{-iq\cdot x}d^4x\langle p, s|T(J_\mu(x)J_\nu(0))|p, s\rangle$$
$$= -\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)T_L + \frac{2}{(q^2)^2}\left[p^\mu p^\nu q^2 + \delta^{\mu\nu}(p\cdot q)^2 - p^\mu q^\nu(p\cdot q) - p^\nu q^\mu(p\cdot q)\right]T_2. \tag{5}$$

The amplitudes T_L and T_2 are functions of q^2 and ω, related to W_1 and νW_2 by

$$W_1 = \frac{2}{\pi m} \operatorname{Im}\left[-T_L + 2\left(\frac{p\cdot q}{q^2}\right)^2 T_2\right], \tag{6a}$$

$$\nu W_2 = -\frac{4}{\pi} \operatorname{Im}\left(\frac{p\cdot q}{q^2}\, T_2\right), \tag{6b}$$

for $\omega \geq 1$. If the Wilson expansion (2) is used to evaluate the left-hand side of Eq. (5) and the Fourier transform performed, we find that for q_μ large[16]

$$T_2(q^2, \omega) = \sum_{n=0}^{N} (\omega)^n \sum_{i=1}^{u_n} \tilde{E}_n^{(i)}(q^2) c_{n+2}^{(i)} + r_N(q^2, \omega), \tag{7a}$$

$$T_L(q^2, \omega) = \sum_{n=0}^{N} (\omega)^n \sum_{i=1}^{u_n} \tilde{F}_n^{(i)}(q^2) c_n^{(i)} + r_N'(q^2, \omega), \tag{7b}$$

where

$$\tilde{E}_n^{(i)}(q^2) = \frac{i}{8} (q^2)^{n+2} \frac{\partial^n}{\partial (q^2)^n} \int d^4x\, e^{-iq\cdot x} E_n^{(i)}(x^2 + i\epsilon), \tag{8a}$$

$$\tilde{F}_n^{(i)}(q^2) = \frac{i}{4} (q^2)^{n+1} \frac{\partial^n}{\partial (q^2)^n} \int d^4x\, e^{-iq\cdot x} \frac{F_n^{(i)}(x^2+i\epsilon)}{x^2 + i\epsilon}, \tag{8b}$$

and

$$\tfrac{1}{2} \sum_{s=\pm 1/2} \langle p, s | O_{\alpha_1 \cdots \alpha_n}^{(i)} | p, s\rangle = c_n^{(i)} p_{\alpha_1} \cdots p_{\alpha_n} (i)^n + (\text{terms containing } \delta_{\alpha_i \alpha_j}). \tag{9}$$

For large q^2 and small ω the remainder terms $r_N(q^2, \omega)$, $r_N'(q^2, \omega)$ are of order ω^{n+1}. Note that only the term proportional to $p_{\alpha_1} \cdots p_{\alpha_n}$ in Eq. (9) yields leading terms in the Wilson expansion (2) on the light cone. The matrix element (9) of $O_{\alpha_1 \cdots \alpha_n}^{(i)}$ is proportional to the single symmetric traceless tensor that can be formed from the four-vector p_μ. All terms in this tensor, other than the $p_{\alpha_1} \cdots p_{\alpha_n}$ term,[10] contain factors of $p^2 = -m^2$ and therefore give contributions to T_1 and T_2 smaller by a factor of $m^2/q^2\omega^2$.

The analyticity of T_L and T_2 in ν for fixed q^2 implies that to any finite order in perturbation theory the limit $N \to \infty$ of the sums in Eq. (7) defines two analytic functions of ω near $\omega = 0$ (Ref. 17):

$$T_2^{\rm AF}(q^2, \omega) = \sum_{n=0}^{\infty} (\omega)^n \sum_{i=0}^{u_n} \tilde{E}_n^{(i)}(q^2) c_{n+2}^{(i)}, \tag{10a}$$

$$T_L^{\rm AF}(q^2, \omega) = \sum_{n=0}^{\infty} (\omega)^n \sum_{i=0}^{u_n} \tilde{F}_n^{(i)}(q^2) c_n^{(i)}. \tag{10b}$$

These asymptotic forms for T_L and T_2 can be continued into the entire ω plane with the exception of branch points at $\omega = \pm 1$, and used in Eq. (6) to compute W_1 and νW_2 for large q^2 and fixed $\omega > 1$. The familiar connection between the large-q^2, fixed-ω behavior of T_L or T_2 and the $x^2 = 0$ singularity of the coefficients $E_n^{(i)}(x^2)$, $F_n^{(i)}(x^2)$ can be seen from Eq. (8).

The relationship between the coefficients $\tilde{E}_n^{(i)}(q^2)$, $\tilde{F}_n^{(i)}(q^2)$ and the asymptotic behavior of the structure functions W_1 and νW_2 implied by Eqs. (6) and (10) can be neatly inverted. Using Cauchy's theorem Eq. (10a) can be written

$$\sum_i c_{n+2}^{(i)} \tilde{E}_n^{(i)}(q^2) = \frac{1}{2\pi i} \int_c d\omega\, \omega^{-n-1} T_2^{\rm AF}(q^2, \omega), \tag{11}$$

where c is a contour circling the origin in a counterclockwise direction. Since $T_2^{\rm AF}$ has branch points in ω at ± 1 and is even in ω, we can open up the contour to obtain

$$\sum_i c_{n+2}^{(i)} \tilde{E}_n^{(i)}(q^2) = \frac{2}{\pi} \int_1^\infty \omega^{-n-1} d\omega \operatorname{Im} T_2^{\rm AF}(q^2, \omega), \tag{12}$$

or using Eq. (6a)

$$\sum_i c_{n+2}^{(i)} \tilde{E}_n^{(i)}(q^2) = \int_1^\infty \omega^{-n-2} \nu W_2^{\rm AF}(q^2, \omega)\, d\omega = \int_0^1 (1/\omega)^n \nu W_2^{\rm AF}(q^2, \omega) d(1/\omega); \tag{13a}$$

likewise

$$\sum_i c_n^{(i)} \tilde{F}_n^{(i)}(q^2) = \int_1^\infty \omega^{-n-1} [\tfrac{1}{2} \omega \nu W_2^{\rm AF}(q^2, \omega) - m W_1^{\rm AF}(q^2, \omega)]\, d\omega. \tag{13b}$$

Equation (13) interprets the Callan-Gross and Cornwall-Norton sum rule[18] in the language of the Wilson expansion. It also identifies $\sum_i c_{n+2}^{(i)} \tilde{E}_n^{(i)}(q^2, \omega)$ as the Mellin transform of $\nu W_2^{\rm AF}(q^2, \omega)$ with respect to the variable $1/\omega$. This transformation can be inverted, giving

$$\nu W_2^{\rm AF}(q^2, \omega) = -\frac{i}{2\pi} \int_{\delta - i\infty}^{\delta + i\infty} dn\, \omega^{n+1} \left[\sum_i c_{n+2}^{(i)} \tilde{E}_n^{(i)}(q^2)\right], \tag{14a}$$

and similarly

$$\tfrac{1}{2} \omega \nu W_2^{\rm AF} - m W_1^{\rm AF} = -\frac{i}{2\pi} \int_{\delta - i\infty}^{\delta + i\infty} dn\, \omega^n \left[\sum_i c_n^{(i)} \tilde{F}_n^{(i)}(q^2)\right], \tag{14b}$$

for sufficiently large, real positive δ.

B. Pseudoscalar Theory

Let us now consider the pseudoscalar case, specified by the Lagrangian

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \tfrac{1}{2} \partial_\mu \phi \partial_\mu \phi - \tfrac{1}{2} \mu^2 \phi^2 + ig \bar{\psi} \gamma_5 \psi \phi + \frac{1}{4!} h \phi^4 + (\text{counter terms}); \tag{15}$$

m and μ are the physical masses of the spin-$\frac{1}{2}$ and the pseudoscalar particles, while g and h are renormalized coupling constants. The renormalization procedure is specified in Appendix A.

The starting point of our derivation is the Callan-Symanzik equations[12] for the matrix elements

$$\Gamma^{(1)}_{\mu\nu}(p,x)=\frac{i}{2}\int e^{ip\cdot(z-y)}d^4z\,d^4y\,(\not p)_{\delta\sigma}\langle 0|T(\psi_\sigma(y)J_\mu(x)J_\nu(0)\bar\psi_\delta(z))|0\rangle_A\,, \tag{16a}$$

$$\Gamma^{(2)}_{\mu\nu}(p,x)=i\int e^{ip\cdot(z-y)}d^4z\,d^4y\,\langle 0|T(\phi(y)J_\mu(x)J_\nu(0)\phi(z))|0\rangle_A\,, \tag{16b}$$

where the subscript A means that the propagators corresponding to external lines have been removed. The Callan-Symanzik equations obeyed by these matrix elements and derived in Appendix A are

$$D_i\Gamma^{(i)}_{\mu\nu}(x,p)=\Delta\Gamma^{(i)}_{\mu\nu}(x,p)\,, \tag{17}$$

for $i=1,2$. The differential operator D_i is given by

$$D_i=m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}+\beta\frac{\partial}{\partial g}+\beta'\frac{\partial}{\partial h}-2\gamma_i\,, \tag{18}$$

while[19]

$$\Delta\Gamma^{(1)}_{\mu\nu}(x,p)=\frac{1}{2}\int e^{ip\cdot(z-y)}d^4z\,d^4y\,(\not p)_{\delta\sigma}\langle 0|T(\psi_\sigma(y)J_\mu(x)J_\nu(0)u\bar\psi_\delta(z))|0\rangle_A\,, \tag{19a}$$

$$\Delta\Gamma^{(2)}_{\mu\nu}(x,p)=\int e^{ip\cdot(z-y)}d^4z\,d^4y\,\langle 0|T(\phi(y)J_\mu(x)J_\nu(0)u\phi(z))|0\rangle_A\,. \tag{19b}$$

The operator u, in the notation of Zimmerman,[20] is

$$u=\frac{1}{2}\int d^4x\{m\delta_1 N[\bar\psi(x)\psi(x)]+\mu^2\delta_2 N[\phi(x)\phi(x)]\}\,, \tag{20}$$

where the symbol N indicates the inclusion of subtraction terms, chosen in a manner specified in Appendix A, so that all matrix elements of u are finite. The dimensionless constants $\beta, \beta', \gamma_1, \gamma_2, \delta_1, \delta_2$ are functions of g, h and m/μ and can be computed to arbitrary order in perturbation theory. The Callan-Symanzik Eqs. (17) are exact, but not very useful as they stand since they relate the behavior of the quantities of interest, (16), to that of two new unknown functions (19). However, if we consider the small x_μ limit of Eq. (17) and substitute the Wilson expansion (2) into both the right- and left-hand sides, then we find that the small x_μ dependence of both sides is determined by the same functions $E^{(i)}_n(x^2)$ and $F^{(i)}_n(x^2)$.[11,21]

The resulting equations are

$$D_i\left[\sum_{n=1}^{\infty}\sum_{j=1}^{u_{2n}}E^{(i)}_{2n-2}(x^2)b^{i,j}_{2n}(x\cdot p)^{2n-2}\right]=\sum_{n=1}^{\infty}\sum_{j=1}^{u_{2n}}E^{(j)}_{2n-2}(x^2)a^{i,j}_{2n}(x\cdot p)^{2n-2}\,, \tag{21}$$

for $i=1,2$ and $p^2=0$. The constants $a^{i,j}_n$ and $b^{i,j}_n$ are related to the relevant matrix elements of $O^{(i)}_{\alpha_1\cdots\alpha_n}$ by

$$\frac{1}{2}\int e^{ip\cdot(z-y)}d^4y\,d^4z\,(\not p)_{\delta\sigma}\langle 0|T(\psi_\sigma(y)O^{(j)}_{\alpha_1\cdots\alpha_n}\bar\psi_\delta(z))|0\rangle_A=b^{1,j}_n p_{\alpha_1}\cdots p_{\alpha_n}(i)^n+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{22a}$$

$$\int e^{ip\cdot(z-y)}d^4y\,d^4z\,\langle 0|T(\phi(y)O^{(j)}_{\alpha_1\cdots\alpha_n}\phi(z))|0\rangle_A=b^{2,j}_n p_{\alpha_1}\cdots p_{\alpha_n}(i)^n+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{22b}$$

$$\frac{1}{2}\int e^{ip\cdot(z-y)}d^4y\,d^4z\,(\not p)_{\delta\sigma}\langle 0|T(\psi_\sigma(y)O^{(j)}_{\alpha_1\cdots\alpha_n}u\bar\psi_\delta(z))|0\rangle_A=a^{1,j}_n p_{\alpha_1}\cdots p_{\alpha_n}(i)^n+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{22c}$$

$$\int e^{ip\cdot(z-y)}d^4y\,d^4z\,\langle 0|T(\phi(y)O^{(j)}_{\alpha_1\cdots\alpha_n}u\phi(z))|0\rangle_A=a^{2,j}_n p_{\alpha_1}\cdots p_{\alpha_n}(i)^n+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{22d}$$

for $p^2=0$ and $1\le\alpha_i\le 3$, $1\le i\le n$, and n even. Bose symmetry and charge-conjugation invariance imply that the left-hand sides of these equations vanish for odd n. In Eq. (22) we use $p^2=0$, so that the quantities $a^{i,j}_n$ and $b^{i,j}_n$ depend only on $m^2/\mu^2, g, h$. Equating the coefficients of equal powers of $(x\cdot p)^n$, we obtain a series of equations diagonal in the index n,

$$D_i\left[\sum_j E^{(j)}_{n-2}(x^2)b^{i,j}_n\right]=\sum_j E^{(j)}_{n-2}(x^2)a^{i,j}_n\,, \tag{23}$$

$2 \leqslant n \leqslant \infty$, $i=1,\ldots,u_n$, for even n. Equations (21) and (23) are also obeyed by $F_n^{(j)}(x^2)$ and $0 \leqslant n \leqslant \infty$. These equations can be Fourier-transformed, yielding identical equations for the quantities $\tilde{E}_n^{(i)}(q^2)$

$$D_i\left[\sum_j \tilde{E}^{(j)}_{n-2}(q^2)b_n^{i,j}\right]=\sum_j \tilde{E}^{(j)}_{n-2}(q^2)a_n^{i,j}\,, \tag{24}$$

which are also obeyed by the functions $\tilde{F}_n^{(i)}(q^2)$.

