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Long-Range-Potential Model of Hadrons*

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A hadron model is proposed in which nonasymptotic quarks interact with a coherent and long-range vector-gluon field, with a resulting linearly rising spectrum. Solutions for $L=0$ pseudoscalar and vector mesons in this model exhibit an SU_6-type structure. The pion decay constant and the ρ-γ coupling constant for the unperturbed ground states satisfy the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation and have the right order of magnitude. The ratio $\sigma(e^+ + e^- \to \text{vector mesons})/\sigma(e^+ + e^- \to \mu^+ + \mu^-)$ decreases like $s^{-1/2}$ in the present model, whereas it is shown that the ratio approaches a constant in the three-dimensional oscillator-potential model.

I. INTRODUCTION

In this section, the basic ideas introduced in the previous paper[1] will be briefly described. We consider a dynamical model in which infinitely (or almost infinitely) heavy quarks interact via the neutral-vector-gluon field $\varphi_\mu(x)$. The equation for the quark field $q(x)$ is

$$(i\not\partial - M_0)q(x) = g\not\varphi(x)q(x), \tag{1}$$

where M_0 is the unrenormalized mass. Since the quark is a nonasymptotic field, the physical mass has no meaning. We define a 4×4 wave function $\chi(\xi)$ for a q-$\bar{q}$ system of total momentum P_μ by

$$\chi_{\alpha\beta}(\xi)e^{-iP\cdot X} = (\Phi_0, T(\bar{q}_\beta(x')q_\alpha(x))\Phi_P) \tag{2}$$

with

$$X = \tfrac{1}{2}(x+x'), \quad \xi = x'-x.$$

Under the assumption that the gluon field is so strong in the hadron that it can be replaced by a c-number field $f_\mu(x)$, we obtain from (1) and (2) our basic equation

$$[\tfrac{1}{2}\not P + i\not\partial - g\not f(\xi) - M_0]\chi(\xi) = 0. \tag{3}$$

Here $f_\mu(\xi)$ is the total field including the self-field of the quarks. Although we may write down an equation to determine f_μ, providing a self-consistent scheme together with (3), such a scheme yields no simple solution. In the present paper we will not take this difficult route, but instead postulate a certain form for f_μ and investigate its consequences. The form we assume is

$$f_\mu = P_\mu V(r), \tag{4}$$

where

$$r^2 = -\xi^2 + (P\cdot\xi)^2/P^2, \tag{5}$$

which reduces to the radial distance in the rest frame. In other words, we are assuming that the proper configuration of a hadron is three-dimensional. However, this assumption is not an absolute necessity, because we may assume instead that V is a function of $R^2 = -\xi^2$, and can obtain essentially the same results. For $V(r)$ we take

$$gV(r) = \tfrac{1}{2} - v(r),$$
$$v(r) = \epsilon + a/2\epsilon r. \tag{6}$$

The constant term $\frac{1}{2}-\epsilon$ in gV represents a self-mass effect, and we will see that $\epsilon=0$ corresponds to the infinite physical mass for the quark. The $1/r$ part may be justified in two aspects. As the singularity at the origin it is certainly field-theoretical. For large r we find that the quarks do not propagate outside a certain definite range, and within this bound a Yukawa potential with a larger range may be approximated by $1/r$. In any event (6) is the basic ingredient of our model, and we may call it a long-range-potential model. It is an extreme opposite of the oscillator-potential model both in the singularity at the origin and in the behavior at larger r. In the following, we will neglect consistently the $1/r^2$ term in v^2 and v'/M_0 compared to v. The approximations may be justified for large r but only for high orbital states near the origin. This asymptotic nature of our approximation and the resulting spectrum should be kept in mind. With this approximation, (6) is equivalent to

$$v = (\epsilon^2 + a/r)^{1/2}. \tag{7}$$

As we see in Eq. (46) below, v^2 represents a kernel of a Bethe-Salpeter-type equation, so that (7) may not be totally unreasonable.

With (4) and (6), Eq. (3) can be written as

$$(\not P v + i\not\partial - M_0)\chi(\xi) = 0, \tag{8}$$

which can be rationalized immediately if we neglect v' to give

$$[P^2(\epsilon^2 + a/r) - \partial^2 - M_0^2]\chi(r) = 0\,. \tag{9}$$

It gives a spectrum

$$\begin{aligned} P^2 &= m_n^2 \\ &= \frac{2M_0^2 n}{\epsilon^2 n + (\epsilon^4 n^2 + M_0^2 a^2)^{1/2}} \quad (n = 1, 2, \ldots), \end{aligned} \tag{10}$$

which converges to

$$m_\infty^2 = M_0^2/\epsilon^2\,.$$

This ionization point defines the physical quark mass as

$$\begin{aligned} M &= m_\infty/2 \\ &= M_0/2\epsilon\,. \end{aligned}$$

For $\epsilon \to 0$, we have a linear trajectory,

$$m_n^2 = (2M_0/a)n \quad (n = 1, 2, \ldots), \tag{11}$$

with a level spacing in squared mass,

$$\Delta = 2M_0/a\,. \tag{12}$$

The normalized wave function for $\epsilon = 0$ is

$$\begin{aligned} \chi_{n,l}(r) = 2(-1)^{n-l-1}\left(\frac{M_0^3(n-l-1)!}{n(n+l)!}\right)^{1/2} \\ \times (2M_0 r)^l e^{-M_0 r} L_{n-l-1}^{(2l+1)}(2M_0 r) Y_{lm}(\theta, \varphi)\,. \end{aligned} \tag{13}$$