Let us now determine explicitly the operators which appear in the Wilson expansion (2) for the particular theory at hand. Because of the requirements of symmetry in the Lorentz indices and the absence of $\delta_{\alpha_i\alpha_j}$ factors, there are only two Nth-rank tensor operators with the smallest dimension which can be formed[22]:

$$O^{(1)}_{\alpha_1\cdots\alpha_n}(y)=-\frac{1}{4n}[1+(-1)^n]\sum_{j=1}^{n}N[\bar\psi(y)\partial_{\alpha_1}\cdots\partial_{\alpha_{j-1}}\gamma_{\alpha_j}\partial_{\alpha_{j+1}}\cdots\partial_{\alpha_n}\psi(y)]+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{25a}$$

$$O^{(2)}_{\alpha_1\cdots\alpha_n}(y)=\tfrac{1}{2}N[\phi(y)\partial_{\alpha_1}\cdots\partial_{\alpha_n}\phi(y)]+(\text{terms containing }\delta_{\alpha_i\alpha_j})\,, \tag{25b}$$

where the symbol N again indicates that sufficient subtractions have been made so that the resulting operator is finite. The subtractions will be chosen so that

$$b_n^{i,j}(g,h,m^2/\mu^2)=\delta_{ij}\,. \tag{26}$$

The equations obeyed by $\tilde{E}_n^{(i)}(q^2)$ for $i=1,2$, $2\leqslant n \leqslant\infty$ [and $\tilde{F}_n^{(i)}(q^2)$, $i=1,2$, $0\leqslant n\leqslant\infty$] then become

$$D_i\tilde{E}^{(i)}_{n-2}(q^2)=\sum_{j=1,2}a_n^{i,j}\tilde{E}^{(j)}_{n-2}(q^2)\,. \tag{27}$$

These two coupled first-order differential equations can be written as uncoupled second-order equations

$$(D_2-a_n^{2,2})\frac{1}{a_n^{1,2}}(D_1-a_n^{1,1})\tilde{E}_n^{(1)}(q^2)=a_n^{2,1}\tilde{E}_n^{(1)}(q^2)\,, \tag{28a}$$

$$(D_1-a_n^{1,1})\frac{1}{a_n^{2,1}}(D_2-a_n^{2,2})\tilde{E}_n^{(2)}(q^2)=a_n^{1,2}\tilde{E}_n^{(2)}(q^2)\,. \tag{28b}$$

These equations [(27) or (28)] are the desired equations for the pseudoscalar theory. They are the generalization of Symanzik's exceptional momentum equation to all the operators in the Wilson expansion on the light cone. These equations will be used in Sec. III to compute the leading logarithmic contribution to νW_2 and in Sec. IV to speculate about the exact asymptotic behavior of W_1 and νW_2.

C. Massive Vector Theory

We now consider the theory of a vector field V_μ of mass μ interacting with a spin-$\frac{1}{2}$ field of mass m, specified by the Lagrangian

$$\mathcal{L}(x)=-\bar\psi(\gamma_\mu\partial_\mu+m)\psi-\tfrac{1}{4}G_{\mu\nu}G_{\mu\nu}+igV_\mu\bar\psi\gamma_\mu\psi-\tfrac{1}{2}\mu^2V_\mu V_\mu+(\text{counter terms})\,, \tag{29}$$

where g is the renormalized coupling constant and

$$G_{\mu\nu}(x)=\partial_\mu V_\nu(x)-\partial_\nu V_\mu(x)\,. \tag{30}$$

In analogy[23] with Eq. (16) we define the amplitudes

$$\Gamma^{(1)}_{\mu\nu}(p,x)=\tfrac{1}{2}i\int e^{ip\cdot(z-y)}d^4z\,d^4y\,(\not p)_{\delta\sigma}\times\langle 0|T(\psi_\sigma(y)J_\mu(x)J_\nu(0)\bar\psi_\delta(z))|0\rangle_A\,, \tag{31a}$$

$$\Gamma^{(2)}_{\mu\nu}(p,x)=\tfrac{1}{3}i\int e^{ip\cdot(z-y)}d^4z\,d^4y\times\langle 0|T(V_\rho(y)J_\mu(x)J_\nu(0)V_\rho(z))|0\rangle_A\,. \tag{31b}$$

As is shown in Appendix A, these amplitudes obey the Callan-Symanzik equation

$$D_i\Gamma^{(i)}_{\mu\nu}(p,x)=\Delta\Gamma^{(i)}_{\mu\nu}(p,x)\,, \tag{32}$$

for $i=1,2$, where

$$D_i=\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}+\beta\frac{\partial}{\partial g}-4\gamma_2-2\gamma_i\right)\gamma_3{}^2 \tag{33}$$

and

$$\Delta\Gamma^{(1)}_{\mu\nu}(x,p)=\tfrac{1}{2}\int e^{ip\cdot(z-y)}d^4z\,d^4y\,(\not p)_{\rho\sigma}\gamma_3{}^2\times\langle 0|T(\psi_\sigma(y)J_\mu(x)J_\nu(0)u\bar\psi_\rho(z))|0\rangle_A\,, \tag{34a}$$

$$\Delta\Gamma^{(2)}_{\mu\nu}(x,p)=\tfrac{1}{3}\int e^{ip\cdot(z-y)}d^4z\,d^4y\,\gamma_3{}^2\times\langle 0|T(V_\rho(y)J_\mu(x)J_\nu(0)uV_\rho(z))|0\rangle_A\,, \tag{34b}$$

for $\beta=g\gamma_2$ and

$$u=\tfrac{1}{2}\int d^4x\,\{m\delta_1N[\bar\psi(x)\psi(x)]+\mu^2\delta_2N[V_\rho(x)V_\rho(x)]\}\,. \tag{35}$$

Substituting the Wilson expansion (2) into Eq. (32) and equating equal powers of $x\cdot p$, we obtain an equation identical in form to Eq. (23):

$$D_i\left(\sum_{j=1}^{u_n}E^{(j)}_{n-2}b_n^{i,j}\right)=\sum_{j=1}^{u_n}E^{(j)}_{n-2}a_n^{i,j}\,, \tag{36}$$

for $i=1,2$, n even, and $2\leqslant n<\infty$. The constants $a_n^{i,j}(m^2/\mu^2,g)$ and $b_n^{i,j}(m^2/\mu^2,g)$ are defined by equations obtained from Eq. (22) by replacing $\phi(y)\phi(z)$ by $\frac{1}{3}V_\rho(y)V_\rho(z)$ and multiplying the left-hand sides of Eqs. (22c) and (22d) by $\gamma_3{}^2$. The longitudinal coefficients $F_n^{(i)}$ also obey Eq. (36) for $0\leqslant n<\infty$.

Just as in the pseudoscalar case there are two types of operators that can contribute:

$$O^{(1)}_{\alpha_1\cdots\alpha_n}=-\frac{1}{4n}\left[1+(-1)^n\right]\sum_{j=1}^{n}N[\bar\psi(\partial_{\alpha_1}-igV_{\alpha_1})\cdots\gamma_{\alpha_j}\cdots(\partial_{\alpha_n}-igV_{\alpha_n})\psi]+(\text{terms containing }\delta_{\alpha_i\alpha_j}), \tag{37a}$$

$$O^{(2)}_{\alpha_1\cdots\alpha_n}=\frac{3}{4}\,\frac{1}{(n-1)n}\sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}N[G_{\gamma\alpha}\partial_{\alpha_1}\cdots\delta_{\alpha_j\beta}\cdots\delta_{\alpha_i\alpha}\cdots\,\partial_{\alpha_n}G_{\beta\gamma}]+(\text{terms containing }\delta_{\alpha_i\alpha_j}). \tag{37b}$$

The number of possible operators is limited to only two, for a given n, by gauge invariance. Both the operator $J_\mu(x)J_\nu(0)$ and the first three terms of our Lagrangian (29) are invariant under the transformation

$$V_\mu(x)\to V_\mu(x)+g\partial_\mu\Lambda(x),\qquad \psi(x)\to e^{ig\Lambda(x)}\psi(x). \tag{38}$$

Although the mass term $-\frac{1}{2}\mu^2V_\rho V_\rho$ breaks this gauge symmetry, the leading terms in Wilson's expansion (2) are independent of μ^2 and hence are left unchanged by the transformation (38).

Thus only two series of functions $E_n^{(1)}(x^2)$ and $E_n^{(2)}(q^2)$ are needed to determine νW_2 in the large q^2 and ν region. If we choose the subtractions required to make the operators (37) finite in such a way that

$$b_n^{i,j}=\delta_{ij} \tag{39}$$

and transform to momentum space, then Eq. (36) becomes

$$D_i\tilde E^{(i)}_{n-2}(q^2)=\sum_{j=1,2}a_n^{i,j}\tilde E^{(j)}_{n-2}(q^2), \tag{40}$$

for $i=1,2$, $2\leqslant n<\infty$, an equation identical in form to that found for the pseudoscalar theory. This equation is also obeyed by the functions $F_n^{(i)}(q^2)$, $i=1,2$, $0\leqslant n<\infty$.

III. PERTURBATION-THEORY CALCULATIONS

In this section we use the equations derived in Sec. II to calculate the inelastic electroproduction structure function νW_2 in a leading logarithmic approximation. Various authors[15,24-26] have performed such calculations by applying infinite-momentum methods directly to specific classes of Feynman graphs. Such approaches require considerable expertise in the art of extracting asymptotic behavior from Feynman amplitudes. We will show how these leading logarithmic results emerge rather trivially from Eqs. (27) and (40). Altogether three specific examples will be considered: (A) the ladder graphs in the pseudoscalar theory calculated by Chang and Fishbane,[24] (B) the complete leading logarithmic behavior in the pseudoscalar theory, first computed by Gribov and Lipatov,[15] and (C) the complete leading logarithmic behavior in the vector theory, also computed by Gribov and Lipatov.[15]

A. Chang-Fishbane Calculation

Chang and Fishbane consider the ladder graphs of Fig. 1 in the leading logarithmic approximation. In our notation this means that they keep all terms in $\tilde E_n^{(1)}(q^2)$ of the form $(g^2)^r(g^2\ln q^2)^l$ with $r=0$. Since no intermediate state containing only two pseudoscalar particles appears in the Feynman diagrams of Fig. 1, the operator $O^{(2)}_{\alpha_1\cdots\alpha_n}$ should be omitted from the Wilson expansion[7] of $J_\mu(x)J_\nu(0)$; therefore, we set $\tilde E_n^{(2)}(q^2)=0$. Furthermore, there are no propagator or vertex corrections included in this set of graphs, so $\beta=\beta'=\gamma_1=\gamma_2=0$. (In Chang and Fishbane's language we are taking only their outer rainbow graphs.) Thus, Eq. (27) becomes simply

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\tilde E_n^{(1)}(q^2)=a^{1,1}_{n+2}E_n^{(1)}(q^2). \tag{41}$$

Since $\tilde E_n^i(q^2)$ is a dimensionless function of q^2, m^2, and μ^2, we may replace $m^2\partial/\partial m^2+\mu^2\partial/\partial\mu^2$ by $-q^2\partial/\partial q^2$, so that Eq. (41) can be rewritten

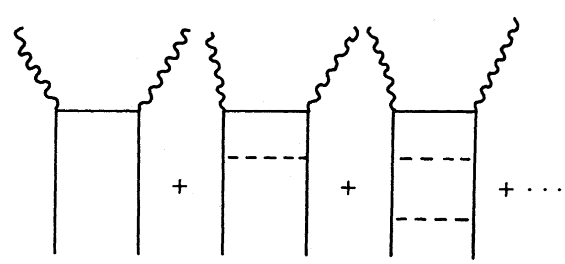

FIG. 1. Ladder graphs representing the "outer rainbow" amplitudes considered in the Chang-Fishbane calculation. The solid lines represent fermion propagators, the dashed lines pseudoscalar propagators, and the wavy lines virtual photons.

$$q^2 \frac{\partial}{\partial q^2} \tilde{E}_n^{(1)}(q^2) = -a_{n+2}^{1,1} E_n^{(1)}(q^2), \tag{42}$$

whose solution is

$$\tilde{E}_n^{(1)}(q^2) = v_n \exp[-a_{n+2}^{1,1} \ln(q^2)] = v_n (q^2)^{-a_{n+2}^{1,1}}. \tag{43}$$

To obtain the leading logarithmic behavior of $\tilde{E}_n^{(1)}(q^2)$, we need only compute the parameter $a_{n+2}^{1,1}$ from Eq. (22c) to lowest order in g and determine the integration constant v_n from the $g^2=0$ Born term. This calculation of the quantities $a_n^{1,1}$ involves the evaluation of a simple lowest-order vertex correction and is carried out in Appendix B, yielding

$$a_{n+2}^{1,1} = -\frac{g^2}{16\pi^2} \frac{1}{(n+2)(n+3)}, \tag{44}$$

for even n. Since to lowest order in g, the $c_n^{(i)}$ of Eq. (9) equals 1 and

$$\nu W_2 = \delta(1 - 1/\omega), \tag{45}$$

Eq. (13a) implies $v_n = 1$, so that in leading logarithmic approximation

$$\int_0^1 d\left(\frac{1}{\omega}\right)\left(\frac{1}{\omega}\right)^{n+1} \nu W_2^{\rm AF}(q^2, \omega) = \tilde{E}_n^{(1)}(q^2) = (q^2)^{\zeta}, \tag{46}$$

where

$$\zeta = \frac{g^2}{16\pi^2} \frac{1}{(n+2)(n+3)}.$$

This is exactly the result of Chang and Fishbane for the set of outer rainbow amplitudes. Thus, the Mellin transform used so judiciously by Chang and Fishbane and by Gribov and Lipatov is nothing other than the index-continued Wilson expansion, the continuation being analogous to the Sommerfeld-Watson continuation of a partial-wave expansion.

B. Gribov-Lipatov Calculation for the Pseudoscalar Theory

We will now find all the leading logarithmic terms in $\tilde{E}_n^{(1)}(q^2)$ for the γ_5 theory. The basic equations for this calculation are given by Eq. (27), which we write in full as

$$\left(-q^2 \frac{\partial}{\partial q^2} + \beta \frac{\partial}{\partial g} + \beta' \frac{\partial}{\partial h} - 2\gamma_1\right) \tilde{E}_n^{(1)}(q^2) = a_{n+2}^{1,1} \tilde{E}_n^{(1)}(q^2) + a_{n+2}^{1,2} \tilde{E}_n^{(2)}(q^2), \tag{47}$$

$$\left(-q^2 \frac{\partial}{\partial q^2} + \beta \frac{\partial}{\partial g} + \beta' \frac{\partial}{\partial h} - 2\gamma_2\right) \tilde{E}_n^{(2)}(q^2) = a_{n+2}^{2,1} \tilde{E}_n^{(1)}(q^2) + a_{n+2}^{2,2} \tilde{E}_n^{(2)}(q^2),$$

for even $n \geq 0$. Following Gribov and Lipatov we set h and therefore β' equal to zero. (In a regularized theory with no ϕ^4 interaction term, h is of order g^4.) The quantities β, γ_1, and γ_2 can be computed to lowest order in g from Eq. (A9) of Appendix A, while in Appendix C the $a_n^{i,j}$ are determined and their connection with various graphs indicated. The results are

$$\beta = \frac{5g^3}{32\pi^2}, \quad \gamma_1 = \frac{g^2}{64\pi^2}, \quad \gamma_2 = \frac{g^2}{16\pi^2},$$
$$a_{n+2}^{1,1} = -\frac{g^2}{16\pi^2} \frac{1}{(n+2)(n+3)}, \quad a_{n+2}^{1,2} = -\frac{g^2}{16\pi^2} \frac{1}{(n+3)},$$
$$a_{n+2}^{2,1} = -\frac{g^2}{4\pi^2} \frac{1}{(n+2)}, \quad a_{n+2}^{2,2} = 0. \tag{48}$$

Since in leading logarithmic approximation $\tilde{E}_n^{(i)}(q^2)$ depends only on $g^2 \ln(q^2)$, it is convenient to introduce the variable

$$\xi = -\tfrac{1}{5} \ln[1 - (5g^2/16\pi^2) \ln(q^2)]. \tag{49}$$

The reader will note that

$$-q^2 \frac{\partial}{\partial q^2} + \beta \frac{\partial}{\partial g} = -\frac{g^2}{16\pi^2} \frac{\partial}{\partial \xi} \tag{50}$$

when acting on a function of ξ alone. Using Eqs. (48) and (50), we can rewrite Eq. (47) as

$$\left(\frac{\partial}{\partial \xi} + \frac{1}{2}\right) \tilde{E}_n^{(1)}(\xi) = \frac{1}{(n+2)(n+3)} \tilde{E}_n^{(1)}(\xi) + \frac{1}{(n+3)} \tilde{E}_n^{(2)}(\xi),$$
$$\left(\frac{\partial}{\partial \xi} + 2\right) \tilde{E}_n^{(2)}(\xi) = \frac{4}{(n+2)} \tilde{E}_n^{(1)}(\xi), \tag{51}$$

which are equivalent to

$$\left[\frac{\partial^2}{\partial \xi^2} + \left(\frac{5}{2} - \frac{1}{(n+2)(n+3)}\right)\frac{\partial}{\partial \xi} + 1 - \frac{6}{(n+2)(n+3)}\right] \tilde{E}_n^{(1)}(\xi) = 0 \tag{52}$$

and

$$\tilde{E}_n^{(2)}(\xi) = (n+3)\left[\frac{\partial}{\partial \xi} + \frac{1}{2} - \frac{1}{(n+2)(n+3)}\right] \tilde{E}_n^{(1)}(\xi). \tag{53}$$

Equation (52) implies that $\tilde{E}_n^{(1)}(\xi)$ has the form

$$E_n^{(1)}(\xi) = C_n e^{\nu_n \xi} + C_n' e^{\nu_n' \xi}, \tag{54}$$

where

$$\nu_n = -\frac{5}{4} + \frac{1}{2(n+2)(n+3)} + \left[\left(\frac{3}{4} + \frac{1}{2(n+2)(n+3)}\right)^2 + \frac{4}{(n+2)(n+3)}\right]^{1/2},$$
$$\nu_n' = -\frac{5}{4} + \frac{1}{2(n+2)(n+3)} - \left[\left(\frac{3}{4} + \frac{1}{2(n+2)(n+3)}\right)^2 + \frac{4}{(n+2)(n+3)}\right]^{1/2}. \tag{55}$$

The integration constants C_n and C_n' are determined from the known $g^2=0$ limit given by the Born terms

$$\tilde{E}_n^{(1)}(\xi)|_{\xi=0}=1\,, \tag{56a}$$

$$\tilde{E}_n^{(2)}(\xi)|_{\xi=0}=0\,, \tag{56b}$$

which requires

$$C_n=\frac{1}{\nu_n-\nu_n'}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2}-\nu_n'\right],$$
$$C_n'=\frac{1}{\nu_n'-\nu_n}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2}-\nu_n\right]. \tag{57}$$

A continuation of Eq. (54) to complex values of the index n, when substituted in Eq. (14a), yields

$$\nu W_2(q^2,\omega)=-\frac{i}{2\pi}\int_{\delta-i\infty}^{\delta+i\infty}dn\,\omega^{n+1}\times(C_n e^{\nu_n\xi}+C_n' e^{\nu_n'\xi})\,, \tag{58}$$

in exact agreement with Gribov and Lipatov for the leading logarithmic behavior of the structure function νW_2 for deep-inelastic scattering of electrons off the ψ field. The reader is referred to the work of Gribov and Lipatov for a discussion of the physical significance, if any, of this result.

C. Gribov-Lipatov Calculation for the Massive Vector Theory

Finally we turn to the calculation of the leading logarithms in νW_2 for the vector theory studied in Sec. II C. Since this calculation proceeds much as in the pseudoscalar case, we will simply outline the procedure for obtaining the result – identical to that of Gribov and Lipatov. The quantities $\beta=2\gamma_2$, γ_1, and γ_3, determined in Appendix A, can be computed in the Feynman gauge to lowest order in g with the result

$$\beta=\frac{g^3}{24\pi^2},\quad \gamma_1=\frac{g^2}{32\pi^2},\quad \gamma_3=1\,. \tag{59}$$

Similarly, the constants $a_n^{i,j}$ are evaluated in Appendix C to order g^2:

$$a_{n+2}^{1,1}=-\frac{g^2}{8\pi^2}\left(\frac{1}{(n+2)(n+3)}-\sum_{l=0}^{n}\frac{2}{l+2}\right),$$
$$a_{n+2}^{1,2}=\frac{3g^2}{16\pi^2}\frac{n^2+5n+8}{(n+1)(n+2)(n+3)}\,,$$
$$a_{n+2}^{2,1}=\frac{g^2}{6\pi^2}\frac{n^2+5n+8}{(n+2)(n+3)(n+4)}\,, \tag{60}$$
$$a_{n+2}^{2,2}=0\,.$$

Introducing the variable

$$\xi=-\tfrac{3}{4}\ln[1-(g^2/12\pi^2)\ln(q^2)]\,,$$

we can rewrite Eq. (40) for $\tilde{E}_n^{(i)}(q^2)$ as

$$\left(\frac{\partial^2}{\partial\xi^2}-[3+\psi_1(n+2)+\psi_3(n+2)]\frac{\partial}{\partial\xi}+\frac{20}{9}+\tfrac{4}{3}[\psi_1(n+1)+\psi_3(n+2)]-\psi_2(n+2)\right)E_n^{(1)}(\xi)=0\,, \tag{61}$$

$$E_n^{(2)}=-\frac{g^2}{16\pi^2}\frac{1}{a_{n+2}^{1,2}}\left[\frac{\partial}{\partial\xi}-\frac{5}{3}-\psi_1(n+2)-\psi_3(n+2)\right]E_n^{(1)},$$

where

$$\psi_1(j)=\frac{2}{j(j+1)}\,,$$
$$\psi_2(j)=\frac{8(j^2+j+2)^2}{(j-1)j^2(j+1)^2(j+2)}\,, \tag{62}$$
$$\psi_3(j)=-4\sum_{l=2}^{j}\frac{1}{l}\,,$$

in the notation of Gribov and Lipatov. These equations, when coupled with the requirement[27] (56), can be explicitly solved as in the preceding section yielding the Gribov-Lipatov result.

IV. GENERAL SOLUTION

We now consider the general solution to the Eqs. (23) and (36) obeyed by the coefficients $E_n^{(i)}(x^2)$, $F_n^{(i)}(x)^2$ appearing in the Wilson expansion (2). We first study the simplified equations which govern the Chang-Fishbane calculation of Sec. III A in which all self-energy corrections, vertex corrections, and amplitudes containing a two-pseudoscalar intermediate state have been omitted. Next, those amplitudes containing a two-pseudoscalar intermediate state are included and the resulting equations solved. In both cases the functions $\tilde{E}_n^{(i)}(q^2)$ show a power dependence on q^2, where the exponent of q^2 depends explicitly on n. Thus, for these examples the operators $O_{\alpha_1\cdots\alpha_n}^{(i)}$ possess an n-dependent anomalous dimension in the sense of Wilson.[6] Finally, the general solution of Eq. (40) for the vector theory is found, determining $\tilde{E}_n^{(i)}(q^2,g)$ in terms of two unknown functions of a single variable and the quantities $\beta(g)$, $\gamma_1(g)$, and $a_n^{i,j}(g)$. If we assume that $\beta(g)$ has a zero at $g=g_\infty$, and that the quantities $\tilde{E}_n^{(i)}(q^2,g)$, $\gamma_i(g)$, and $a_n^{i,j}(g)$ are regular at $g=g_\infty$, then this solution also shows power dependence in q^2, with the power depending on n. Although in each of these three cases we find or hypothesize solutions which display a power behavior in q^2, we see no suggestion that these powers should be identically zero for all n as is required if the structure function $\nu W_2(q^2,\omega)$ is to be independent of q^2 for large q^2.

A. Chang-Fishbane Amplitudes

We begin by examining the set of amplitudes first studied by Chang and Fishbane. These amplitudes contain no self-energy corrections, no vertex corrections, and no intermediate state composed of only two pseudoscalar particles. As was shown in Sec. III A, the resulting functions $\tilde{E}_n^{(1)}(q^2)$ have the form

$$\tilde{E}_n^{(1)}(q^2)=v_n\times(q^2)^{-a_{n+2}^{1,1}}, \tag{63}$$

where the constants v_n and $a_{n+2}^{1,1}$ can be computed in perturbation theory:

$$\begin{aligned} v_n&=1+O(g^2),\\ a_{n+2}^{1,1}&=\frac{-g^2}{16\pi^2(n+2)(n+3)}+O(g^4). \end{aligned} \tag{64}$$

The position-space function $E_n^{(i)}(x^2)$ follows from Eqs. (8a), (23), and (63)[28]:

$$\begin{aligned} E_n^{(1)}(x^2)&=-\frac{v_{n+2}}{2\pi^2a_{n+2}^{1,1}}\frac{\Gamma(1-a_{n+2}^{1,1})}{\Gamma(n+a_{n+2}^{1,1}+2)}(\tfrac{1}{4}x^2)^{a_{n+2}^{1,1}}\\ &=\frac{V_{n+2}}{a_{n+2}^{1,1}}(x^2)^{a_{n+2}^{1,1}}, \end{aligned} \tag{65a}$$

where $\Gamma(z)$ is Euler's Γ function. A similar argument yields the longitudinal coefficients $F_n^{(1)}(x^2)$,

$$F_n^{(1)}(x^2)=\frac{V_n'}{a_n^{1,1}-1}(x^2)^{a_n^{1,1}}. \tag{65b}$$

If these expressions are substituted into the Wilson expansion (2), we find

$$\begin{aligned} J_\mu\left(\frac{x+y}{2}\right)J_\nu\left(\frac{-x+y}{2}\right)\simeq 4\sum_{n=0}^{\infty}\Big\{&\delta_{\mu\nu}x_{\alpha_1}x_{\alpha_2}[(a_n^{1,1}-1)V_n-(a_n^{1,1}+n-\tfrac{1}{2})V_n']\\ &+\delta_{\mu\alpha_1}\delta_{\nu\alpha_2}x^2\left[a_n^{1,1}V_n-\frac{1}{4}\frac{n(n-1)}{a_n^{1,1}-1}V_n'\right]\\ &-(x_\nu x_{\alpha_2}\delta_{\mu\alpha_1}+x_\mu x_{\alpha_2}\delta_{\nu\alpha_1})[(a_n^{1,1}-1)V_n+\tfrac{1}{2}nV_n']\\ &-x_\mu x_\nu x_{\alpha_1}x_{\alpha_2}[(a_n^{1,1}-2)V_n'x^{-2}]\Big\}(x^2)^{a_n^{1,1}}O^{(1)}_{\alpha_1\cdots\alpha_n}x_{\alpha_3}\cdots x_{\alpha_n}\\ &+(\text{gauge-dependent terms}), \end{aligned} \tag{66}$$

for $V_0=V_1=0$.

Thus, if we consider only amplitudes containing no self-energy or vertex corrections and no two-pseudoscalar intermediate states, the operators $O^{(1)}_{\alpha_1\cdots\alpha_n}$ possess an anomalous dimension d_n,

$$d_n=2+n+2a_n^{1,1}, \tag{67}$$

in the sense of Wilson. Here d_n is just the dimension (in units of mass) of the current × current product on the left-hand side of Eq. (66), minus the dimension of the singular coefficient of the operator $O^{(1)}_{\alpha_1\cdots\alpha_n}$ on the right-hand side of that equation. The dimension d_n clearly depends on n in a rather complicated way since to order g^2

$$d_n=2+n-\frac{g^2}{8\pi^2n(n+1)}. \tag{68}$$

B. Amplitudes with Self-Energy and Vertex Corrections Omitted

Next we study all the amplitudes of the pseudoscalar theory which do not contain self-energy or vertex corrections. The resulting functions $\tilde{E}_n^{(i)}(q^2)$ obey Eq. (27) with $\beta=\gamma_1=\gamma_2=0$. Thus,

$$\begin{aligned} -q^2\frac{\partial}{\partial q^2}\tilde{E}_n^{(1)}&=a_{n+2}^{1,1}\tilde{E}_n^{(1)}+a_{n+2}^{1,2}\tilde{E}_n^{(2)},\\ -q^2\frac{\partial}{\partial q^2}\tilde{E}_n^{(2)}&=a_{n+2}^{2,1}\tilde{E}_n^{(1)}+a_{n+2}^{2,2}\tilde{E}_n^{(2)}. \end{aligned} \tag{69}$$

The general solution to this set of coupled first-order differential equations is

$$\begin{aligned} \tilde{E}_n^{(1)}(q^2)&=v_{n+2}^{(1)}\times(q^2)^{-\nu_{n+2}^{(1)}}+v_{n+2}^{(2)}\times(q^2)^{-\nu_{n+2}^{(2)}},\\ \tilde{E}_n^{(2)}(q^2)&=v_{n+2}^{(1)}(\nu_{n+2}^{(1)}-a_{n+2}^{1,1})\frac{1}{a_{n+2}^{1,2}}(q^2)^{-\nu_{n+2}^{(1)}}+v_{n+2}^{(2)}(\nu_{n+2}^{(2)}-a_{n+2}^{1,1})\frac{1}{a_{n+2}^{1,2}}(q^2)^{-\nu_{n+2}^{(2)}}, \end{aligned} \tag{70}$$

where the $v_n^{(i)}$ are integration constants and

$$\nu_n^{(i)}=\tfrac{1}{2}(a_n^{1,1}+a_n^{2,2})+(2i-3)[\tfrac{1}{4}(a_n^{1,1}-a_n^{2,2})^2+a_n^{1,2}a_n^{2,1}]^{1/2}, \tag{71}$$

for $i=1,2$. As in the previous case, we can obtain the position-space functions $E_n^{(i)}(x^2)$ and $F_n^{(i)}(x^2)$ and substitute them into the Wilson expansion (2), with the result

$$J_\mu\left(\frac{x+y}{2}\right)J_\nu\left(\frac{-x+y}{2}\right) \simeq 4\sum_{j=1,2}\sum_{n=0}^{\infty}\Big\{\delta^{\mu\nu}x_{\alpha_1}x_{\alpha_2}[(\nu^{(j)}-1)V_n^{(j)}-(a_n^{1,1}+n-\tfrac{1}{2})V_n'^{(j)}]$$
$$+\delta_{\mu\alpha_1}\delta_{\nu\alpha_2}x^2\left[\nu_n^{(j)}V_n^{(j)}-\frac{1}{4}\frac{n(n-1)}{\nu_n^{(j)}-1}V_n'^{(j)}\right]$$
$$-(x_\nu x_{\alpha_2}\delta_{\mu\alpha_1}+x_\mu x_{\alpha_2}\delta_{\nu\alpha_2})[(\nu_n^{(j)}-1)V_n^{(j)}+\tfrac{1}{2}nV_n'^{(j)}]$$
$$-x_\mu x_\nu x_{\alpha_1}x_{\alpha_2}(\nu_n^{(j)}-2)V_n'^{(j)}x^{-2}\Big\}(x^2)^{\nu_n^{(j)}}$$
$$\times\left[O^{(1)}_{\alpha_1\cdots\alpha_n}(y)+\frac{\nu_n^{(j)}-a_n^{1,1}}{a_n^{1,2}}O^{(2)}_{\alpha_1\cdots\alpha_n}(y)\right]x_{\alpha_3}\cdots x_{\alpha_n}$$
$$+(\text{gauge-dependent terms}). \tag{72}$$

Here the constants $V_n^{(i)}$, $V_n'^{(i)}$ can be obtained from the $v_n^{(i)}$, $v_n'^{(i)}$ by using Eq. (8), where $v_n'^{(j)}$ is the integration constant multiplying $(q^2)^{-\nu_n^{(j)}}$ in the expression for $\tilde{F}_n^{(i)}(q^2)$ analogous to Eq. (70). Equation (72) implies that the operator

$$O^{(1)}_{\alpha_1\cdots\alpha_n}+\frac{\nu_n^{(j)}-a_n^{1,1}}{a_n^{1,2}}O^{(2)}_{\alpha_1\cdots\alpha_n} \tag{73}$$

has anomalous dimension

$$d_n^{(j)}=n+2+2\nu_n^{(j)}, \tag{74}$$

for $j=1,2$.

C. General Solution

Finally, we solve the exact equations (40) obeyed by the functions $\tilde{E}_n^{(i)}(q^2)$ in the vector theory. Equation (40) can be rewritten as

$$\left[-q^2\frac{\partial}{\partial q^2}+\beta\frac{\partial}{\partial g}+A_n^{(1)}(g)\right]\tilde{E}_n^{(1)}(q^2,g)=B_n^{(2)}(g)\tilde{E}_n^{(2)}(q^2,g),$$
$$\left[-q^2\frac{\partial}{\partial q^2}+\beta\frac{\partial}{\partial g}+A_n^{(2)}(g)\right]\tilde{E}_n^{(2)}(q^2,g)=B_n^{(1)}(g)\tilde{E}_n^{(1)}(q^2,g), \tag{75}$$

where $A_n^{(i)}$ and $B_n^{(i)}$ are linear combinations of γ_i and $a_n^{j,k}$. Now define the new independent variables

$$\rho(g)=\int_{g_0}^{g}\frac{dg'}{\beta(g')}, \tag{76a}$$

$$z(q^2,g)=\ln(q^2/q_0^{\,2})+\rho(g), \tag{76b}$$

for some fixed values g_0, $q_0^{\,2}$. Let $G(\rho)$ be the inverse of the function $\rho(g)$ defined by Eq. (76a). In terms of these new variables Eq. (75) becomes

$$\left[\frac{\partial}{\partial\rho}+A_n^{(1)}(G(\rho))\right]\tilde{E}_n^{(1)}=B_n^{(2)}(G(\rho))\tilde{E}_n^{(2)},$$
$$\left[\frac{\partial}{\partial\rho}+A_n^{(2)}(G(\rho))\right]\tilde{E}_n^{(2)}=B_n^{(1)}(G(\rho))\tilde{E}_n^{(1)}, \tag{77}$$

where the functions $\tilde{E}_n^{(i)}(q_0^{\,2}\exp(z-\rho),G(\rho))$ are to be treated as functions of z and ρ. This set of two first-order coupled differential equations in the single variable ρ has a general solution of the form

$$\tilde{E}_n^{(1)}(q^2,g)=v_n^{(1)}(\ln(q^2/q_0^{\,2})+\rho(g))L_n^{(1)}(\rho(g))+v_n^{(2)}(\ln(q^2/q_0^{\,2})+\rho(g))L_n^{(2)}(\rho(g)),$$
$$\tilde{E}_n^{(2)}(q^2,g)=\frac{v_n^{(1)}(\ln(q^2/q_0^{\,2})+\rho(g))}{B_n^{(2)}(g)}\left[\frac{d}{d\rho}L_n^{(1)}(\rho(g))+A_n^{(1)}(g)L_n^{(1)}(\rho(g))\right]$$
$$+\frac{v_n^{(2)}(\ln(q^2/q_0^{\,2})+\rho(g))}{B_n^{(2)}(g)}\left[\frac{d}{d\rho}L_n^{(2)}(\rho(g))+A_n^{(1)}(g)L_n^{(2)}(\rho(g))\right], \tag{78}$$

where $v_n^{(1)}(z)$ and $v_n^{(2)}(z)$ are integration "constants" which can depend on $z=\ln(q^2/q_0^2)+\rho(g)$, while $L_n^{(1)}(\rho)$ and $L_n^{(2)}(\rho)$ are the two independent solutions of the second-order differential equation

$$\left[\frac{d}{d\rho}+A_n^{(2)}(G(\rho))\right]\frac{1}{B_n^{(2)}(G(\rho))}\left[\frac{d}{d\rho}+A_n^{(1)}(G(\rho))\right]L_n^{(i)}(\rho)-B_n^{(1)}(G(\rho))L_n^{(i)}(\rho)=0\,. \tag{79}$$

Thus, the original equations (40) allow the two functions $\tilde{E}_n^{(i)}(q^2,g)$ which depend on two variables to be determined in terms of the two unknown functions $v_n^{(i)}(z)$ of a single variable.

Now let us speculate about a possible large-q^2 behavior of the solutions $\tilde{E}_n^{(i)}(q^2,g)$ given by Eq. (78). Since the unknown functions $v_n^{(i)}(z)$ appearing in Eq. (78) depend only on the sum of $\ln(q^2/q_0^2)$ and $\rho(g)$, the large-q^2 behavior and the large-ρ behavior of the functions $\tilde{E}_n^{(i)}(q^2,G(\rho))$ are directly related once the large-ρ behavior of $A_n^{(i)}(G(\rho))$, $B_n^{(i)}(G(\rho))$, and $L_n^{(i)}(\rho)$ is known. In fact, carrying out the algebraic steps outlined in Appendix D, we find

$$\tilde{E}_n^{(i)}(q^2,g)=\sum_{j=1,2} w_n^{i,j}(\ln(q^2/q_0^2)+\rho(g),g)\times\tilde{E}_n^{(j)}(q_0^2,G(\ln(q^2/q_0^2)+\rho(g)))\,, \tag{80}$$

where the quantities $w_n^{i,j}(z,g)$, defined in Appendix D, are rational functions of $A_n^{(k)}$, $B_n^{(k)}$, and $L_n^{(k)}$. Thus the large-ρ behavior of $\tilde{E}_n^{(j)}(q_0^2,G(\rho))$ determines, through Eq. (80), the large-q^2 dependence of $\tilde{E}_n^{(i)}(q^2,g)$. Following Gell-Mann and Low, we consider the possibility that $\beta(g)$ has a root, g_∞, so that

$$\lim_{g\to g_\infty}\rho(g)=\infty\,. \tag{81}$$

If we assume that the quantities $\tilde{E}_n^{(i)}(q_0^2,g)$ are well defined and finite at the point $g=g_\infty$, then Eq. (78) determines the large-q^2 behavior of $\tilde{E}_n^{(i)}(q^2,g)$ in terms of the functions $\rho(g)$, $A_n^{(i)}(g)$, and $B_n^{(i)}(g)$ which appear in our equation.

A particularly simple asymptotic q^2 behavior of $\tilde{E}_n^{(i)}(q^2)$ results if we assume that g_∞ is a simple root of $\beta(g)$ and that $A_n^{(i)}(g)$ and $B_n^{(i)}(g)$ are regular at g_∞. As is shown in Appendix D, these assumptions when combined with Eqs. (79) and (80) imply a simple power behavior for $\tilde{E}_n^{(i)}(q^2)$.

V. CONCLUSIONS

Using a technique of Symanzik and the Callan-Symanzik equations, we obtain a series of equations obeyed to arbitrary order in perturbation theory by all the c-number coefficients of the operators appearing in the light-cone expansion of $J_\mu(x)J_\nu(0)$. These equations are used to determine the leading logarithmic behavior of νW_2 for two specific field theories, giving results in agreement with previous, more laborious calculations. For simplified classes of amplitudes in which no coupling-constant renormalization is required, the equations predict a power-law behavior of the coefficients $E_n^{(i)}(x^2)$ and a corresponding anomalous dimension $d_n=2+n+2\nu_n$ for linear combinations of the operators $O^{(i)}_{\alpha_1\cdots\alpha_n}$ appearing in the Wilson expansion. In general, the added quantity ν_n depends in a nontrivial fashion on n. Since the same operators $O^{(i)}_{\alpha_1\cdots\alpha_n}$ enter both the transverse and longitudinal terms in the Wilson expansion, the functions $\tilde{E}_n^{(i)}(x^2)$ and $F_n^{(i)}(x^2)$ both obey the same set of equations. Thus, in this formalism only the presence of different integration constants distinguishes the small-x^2 behavior of the transverse and longitudinal components of the product $J_\mu(x)J_\nu(0)$. Finally, these equations allow us to speculate about the large-q^2 and large-ν behavior of W_1 and W_2, following the path previously considered by Gell-Mann and Low, Wilson,[6] and Symanzik.[11]

APPENDIX A

In this appendix we provide a derivation[12,29] of the Callan-Symanzik equations used in Sec. II. Let us begin by considering the pseudoscalar theory specified by the Lagrangian (13). The complete Lagrangian, including counterterms, is

$$\begin{aligned}\mathcal{L}=&-\bar\psi(\gamma_\mu\partial_\mu+m)\psi-\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi-\tfrac{1}{2}\mu^2\phi^2+\frac{h}{4!}\phi^4+ig\phi\bar\psi\gamma_5\psi-\delta mZ_2\bar\psi\psi-\tfrac{1}{2}\delta\mu^2Z_3\phi^2\\&-(Z_2-1)\bar\psi(\gamma_\mu\partial_\mu+m)\psi-\tfrac{1}{2}(Z_3-1)(\partial_\mu\phi\partial_\mu\phi+\mu^2\phi^2)+i(Z_1-1)g\phi\bar\psi\gamma_5\psi+(Z_4-1)\frac{h}{4!}\phi^4\,.\end{aligned} \tag{A1}$$

In order to specify the subtraction procedure represented by the above counterterms, we consider the propagators $S(\not{p},m,\mu)$, $\Delta(k^2,m,\mu)$, and the amputated vertex functions $\Gamma^5(p_1,p_2)$, $\Box(k_1,k_2,k_3)$ defined by

$$S(\not p, m, \mu)=i\int e^{ip\cdot x}\, d^4x\langle 0|\, T(\psi(0)\overline{\psi}(x))\, |0\rangle,$$

$$\Delta(k^2, m, \mu)=i\int e^{ik\cdot x}\, d^4x\langle 0|\, T(\phi(0)\phi(x))\, |0\rangle,$$

$$\Gamma^5(p_1, p_2, m, \mu)=-i\int e^{i(p_1\cdot x-p_2\cdot y)}d^4x\, d^4y\langle 0|T(\psi(y)\phi(0)\overline{\psi}(x))|0\rangle_A, \tag{A2}$$

$$\Box(k_1, k_2, k_3, m, \mu)=-i\int e^{i(k_1\cdot x+k_2\cdot y+k_3\cdot z)}\, d^4x\, d^4y\, d^4z\langle 0|\, T(\phi(x)\phi(y)\phi(z)\phi(0))|0\rangle_A.$$

The subtraction constants Z_1, Z_2, Z_3, Z_4, $\delta\mu^2$, and δm^2 are chosen so that the following conditions are satisfied:

$$S^{-1}(\not p, m, \mu)=0,\quad \frac{\partial}{\partial \not p}S^{-1}(\not p)=-1 \text{ at } \not p=m,$$

$$\Delta^{-1}(k^2, m, \mu)=0,\quad \frac{\partial}{\partial k^2}\Delta^{-1}(k^2)=1 \text{ at } k^2=-\mu^2,$$

$$\Gamma^5(p_1, p_2, m, \mu)=i\gamma_5 g \text{ at } \not p_1=\not p_2=m,\quad (p_1+p_2)^2=-\mu^2, \tag{A3}$$

$$\Box(k_1, k_2, k_3, m, \mu)=h \text{ at } k_1^2=k_2^2=k_3^2=-\mu^2,$$

$$(k_1+k_2)^2=(k_1+k_3)^2=(k_2+k_3)^2=-\tfrac{4}{3}\mu^2.$$

Having made this choice of subtraction constants we now calculate order by order in perturbation theory each Green's function $\Gamma(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho)$ for

$$\Gamma(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho) = \prod_{i=1}^{2n}\int d^4x_i\, e^{ip_i\cdot x_i}\prod_{i=1}^{\rho}\int d^4y_i\, e^{ik_i\cdot y_i}\langle 0|\, T(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(x_{n+1})\cdots\overline{\psi}(x_{2n})\phi(y_1)\cdots\phi(y_\rho))\, |0\rangle, \tag{A4}$$

as a function of g, h, m, and μ.

In order to derive the Callan-Symanzik equations we consider a second procedure for computing the Green's functions of this theory in which the subtractions are carried out at arbitrary points λ_1 and λ_2. We rewrite the Lagrangian $\mathcal{L}$ in terms of fields ψ_λ, ϕ_λ and coupling constants g_λ, h_λ normalized at these new points:

$$\mathcal{L}=-\overline{\psi}_\lambda(\gamma_\mu\partial_\mu+m)\psi_\lambda-\tfrac{1}{2}\partial_\mu\phi_\lambda\partial_\mu\phi_\lambda-\tfrac{1}{2}\mu^2\phi_\lambda^2+\frac{h_\lambda}{4!}\phi_\lambda^4+ig_\lambda\phi_\lambda\overline{\psi}_\lambda\gamma_5\psi_\lambda-\delta m Z_{2,\lambda}\overline{\psi}_\lambda\psi_\lambda-\tfrac{1}{2}\delta\mu^2 Z_{3,\lambda}\phi^2$$
$$-(Z_{2,\lambda}-1)\overline{\psi}_\lambda(\gamma_\mu\partial_\mu+m)\psi_\lambda-\tfrac{1}{2}(Z_{3,\lambda}-1)(\partial_\mu\phi_\lambda\partial_\mu\phi_\lambda+\mu^2\phi_\lambda^2)+i(Z_{1,\lambda}-1)g_\lambda\phi_\lambda\overline{\psi}_\lambda\gamma_5\psi_\lambda+(Z_{4,\lambda}-1)\frac{h_\lambda}{4!}\phi_\lambda^4. \tag{A5}$$

The subtraction constants $Z_{1,\lambda}$, $Z_{2,\lambda}$, $Z_{3,\lambda}$, $Z_{4,\lambda}$, δm, and $\delta\mu$ are so chosen that the functions S_λ, Δ_λ, $\Gamma_\lambda{}^5$, and $\Box_\lambda$ defined from Eq. (A2) by replacing the fields ψ, ϕ by ψ_λ, ϕ_λ satisfy the following normalization conditions:

$$S_\lambda{}^{-1}(\not p, m, \mu)=0 \text{ at } \not p=m,\quad \frac{\partial}{\partial\not p}S_\lambda{}^{-1}(\not p, \lambda_1, \lambda_2)=-1 \text{ at } \not p=\lambda_1, \tag{A6a}$$

$$\Delta_\lambda{}^{-1}(k^2, m, \mu)=0 \text{ at } k^2=-\mu^2,\quad \frac{\partial}{\partial k^2}\Delta_\lambda{}^{-1}(k^2, \lambda_1, \lambda_2)=1 \text{ at } k^2=-\lambda_2{}^2, \tag{A6b}$$

$$\Gamma^5_\lambda(p_1, p_2, \lambda_1, \lambda_2)=i\gamma_5 g_\lambda \text{ at } \not p_1=\not p_2=\lambda_1,\quad (p_1+p_2)^2=-\lambda_2{}^2, \tag{A6c}$$

$$\Box_\lambda(k_1, k_2, k_3, \lambda_1, \lambda_2)=h_\lambda \text{ at } k_1{}^2=k_2{}^2=k_3{}^2=-\lambda_2{}^2,\quad (k_1+k_2)^2=(k_1+k_3)^2=(k_2+k_3)^2=-\tfrac{4}{3}\lambda_2{}^2. \tag{A6d}$$

The Lagrangians (A1) and (A5) are equal, the quantities g_λ, h_λ being functions of $g, h, m, \mu, \lambda_1, \lambda_2$. The Green's functions $\Gamma_\lambda(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho)$ computed by replacing the fields ψ, ϕ by ψ_λ, ϕ_λ in Eq. (A4) are proportional to the original $\Gamma(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho)$,

$$\Gamma(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho)=Z_{2,\lambda}{}^n Z_{3,\lambda}{}^{\rho/2}\Gamma_\lambda(p_1, \ldots, p_{2n}, k_1, \ldots, k_\rho). \tag{A7}$$

The Callan-Symanzik equations can be obtained by differentiating Eq. (A7) with respect to m and μ and then setting $\lambda_1=m$, $\lambda_2=\mu$,

$$D\Gamma=\left\{n(DZ_{2,\lambda})+\frac{\rho}{2}(DZ_{3,\lambda})+(Dg_{\lambda})\frac{\partial}{\partial g_{\lambda}}+(Dh_{\lambda})\frac{\partial}{\partial h_{\lambda}}+m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right\}\Gamma_{\lambda}\Big|_{\lambda_1=m;\lambda_2=\mu} \quad \text{(A8)}$$

where

$$D\equiv m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\,.$$

When computed in perturbation theory from the Lagrangian (A5), the Green's function Γ_λ is determined as a function of $g_\lambda, h_\lambda, m, \mu, \lambda_1, \lambda_2$; the last two partial derivatives in Eq. (A8) of this function Γ_λ, $\partial/\partial m^2$ and $\partial/\partial\mu^2$, are to be performed with g_λ and h_λ held fixed. This can be recognized as just the Callan-Symanzik Eq. (18), if we (a) identify

$$\gamma_1=\tfrac{1}{2}DZ_{2,\lambda}\,,\quad \gamma_2=\tfrac{1}{2}DZ_{3,\lambda}\,,$$
$$\beta=-Dg_\lambda\,,\quad \beta'=-Dh_\lambda\,, \quad \text{(A9a)}$$

$$\Delta\Gamma=\left[m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right]\Gamma_\lambda\,, \quad \text{(A9b)}$$

all evaluated at $\lambda_1=m$, $\lambda_2=\mu$; (b) set $n=1$, $\rho=0$ or $n=0$, $\rho=2$; and (c) insert $J_\mu(x)\,J_\nu(0)$ into the time-ordered product defining Γ and Γ_λ. We need only show that the amplitude $\Delta\Gamma$ can be obtained by inserting the mass operator u of Eq. (20) into the time-ordered product (A4) defining Γ. Since only the normalization conditions for S_λ and Δ_λ involve the masses m and μ, the operation $m^2\,\partial/\partial m^2+\mu^2\partial/\partial\mu^2$ when applied to the amplitude Γ_λ yields a series of terms, each obtained from Γ_λ by (a) replacing a spinor propagator $S_\lambda(\not p)$ by

$$-S_\lambda(\not p)\left[\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)S_\lambda^{-1}(\not p)\right]S_\lambda(\not p)\,, \quad \text{(A10a)}$$

or (b) replacing a pseudoscalar propagator $\Delta_\lambda(k^2)$ by

$$-\Delta_\lambda(k^2)\left[\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\Delta_\lambda^{-1}(k^2)\right]\Delta_\lambda(k^2)\,. \quad \text{(A10b)}$$

On the other hand, the effect of inserting $-iu$ into the time-ordered product defining Γ is similar, yielding a sum of terms obtained from Γ by (a) replacing a spinor propagator $S(\not p)$ by

$$\int e^{ip\cdot x}\,d^4x\langle 0|\,T(\psi(0)\,u\,\overline{\psi}(x))\,|0\rangle\equiv -S(\not p)U_1(\not p)S(\not p)\,, \quad \text{(A11a)}$$

or (b) changing a pseudoscalar propagator $\Delta(k^2)$ to

$$\int e^{ik\cdot x}\,d^4x\langle 0|\,T(\phi(0)\,u\,\phi(x))\,|0\rangle$$
$$\equiv -\Delta(k^2)U_2(k^2)\Delta(k^2)\,. \quad \text{(A11b)}$$

It is not difficult to see that $U_2(k^2)$ and

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\Delta_\lambda^{-1}(k^2)\Big|_{\lambda_1=m;\;\lambda_2=\mu^2} \quad \text{(A12)}$$

obey Dyson integral equations with the same kernel. Since the normalization condition for $(\partial/\partial k^2)\Delta_\lambda^{-1}(k^2)$ in Eq. (A6b) does not involve m or μ, $Z_{3,\lambda}$ depends on g_λ and h_λ, but not on m or μ. Thus, the Dyson equations obeyed by both quantities contain only a constant inhomogeneous term. Therefore, the two functions of k^2 must be proportional. If we let

$$\delta_2=\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\Delta_\lambda^{-1}(k^2)\Big|_{\lambda_1=m;\;\lambda_2=\mu;\;k^2=-\mu^2}\,, \quad \text{(A13a)}$$

and normalize the finite operators $N[\overline{\psi}(x)\psi(x)]$, $N[\phi(x)\phi(x)]$ so that

$$\langle p,s|N[\overline{\psi}(x)\psi(x)]|p,s\rangle=\langle k|N[\phi(x)\phi(x)]|\,k\rangle=1\,,$$
$$\langle p,s|N[\phi(x)\phi(x)]|\,p,s\rangle=\langle k|N[\overline{\psi}(x)\psi(x)]|k\rangle=0\,, \quad \text{(A14)}$$

where the state $|\,k\rangle$ contains a single pseudoscalar particle of momentum k, then

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\Delta_\lambda^{-1}(k^2)\Big|_{\lambda_1=m;\;\lambda_2=\mu}=U_2(k^2)\,. \quad \text{(A15a)}$$

Similar arguments imply

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)S_\lambda^{-1}(\not p)\Big|_{\lambda_1=m;\;\lambda_2=\mu}=U_1(\not p)\,, \quad \text{(A15b)}$$

if

$$\delta_1=2\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)S_\lambda^{-1}(\not p)\Big|_{\lambda_1=m;\lambda_2=\mu;\;\not p=m} \quad \text{(A13b)}$$

Thus Eq. (A9b) is justified and the Callan-Symanzik equations proved for the neutral pseudoscalar theory.

Let us now consider the vector theory. The complete Lagrangian, including counterterms, for this theory is

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \tfrac{1}{4} G_{\mu\nu} G_{\mu\nu} - \tfrac{1}{2}\mu^2 V_\rho V_\rho + igV_\mu \bar{\psi}\gamma_\mu \psi - \delta m Z_2 \bar{\psi}\psi - \tfrac{1}{2}\delta\mu^2 Z_3 V_\rho V_\rho$$
$$- (Z_2 - 1)\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \tfrac{1}{2}(Z_3 - 1)(\tfrac{1}{2}G_{\mu\nu} G_{\mu\nu} + \mu^2 V_\rho V_\rho) + i(Z_1 - 1)g\bar{\psi}\gamma_\mu \psi V_\mu . \quad \text{(A16)}$$

Introducing the propagators[30] and vertex functions

$$S(\not{p}, m, \mu) = i\int e^{ip\cdot x}\, d^4x \langle 0|\, T(\psi(0)\bar{\psi}(x))\,|0\rangle ,$$
$$\Delta(k^2, m, \mu)\delta_{\mu\nu} + \tilde{\Delta}(k^2, m, \mu)k_\mu k_\nu = i\int e^{ik\cdot x}\, d^4x \langle 0|\, T(V_\mu(0)V_\nu(x))|\, 0\rangle ,$$
$$\Gamma_\mu(p_1, p_2, m, \mu) = -i\int e^{ip_1\cdot x - ip_2\cdot y}\, d^4x\, d^4y \langle 0|\, T(\psi(y)V_\mu(0)\bar{\psi}(x))\,|0\rangle_A , \quad \text{(A17)}$$

we choose the subtraction constants $Z_1, Z_2, Z_3, \delta m$, $\delta\mu^2$ so that

$$S^{-1}(\not{p}, m, \mu)|_{\not{p}=m} = 0 ,$$
$$\frac{\partial}{\partial \not{p}} S^{-1}(\not{p}, m, \mu)\bigg|_{\not{p}=m} = -1 ,$$
$$\Delta^{-1}(k^2, m, \mu)|_{k^2=-\mu^2} = 0 , \quad \text{(A18)}$$
$$\frac{\partial}{\partial k^2} \Delta^{-1}(k^2, m, \mu)\bigg|_{k^2=-\mu^2} = 1 ,$$
$$\Gamma_\mu(p_1, p_2, m, \mu)|_{\not{p}_1=\not{p}_2=m;(p_1+p_2)^2=-\mu^2} = ig\gamma_\mu .$$

The electromagnetic interaction of the charged spinor field is now included by adding the interaction term

$$\mathcal{L}_\gamma = +ieA_\mu \bar{\psi}\gamma_\mu \psi + \tfrac{1}{2} f F_{\mu\nu} G_{\mu\nu} + \tfrac{1}{2}(\delta f) Z_3^{1/2} F_{\mu\nu} G_{\mu\nu}$$
$$+ \tfrac{1}{2} f(Z_3^{1/2} - 1)F_{\mu\nu} G_{\mu\nu} + ie(Z_1 - 1)A_\mu \bar{\psi}\gamma_\mu \psi , \quad \text{(A19)}$$

where the subtraction constant δf is so defined that

$$\Delta_1(k^2, m, \mu)|_{k^2=-\mu^2} = f\mu^2$$

for

$$\Delta_1(k^2, m, \mu)[\delta_{\mu\nu} - k_\mu k_\nu / k^2]$$
$$= i\int e^{ik\cdot x}\, d^4x \langle 0|\, T(V_\mu(0)A_\nu(x))|\, 0\rangle_A \quad \text{(A20)}$$

We have not included counterterms of order e^2 or higher in Eq. (A19). Just as in the pseudoscalar case we can consider fields ψ_λ, $V_{\lambda,\rho}$ and coupling constants g_λ, f_λ defined according to a second normalization scheme:

$$S_\lambda^{-1}(\not{p}, m, \mu)|_{\not{p}=m} = 0 ,$$
$$\frac{\partial S^{-1}}{\partial \not{p}}(\not{p}, \lambda_1, \lambda_2)\bigg|_{\not{p}=\lambda_1} = -1 , \quad \text{(A21a)}$$

$$\Delta_\lambda^{-1}(k^2, m, \mu)|_{k^2=-\mu^2} = 0 ;$$
$$\frac{\partial \Delta_\lambda^{-1}}{\partial k^2}(k^2, \lambda_1, \lambda_2)\bigg|_{k^2=-\lambda_2^2} = 1 , \quad \text{(A21b)}$$

$$\Gamma_{\lambda,\mu}(p_1, p_2, \lambda_1, \lambda_2)|_{\not{p}_1=\not{p}_2=\lambda_1;\,(p_1-p_2)^2=-\lambda_2^2} = ig_\lambda \gamma_\mu , \quad \text{(A21c)}$$

$$\Delta_{1,\lambda}(k^2, \lambda_1, \lambda_2)|_{k^2=-\lambda_2^2} = f_\lambda \lambda_2^2 . \quad \text{(A21d)}$$

Differentiating Eq. (A7), rewritten for the vector case, we find

$$D\Gamma = \left[n(DZ_{2,\lambda}) + \frac{\rho}{2}(DZ_{3,\lambda}) + (Dg_\lambda)\frac{\partial}{\partial g_\lambda} + (Df_\lambda)\frac{\partial}{\partial f_\lambda} + m^2 \frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2} \right] \Gamma_\lambda \bigg|_{\lambda_1=m;\,\lambda_2=\mu} . \quad \text{(A22)}$$

This is the complete Callan-Symanzik equation obeyed by the amputated time-ordered product of $2n$ spinor fields, ρ vector fields, and r electromagnetic currents,

$$eJ_\mu = \partial_\nu F_{\nu\mu} . \quad \text{(A23)}$$

The quantities $DZ_{2,\lambda}$, $DZ_{3,\lambda}$, and Dg_λ can be identified with $2\gamma_1$, $2\gamma_2$, and $-\beta$ of Eq. (33), respectively, while an argument similar to that given in the preceding pseudoscalar case shows that

$$\gamma_3^{-r}\Delta\Gamma = \left(m^2 \frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2} \right) \Gamma_\lambda \bigg|_{\lambda_1=m;\,\lambda_2=\mu} . \quad \text{(A24)}$$

Finally we can directly compute Df_λ by noting that $\Delta_1(k^2, m, \mu)$ and $(1/g)\Delta^{-1}(k^2, m, \mu)$ can, by definition, differ only by a first-degree polynomial in k^2, so that Eq. (A20) and current conservation $[\Delta_1(0, m, \mu) = 0]$ imply

$$\Delta_1(k^2, m, \mu) = -fk^2 - \frac{k^2 + \mu^2}{\mu^2}\frac{1}{g}\Delta^{-1}(0, m, \mu) + \frac{1}{g}\Delta^{-1}(k^2, m, \mu) . \quad \text{(A25a)}$$

Likewise,

$$\Delta_{1,\lambda}(k^2, m, \mu) = -f_\lambda k^2 - \frac{k^2}{\lambda_2^2}\frac{1}{g_\lambda}\Delta_\lambda^{-1}(0, \lambda_1, \lambda_2) - \frac{1}{g_\lambda}\Delta_\lambda^{-1}(0, m, \mu) + \frac{1}{g_\lambda}\Delta_\lambda^{-1}(k^2, m, \mu), \quad \text{(A25b)}$$

where the coefficient of k^2 is guaranteed by our subtraction procedure to be independent of m and μ for fixed f_λ and is therefore determined by the condition (A21d). In analogy with Eq. (A7) of the pseudoscalar case we have

$$\Delta_{1,\lambda}(k^2, m, \mu) = Z_{3,\lambda}{}^{-1/2}\,\Delta_1(k^2, m, \mu)\,, \qquad \text{(A26a)}$$

$$\Delta_\lambda{}^{-1}(k^2, m, \mu) = Z_{3,\lambda}{}^{-1}\,\Delta^{-1}(k^2, m, \mu)\,, \qquad \text{(A26b)}$$

$$S_\lambda{}^{-1}(\not p, m, \mu) = Z_{2,\lambda}{}^{-1} S^{-1}(\not p, m, \mu)\,, \qquad \text{(A26c)}$$

$$\Gamma_{\lambda,\mu}(p_1, p_2, m, \mu) = Z_{3,\lambda}{}^{-1/2} Z_{2,\lambda}{}^{-1}\Gamma_\mu(p_1, p_2, m, \mu). \qquad \text{(A26d)}$$

Equations (A26c) and (A26d) together with the Ward identities

$$\frac{1}{g}(p_1 - p_2)_\mu \Gamma_\mu(p_1, p_2) = S^{-1}(p_1) - S^{-1}(p_2)\,,$$
$$\text{(A27)}$$
$$\frac{1}{g_\lambda}(p_1 - p_2)_\mu \Gamma_{\lambda,\mu}(p_1, p_2) = S_\lambda{}^{-1}(p_1) - S_\lambda{}^{-1}(p_2)\,,$$

implied by current conservation and our normalization procedures (A18) and (A21), yield

$$g_\lambda = Z_{3,\lambda}{}^{-1/2} g\,. \qquad \text{(A28)}$$

Combining Eqs. (A25), (A26a), (A26b), and (A28) we obtain

$$f_\lambda = Z_{3,\lambda}{}^{-1/2} f - \frac{1}{g_\lambda}\frac{\Delta_\lambda{}^{-1}(0, \lambda_1, \lambda_2)}{\lambda_2{}^2} + \frac{1}{g} Z_{3,\lambda}{}^{-1/2}\frac{\Delta^{-1}(0, m, \mu)}{\mu^2}\,, \qquad \text{(A29)}$$

or

$$Df_\lambda|_{\lambda_1 = m;\ \lambda_2 = \mu} = -\gamma_2 f + \left(\beta\frac{\partial}{\partial g} - \gamma_2\right)\frac{1}{g}\frac{\Delta^{-1}(0, m, \mu)}{\mu^2}\,, \qquad \text{(A30)}$$

so that our complete Callan-Symanzik equation reads

$$\left\{m^2\frac{\partial}{\partial m^2} + \mu^2\frac{\partial}{\partial \mu^2} + \beta\frac{\partial}{\partial g} - 2n\gamma_1 - \rho\gamma_2 + \left[\gamma_2 f - \left(\beta\frac{\partial}{\partial g} - \gamma_2\right)\frac{1}{g}\frac{\Delta^{-1}(0, m, \mu)}{\mu^2}\right]\frac{\partial}{\partial f}\right\}\Gamma = \gamma_3{}^{-r}\Delta\Gamma\,. \qquad \text{(A31)}$$

If Γ is computed to lowest order in e, the dependence on f is known, allowing the partial derivative with respect to f in Eq. (A31) to be explicitly carried out. If we assume that each electromagnetic current carries a momentum transfer squared much greater than μ^2, then Eq. (A25a) implies that if each current $J_\mu(x)$ is replaced by

$$\left[f + \frac{1}{g\mu^2}\Delta^{-1}(0, m, \mu)\right]\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha} V_\mu(x)\,,$$

the Green's function Γ is not changed. Thus,

$$\left[gf + \frac{1}{\mu^2}\Delta^{-1}(0, m, \mu)\right]^{-r}\Gamma$$

is independent of f, and Eq. (A31) can be rewritten

$$\left(m^2\frac{\partial}{\partial m^2} + \mu^2\frac{\partial}{\partial \mu^2} + \beta\frac{\partial}{\partial g} - 2n\gamma_1 - \rho\gamma_2 + 2r\gamma_2\right) \times\left[gf + \frac{1}{\mu^2}\Delta^{-1}(0, m, \mu)\right]^{-r}\Gamma = \Delta\Gamma\,, \qquad \text{(A32)}$$

for

$$\gamma_3 = \left[gf + \frac{1}{\mu^2}\Delta^{-1}(0, m, \mu)\right]^{-1}\,, \qquad \text{(A33)}$$

which is just Eq. (33). In obtaining the form (A32) we have used the relationship

$$\beta = g\gamma_2 \qquad \text{(A34)}$$

implied by Eq. (A28).

APPENDIX B

In this appendix a detailed calculation of $a_n^{1,1}$ to order g^2 is presented for the pseudoscalar theory. Recall that

$$\int e^{ip\cdot(z-y)} d^4y\, d^4z\, \tfrac{1}{2}(\not p)_{\delta\sigma}\langle 0|T(\psi_\sigma(y) O^{(1)}_{\alpha_1\cdots\alpha_n}(0) u\bar\psi_\delta(z))|0\rangle|_{p^2=0} = (i)^n a_n^{1,1} p_{\alpha_1}\cdots p_{\alpha_n} + (\text{terms containing } \delta_{\alpha_i\alpha_j})\,, \qquad \text{(B1)}$$

for $1 \le a_i \le 3$. Since to lowest order in g^2 no counterterms must be added to make the operator $O^{(1)}_{\alpha_1\cdots\alpha_n}$ finite,

$$O^{(1)}_{\alpha_1\cdots\alpha_n}(x) = -\frac{1}{4n}[1 + (-1)^n]\sum_{j=1}^{n}\bar\psi(x)\partial_{\alpha_1}\cdots\gamma_{\alpha_j}\cdots\partial_{\alpha_n}\psi(x) + (\text{terms containing } \delta_{\alpha_i\alpha_j})\,. \qquad \text{(B2)}$$

Figure 2(a) illustrates the three graphs contributing to $a_n^{1,1}$ to order g^2. In fact, to order g^2, $a_n^{1,1}$ requires no renormalization of any sort, either within the operator $O^{(1)}_{\alpha_1\cdots\alpha_n}$ or of propagators or other vertices. Consequently, the effect of the operator u is simply to differentiate the order-g^2 matrix element of $O^{(1)}_{\alpha_1\cdots\alpha_n}$ represented by Fig. 2(b) with respect to the internal masses:

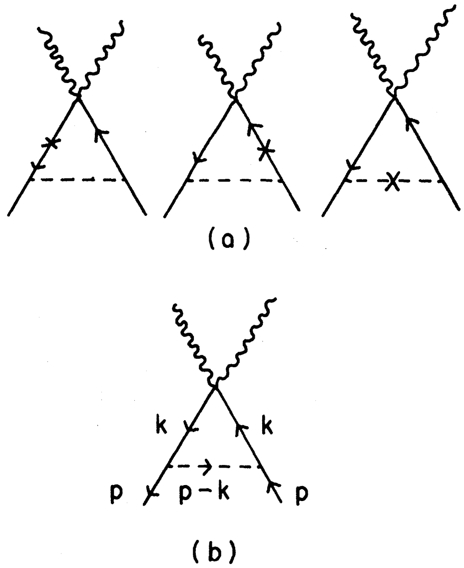

FIG. 2. (a) Diagrams representing the matrix element which determines $a_n^{1,1}$ to order g^2 in the pseudoscalar theory. The cross indicates insertion of the mass operator u, while the two-photon–two-fermion vertex represents the factor given in (C2). (b) The diagram representing the order-g^2, two-fermion matrix element of the operator specified by Eq. (B2).

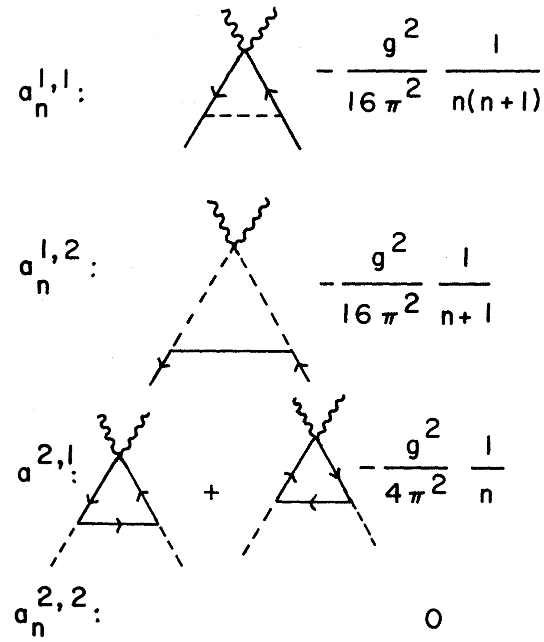

FIG. 3. The results of a calculation of $a_n^{i,j}$ to order g^2 in the pseudoscalar theory and those Feynman diagrams described in Appendix C, from which their values were obtained.

$$a_n^{1,1}p_{\alpha_1}\cdots p_{\alpha_n}=\frac{1}{4n}\sum_{j=1}^{n}\frac{g^2}{(2\pi)^4}\int d^4k\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\mathrm{tr}\frac{[\not p(\not k+m)(k_{\alpha_1}\cdots\gamma_{\alpha_j}\cdots k_{\alpha_n})(\not k+m)]}{[k^2+m^2-i\epsilon]^2[(k-p)^2+\mu^2-i\epsilon]}$$
$$+(\text{terms containing }\delta_{\alpha_i\alpha_j}), \tag{B3}$$

for $p^2=0$. [The quantities δ_1 and δ_2 appearing in the definition (20) of u are both unity to lowest order in g.] It is useful to observe that the mass terms in the numerator do not contribute to the $p_{\alpha_1}\cdots p_{\alpha_n}$ term since, if the differentiation $m^2\partial/\partial m^2+\mu^2\partial/\partial\mu^2$ were performed after the integration over k, then upon integration such terms would yield finite functions of μ^2/m^2 which would be annihilated by the derivatives.

The $p_{\alpha_1}\cdots p_{\alpha_n}$ term in the above integral can be easily evaluated if the integration variables are changed to those of Sudakov. Let

$$p=(0,0,P,iP),\quad \bar p=(0,0,+P,-iP),$$

and

$$k=\alpha\bar p+\beta p+k_\perp,$$

for

$$k_\perp=(k_1,k_2,0,0). \tag{B4}$$

In terms of the variables α, β, k_1, k_2, Eq. (B3) becomes

$$a_n^{1,1}=\frac{2P^2g^2}{(2\pi)^4}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\int d^2k\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\frac{-i\beta^{n-1}k_\perp{}^2}{[4P^2\alpha\beta+k_\perp{}^2+m^2-i\epsilon]^2[4P^2\alpha(\beta-1)+k_\perp{}^2+\mu^2-i\epsilon]}, \tag{B5}$$

where we have equated the coefficients of $p_{\alpha_1}\cdots p_{\alpha_n}$ and evaluated the trace in Eq. (B3) according to

$$\mathrm{tr}(\not p\not k\gamma_{\alpha_i}\not k)=-4ik_\perp{}^2p_{\alpha_i}+[\text{terms with }\bar p_{\alpha_i}\text{ or }(k_\perp)_{\alpha_i}]. \tag{B6}$$

The integral over α can be evaluated using contour integration so that

$$a_n^{1,1} = \frac{g^2}{2(2\pi)^3}\int_0^1 d\beta \int d^2k \left(m^2\frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2}\right) \times \frac{(1-\beta)\beta^{n-1}k_\perp{}^2}{[\beta\mu^2 + (1-\beta)m^2 + k_\perp{}^2]^2}$$

$$= -\frac{g^2}{16\pi^2}\int_0^1 d\beta\, \beta^{n-1}(1-\beta) = -\frac{g^2}{16\pi^2}\frac{1}{n(n+1)}\,. \tag{B7}$$

APPENDIX C

We now give the results of a calculation of all the constants $a_n^{i,j}$ to order g^2 in both the pseudoscalar and vector theories. The values of $a_n^{i,j}$ found for the pseudoscalar theory are shown in Fig. 3. Also shown are Feynman diagrams representing those amplitudes which when differentiated with respect to the internal masses give the adjacent values of $a_n^{i,j}$. The vertex joining two photon lines and two fermion lines represents the factor

$$-\frac{1}{2}\frac{1}{n}\sum_{j=1}^{n} p_{\alpha_1}\cdots\gamma_{\alpha_j}\cdots p_{\alpha_n} \tag{C1}$$

in the corresponding Feynman amplitudes, where p is the four-momentum carried by the incoming fermion line. Likewise, the vertex connecting two photon and two pseudoscalar lines represents the factor

$$k_{\alpha_1}\cdots k_{\alpha_n}\,, \tag{C2}$$

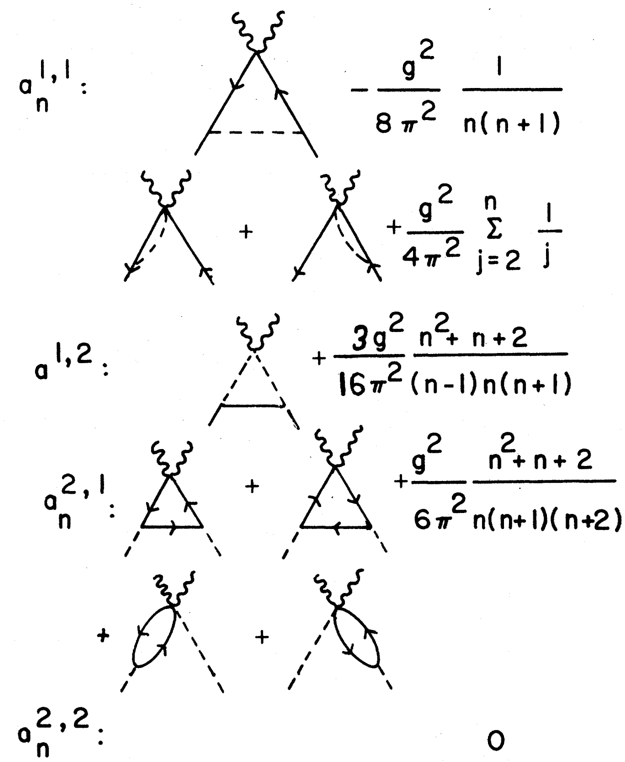

FIG. 4. The quantities $a_n^{i,j}$ computed to order g^2 in the vector theory accompanied by the corresponding graphs as described in Appendix C. Here the dashed lines represent vector-particle propagators.

where k is the momentum carried by one of the pseudoscalar particles.

The results in the vector theory, shown with their corresponding graphs in Fig. 4, are somewhat more complicated. The presence of the vector fields V_{α_i} in the operator $O^{(1)}_{\alpha_1\cdots\alpha_n}$ defined in Eq. (37a) implies that this operator not only contributes the two-photon–two-fermion vertex found in Fig. 4, representing the factor (C1), but also gives a two-photon–two-fermion–vector-particle vertex contribution to $a_n^{1,1}$ and $a_n^{2,1}$. This vertex represents the factor

$$\frac{ig}{2n(n-1)}\sum_{l=1}^{n}\sum_{\substack{j=1\\ j\neq 1}}^{n}(p+k)_{\alpha_1}\cdots(p+k)_{\alpha_{l-1}}\delta_{\rho\alpha_l}p_{\alpha_{l+1}}\cdots\gamma_{\alpha_j}\cdots p_{\alpha_n}\,, \tag{C3}$$

where p and k are the momenta carried in by the spinor and vector particles, respectively, while ρ is the vector particle's polarization index. Finally the two-photon–two-vector-particle vertex in Fig. 4 represents the factor

$$\frac{3}{2n(n-1)}\sum_{l=1}^{n}\sum_{\substack{j=1\\ j\neq 1}}(k_{\alpha_1}\cdots\delta_{\alpha_l\mu}\cdots\delta_{\alpha_j\nu}\cdots k_{\alpha_n})(k_\mu k_\nu\delta_{\sigma\rho} + k^2\delta_{\rho\mu}\delta_{\sigma\nu} - k_\rho k_\nu\delta_{\sigma\mu} - k_\sigma k_\nu \delta_{\rho\mu})\,, \tag{C4}$$

where k is the momentum carried by the vector line and ρ, σ are the vector particles' polarization indices.

APPENDIX D

Finally, we investigate the large-q^2 behavior of the solutions (78) to Eq. (75). First the large-q^2 limit of $\tilde{E}_n^{(i)}(q^2,g)$ is related to the large-ρ limit of $A_n^{(i)}(G(\rho))$, $B_n^{(i)}(G(\rho))$, and $\tilde{E}_n^{(i)}(q_0{}^2, G(\rho))$. Then we consider the possibility, first identified by Gell-Mann and Low, that the function $\beta(g)$ has a zero at $g = g_\infty$. In that case, if $\tilde{E}_n(q^2,g)$ is well defined and nonzero at $g=g_\infty$, then the asymptotic behavior of $\tilde{E}_n^{(i)}(q^2,g)$ for large q^2 is determined by the functions $A_n^{(i)}(g)$, $B_n^{(i)}(g)$, and $\rho(g)$. If in addition g_∞ is a simple zero of β and $A_n^{(i)}(g)$, $B_n^{(i)}(g)$ are regular at g_∞, then a power behavior in q^2 for $\tilde{E}_n^{(i)}(q^2, g)$ is deduced for large q^2.

First, q^2 is replaced by $q_0{}^2$ in Eq. (78) so that $z=\rho(g)$, or $g=G(z)$, and the resulting equation solved for the functions $v_n^{(i)}(z)$:

$$v_n^{(1)}(z) = \frac{1}{w(z)}\left\{\left[\frac{d}{dz}L_n^{(2)}(z) + A_n^{(1)}(G(z))L_n^{(2)}(z)\right]E_n^{(1)}(q_0{}^2, G(z)) - B_n^{(2)}(G(z))L_n^{(2)}(z)E_n^{(2)}(q_0{}^2, G(z))\right\},$$
$$v_n^{(2)}(z) = -\frac{1}{w(z)}\left\{\left[\frac{d}{dz}L_n^{(1)}(z) + A_n^{(1)}(G(z))L_n^{(1)}(z)\right]E_n^{(1)}(q_0{}^2, G(z)) - B_n^{(2)}(G(z))L_n^{(1)}(z)E_n^{(2)}(q_0{}^2, G(z))\right\}, \quad \text{(D1)}$$

where

$$w(z) = \left[\frac{d}{dz}L_n^{(2)}(z)\right]L_n^{(1)}(z) - \left[\frac{d}{dz}L_n^{(1)}(z)\right]L_n^{(2)}(z). \quad \text{(D2)}$$

The equations can now be used to eliminate the functions $v_n^{(i)}(z)$ from Eq. (78), yielding an expression for $\tilde{E}_n^{(i)}(q^2, g)$ in terms of $E_n^{(i)}(q_0{}^2, G(\ln(q^2/q_0{}^2) + \rho(g)))$ so that the large-q^2 and the large-ρ behavior of $\tilde{E}_n^{(i)}(q^2, G(\rho))$ are related:

$$\tilde{E}_n^{(i)}(q^2, g) = \sum_{j=1,2} w_n^{i,j}(z, g)\tilde{E}_n^{(j)}(q_0{}^2, G(z)), \quad \text{(D3)}$$

for $z = \ln(q^2/q_0{}^2) + \rho(g)$ and

$$w_n^{1,1}(z,g) = \frac{1}{w(z)}\left\{\left[\frac{d}{dz}L_n^{(2)}(z) + A_n^{(1)}(G(z))L_n^{(2)}(z)\right]L_n^{(1)}(\rho(g)) - \left[\frac{d}{dz}L_n^{(1)}(z) + A_n^{(1)}(G(z))L_n^{(1)}(z)\right]L_n^{(2)}(\rho(g))\right\},$$
$$w_n^{1,2}(z,g) = \frac{1}{w(z)}\left[L_n^{(1)}(z)L_n^{(2)}(\rho(g)) - L_n^{(2)}(z)L_n^{(1)}(\rho(g))\right]B_n^{(2)}(G(z)),$$
$$w_n^{2,1}(z,g) = \frac{1}{w(z)}\left\{\left[\frac{d}{dz}L_n^{(2)}(z) + A_n^{(1)}(G(z))L_n^{(2)}(z)\right]\left[\frac{d}{d\rho}L_n^{(1)}(\rho(g)) + A_n^{(1)}(g)L_n^{(1)}(\rho(g))\right]\right.$$
$$\left. - \left[\frac{d}{dz}L_n^{(1)}(z) + A_n^{(1)}(G(z))L_n^{(1)}(z)\right]\left[\frac{d}{d\rho}L_n^{(2)}(\rho(g)) + A_n^{(1)}(g)L_n^{(2)}(\rho(g))\right]\right\}[B_n^{(2)}(g)]^{-1}, \quad \text{(D4)}$$
$$w_n^{2,2}(z,g) = \frac{B_n^{(2)}(G(z))}{B_n^{(2)}(g)}\,\frac{w(\rho(g))}{w(z)}\,w_n^{1,1}(\rho(g), G(z)).$$

If we assume that $\beta(g)$ has a root g_∞ and that $\tilde{E}_n^{(i)}(q_0{}^2, g)$ is regular and nonzero at $g=g_\infty$, then Eq. (D3) determines the asymptotic form of $\tilde{E}_n^{(i)}(q^2, g)$, for $g< g_\infty$, in terms of the functions $A_n^{(i)}(g)$, $B_n^{(i)}(g)$, and $\rho(g)$ appearing in our Eq. (75). In particular, if we assume that $A_n^{(i)}(g)$, $B_n^{(i)}(g)$ are regular at g_∞ and that g_∞ is a simple zero of $\beta(g)$, then a power behavior for $\tilde{E}_n^{(i)}(q^2)$ is implied by Eq. (D3).[31] In order to show this, we must determine the large-z behavior of $L_n^{(i)}(z)$ and hence of $w_n^{i,j}(z,g)$. It is not difficult to see from Eq. (79) that, under these conditions on $A_n^i(g)$, $B_n^i(g)$, and $\rho(g)$, the functions $L_n^{(i)}(z)$ can be so chosen that

$$L_n^{(i)}(\ln y) \sim y^{\nu_i}[1 + O(1/y)], \quad \text{(D5)}$$

for y large and

$$\nu_n^{(i)} = -\tfrac{1}{2}[A_n^{(1)}(g_\infty) + A_n^{(2)}(g_\infty)] + (2i-3)\{\tfrac{1}{4}[A_n^{(1)}(g_\infty) - A_n^{(2)}(g_\infty)]^2 + B_n^{(1)}(g_\infty)B_n^{(2)}(g_\infty)\}^{1/2}. \quad \text{(D6)}$$

This asymptotic form for $L_n^{(i)}(z)$ can then be substituted into Eq. (D3), yielding

$$\tilde{E}_n^{(1)}(q^2, g) = \sum_{i=1,2}\sigma_n^{(i)}B_n^{(2)}(g)L_n^{(i)}(\rho(g))(q^2)^{-\nu_n^{(i)}},$$
$$\tilde{E}_n^{(2)}(q^2, g) = \sum_{i=1,2}\sigma_n^{(i)}\left[\frac{d}{d\rho}L_n^{(i)}(\rho(g)) + A_n^{(1)}(g)L_n^{(i)}(\rho(g))\right](q^2)^{-\nu_n^{(i)}}, \quad \text{(D7)}$$

for

$$\sigma_n^{(1)} = \frac{(q_0{}^2)^{\nu_n^{(1)}}}{e^{\rho(g)\nu_n^{(1)}}}\;\frac{B_n^{(2)}(g_\infty)E_n^{(2)}(q_0{}^2, g_\infty) - [\nu_n^{(2)} + A_n^{(1)}(g_\infty)]E_n^{(1)}(q_0{}^2, g_\infty)}{(\nu_n^{(1)} - \nu_n^{(2)})B_n^{(2)}(g)}$$

and

$$\sigma_n^{(2)} = \frac{(q_0^2)^{\nu_n^{(2)}}}{e^{\rho(g)\nu_n^{(2)}}} \frac{B_n^{(2)}(g_\infty)E_n^{(2)}(q_0^2, g_\infty) - [\nu_n^{(1)} + A_n^{(1)}(g_\infty)]E_n^{(1)}(q_0^2, g_\infty)}{(\nu_n^{(2)} - \nu_n^{(1)})B_n^{(2)}(g)} . \quad \text{(D8)}$$

This would imply that the operator

$$B_n^{(2)}(g)L_n^{(j)}(\rho(g))O^{(1)}_{\alpha_1\cdots\alpha_n} + \left[\frac{d}{d\rho} L_n^{(j)}(\rho(g)) + A_n^{(j)}(g)L_n^{(j)}(\rho(g))\right] O^{(2)}_{\alpha_1\cdots\alpha_n} \quad \text{(D9)}$$

has anomalous dimension

$$d_n^{(j)} = n + 2 + 2\nu_n^{(j)}, \quad \text{(D10)}$$

for $j = 1, 2$.

Note added in proof. The solution (D5) to Eq. (79) implicitly assumes that the two roots $\nu^{(1)}$ and $\nu^{(2)}$ defined in Eq. (D6) are different. If the quantities $\nu^{(1)}$ and $\nu^{(2)}$ happen to be equal then the solution need not have the form (D5) but may contain an additional logarithmic singularity. This is the case considered by Dell'Antonio[32]; we thank B. Schroer for bringing this case to our attention.

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[1]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[2]The structure functions W_1 and W_2 are defined following the convention of Ref. 1. The variable $\nu = -p\cdot q/m$, where m is the nucleon mass.

[3]E. D. Bloom *et al.*, Phys. Rev. Letters 23, 930 (1969); M. Breidenbach *et al.*, *ibid.* 23, 935 (1969).

[4]Throughout this paper we specify a four-vector p_μ by three spatial components p_1, p_2, p_3 and an imaginary time component $p_4 = ip_0$; $p^2 = p_1^2 + p_2^2 + p_3^2 - p_0^2$. We use $\not{p} = -i\gamma_\mu p_\mu$, and for Dirac spinors $\overline{U} = U^\dagger\gamma_4$. Single-particle momentum eigenstates $|\vec{p}\rangle$ are normalized as $\langle\vec{p}|\vec{p}'\rangle = 2E(2\pi)^3F(\vec{p} - \vec{p}')$.

[5]L. S. Brown, in *Proceedings of the Boulder Conference on High Energy Physics,* edited by K. T. Mahanthappa, W. D. Walker, and W. E. Brittin (Colorado Associated Univ. Press, Boulder, Colo., 1970); H. Leutwyler and J. Stern, Nucl. Phys. B20, 77 (1970); R. Brandt, Phys. Rev. D 1, 2808 (1970).

[6]K. Wilson, Cornell Report No. LNS-64-15, 1964 (unpublished); Phys. Rev. 179, 1499 (1969).

[7]The validity of this expansion in perturbation theory has been established by R. Brandt, Ann. Phys. (N.Y.) 44, 221 (1967); W. Zimmermann in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.* (MIT Press, Cambridge, Mass., 1971), Vol. I, p.397.

[8]We have omitted from Eq. (2) a third type of term containing operators of the form $O^{\mu\nu}_{\mu_1\cdots\mu_n}$, antisymmetric under interchange of μ or ν with μ_j. Such terms do not contribute to the spin-averaged matrix elements under consideration.

[9]Since we will be interested in the leading light-cone singularities, those operators with coefficients whose behavior at $x^2 = 0$ is less singular than shown in Eq. (3) have been lumped into $R_N^{(i)}$ and $R_N'^{(i)}$.

[10]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); Y. Frishman, Phys. Rev. Letters 25, 966 (1970). The first of these references also gives a method, which we shall use, of incorporating asymptotic gauge invariance into the light-cone expansion.

[11]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[12]C. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[13]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[14]N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).

[15]V. N. Gribov and L. N. Lipatov, Phys. Letters 37B, 78 (1971), and to be published.

[16]All terms in these sums with odd n must vanish because of the crossing relations $T_2(q^2, \omega) = T_2(q^2, -\omega)$, $T_L(q^2, \omega) = T_L(q^2, -\omega)$.

[17]Following Symanzik (Ref. 11) we shall use the superscript AF on a function, $f(q^2, \omega)$, to indicate that only those terms containing the highest power of q^2 are retained in each order of perturbation theory. That is, if

$$f(q^2, \omega) = (q^2)^n \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f_{k,l}(\omega)(q^2)^{-k}(\ln q^2)^l ,$$

where $f_{k,l}(\omega)$ is a power series in g, then $f^{\text{AF}}(q^2, \omega)$ contains only the $k=0$ terms

$$f^{\text{AF}}(q^2, \omega) = (q^2)^n \sum_{l=0}^{\infty} f_{0,l}(\omega)(\ln q^2)^l .$$

[18]C. Callan and D. Gross, Phys. Rev. Letters 22, 156 (1969); J. Cornwall and R. Norton, Phys. Rev. 177, 2584 (1969).

[19]When appearing on a matrix element containing the mass insertion operator u, the subscript A indicates that all propagators corresponding to external lines have been removed and that all amplitudes in which u acts on an external line have been dropped.

[20]Zimmermann, see Ref. 7.

[21]It should be noted that the Wilson expansion (2) is valid for matrix elements containing the mass operator u. In a renormalizable theory all subgraphs appearing in the matrix elements of a traceless, symmetric operator $O^{(i)}_{\alpha_1\cdots\alpha_n}$ of lowest canonical dimension will have degree of divergence less than or equal to zero. There-

fore, the insertion of the operator u will require no additional subtractions.

[22]These are the only two operators for a given n which are charge conjugation even and may have nonvanishing, spin-averaged matrix elements between two identical pseudoscalar or fermion states.

[23]For simplicity we use the same notation to represent analogous quantities in the pseudoscalar and vector theories. It should always be clear from the context to which theory a given symbol refers.

[24]S.-J. Chang and P. Fishbane, Phys. Rev. D 2, 1084 (1970). See also M. Kugler and S. Nussinov, Nucl. Phys. B28, 97 (1971); R. Gatto and P. Menotti, Nuovo Cimento 2A, 881 (1971).

[25]P. Fishbane and J. Sullivan, Phys. Rev. D 4, 2516 (1971).

[26]A. Mason (unpublished).

[27]The requirement (56a) is valid only if we set equal to zero the direct, renormalized photon coupling constant f, introduced in Eqs. (A19) and (A20).

[28]If the right-hand side of Eq. (65a) is expanded in powers of g^2, we find a term behaving as $1/g^2$. This term is independent of x^2 and is therefore annihilated by the derivatives appearing in Eq. (2).

[29]For a derivation of the Callan-Symanzik equations in quantum electrodynamics that follows similar lines, see A. Sirlin, Phys. Rev. D 5, 2132 (1972).

[30]Throughout our discussion of the vector theory we work in the Feynman gauge using $\delta_{\mu\nu}\Delta(k^2, m, \mu)$ for the photon propagator.

[31]A somewhat different asymptotic behavior is implied if g_∞ is a multiple root or an essential singularity of β. For a discussion of these various possibilities see S. Adler, IAS report (unpublished).

[32]G. F. Dell'Antonio (unpublished).

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Eikonal Cancellations in a Solvable Model*

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Two versions of a high-energy field-theoretic eikonal amplitude are examined in a special limit of large internal mass, where infinite sums over all n-point, connected, eikonal graphs become calculable. Both examples exhibit cancellations which tend to reduce the energy dependence of σ_T below the Froissart bound.

I. INTRODUCTION

Recent very-high-energy experiments displaying a constant pp total cross section[1] have acted as a spur to the estimation of σ_T and related multiplicity distributions. In particular, one would like to understand how the eikonal "tower graph" calculations of Cheng and Wu,[2] and the strong-coupling eikonal Regge calculation of Chang and Yan,[3] which generate $\sigma_T \sim \ln^2 s$, might be improved; and it has been suggested[4,5] that neglected crossed-channel multiparticle (connected) amplitudes can provide sufficient cancellations to remove the unobserved energy dependence. The purpose of this note is to describe a special version of a field-theoretic model previously discussed in an approximate way[5]; and to exhibit in an exact way two distinct forms of such cancellation in the special limit of large vector-meson mass (while the mass of emitted scalar "pions" remains finite). The first computation displays deviations from the form of a previous result of Aviv, Sugar, and Blankenbecler,[4] which arise from the inclusion of the next, more complicated set of fundamental graphs employed in the construction of the eikonal. The result of the second calculation, exact in its model context, sums over all contributing, nontrivial graphs, and produces an eikonal function independent of incident particle energy. While this does agree with the experimental $\sigma_T \sim$ const, the main value of these computations lies in the construction of explicit examples which exhibit eikonal cancellations.

The starting point of the analysis is the specification of an interaction Lagrangian, coupling nucleon, neutral vector meson (NVM), and scalar pion fields,

$$\mathcal{L}' = ig\bar{\psi}\sum_\mu \gamma_\mu W_\mu \psi + \tfrac{1}{2}\lambda\Pi \sum_\mu W_\mu{}^2 \ . \tag{1}$$

A formal construction of the eikonal amplitude in