Contrary to the case of a hydrogen atom, the Bohr radius here is $1/M_0$, independent of the principal quantum number n. The effect of this on the transition form factors was discussed in Ref. 1. The wave function for the case $\epsilon \neq 0$ can be obtained from (12) by replacing M_0 by

$$M_0' = (M_0^2 - \epsilon^2 m_n^2)^{1/2}\,. \tag{14}$$

II. SOLUTIONS FOR 0^- AND 1^- STATES

Equation (9) suggests a spin-independent spectrum. We will obtain the spin structure of χ for π and ρ (and their sisters) in this section. Both are $L = 0$ states in the nonrelativistic limit, and we may try to solve (3) with χ given by

$$\begin{aligned} \chi_P^a(\xi) = [\psi_1(r) + \not{P}\psi_2(r) + i\not{\zeta} r^{-1}\psi_3(r) \\ + i\not{P}\not{\zeta} r^{-1}\psi_4(r)]\begin{pmatrix} i\gamma_5 \\ \not{\epsilon} \end{pmatrix}\lambda_a/\sqrt{2}\,, \end{aligned} \tag{15}$$

with

$$\zeta_\mu = \tfrac{1}{2} dr^2/d\xi^\mu\,.$$

The upper line is for 0^- and the lower one is for 1^- with polarization ϵ_μ. The unitary spin is incorporated by the last factor. The SU_6 structure of the first two amplitudes is apparent. Introducing (15) into (3), we find a set of equations for ψ's, which are common for both 0^- and 1^- cases:

$$\begin{aligned} -M_0\psi_1 + P^2 v\psi_2 + \psi_3' + 2r^{-1}\psi_3 &= 0\,, \\ v\psi_1 - M_0\psi_2 - \psi_4' - 2r^{-1}\psi_4 &= 0\,, \\ \psi_1' - M_0\psi_3 + P^2 v\psi_4 &= 0\,, \\ -\psi_2' + v\psi_3 - M_0\psi_4 &= 0\,. \end{aligned} \tag{16}$$

From (16) we obtain

$$\begin{aligned} \psi_1'' + 2r^{-1}\psi_1' - (M_0^2 - P^2 v^2)\psi_1 &= -P^2 v'\psi_4\,, \\ \psi_2'' + 2r^{-1}\psi_2' - (M_0^2 - P^2 v^2)\psi_2 &= v'\psi_3 \end{aligned} \tag{17}$$

and

$$\begin{aligned} \psi_3 &= \frac{M_0\psi_1' - P^2 v\psi_2'}{M_0^2 - P^2 v^2}\,, \\ \psi_4 &= \frac{v\psi_1' - M_0\psi_2'}{M_0^2 - P^2 v^2}\,. \end{aligned} \tag{18}$$

If we set $v' = 0$, (17) is equal to (9) for S wave, so that ψ_1 and ψ_2 are given by $\chi_{n,0}$ of (13). By requiring that ψ_3 and ψ_4 have no singularity at $M_0 = v(P^2)^{1/2}$, we arrive at the relative ratio $\psi_1/\psi_2 = (P^2)^{1/2}$. Thus, we obtain an approximate solution

$$\begin{aligned} \chi_P^a(\xi) = A_n\Big[\left(1 + \frac{\not{P}}{m_n}\right)\chi_{n,0}(r) \\ + i\left(1 - \frac{\not{P}}{m_n}\right)\not{\zeta} r^{-1} C_n(r)\chi_{n,0}'(r)\Big]\begin{pmatrix} i\gamma_5 \\ \not{\epsilon} \end{pmatrix}\frac{\lambda_a}{\sqrt{2}}\,, \end{aligned} \tag{19}$$

where

$$C_n(r) = [M_0 + (m_n^2 a r^{-1})^{1/2}]^{-1} \tag{20}$$

and

$$m_n^2 = (2M_0/a)n \quad (n = 1, 2, \ldots)\,. \tag{21}$$

We have taken $\epsilon = 0$ and (7) for v for definiteness. We see that ψ_1 and ψ_2 are large components and they are the boosted SU_6 wave functions.[2] Unfortunately, the solution is not completely satisfactory because some of the small components do not satisfy the proper charge-conjugation condition. From the defining equation for χ, Eq. (2), we find

$$C^{-1}\chi(\xi)C = \pm\chi^T(-\xi)\,, \tag{22}$$

where C is the ordinary charge-conjugation matrix and $\pm$ refers to the C parity of the states. Thus, ψ_3 for 0^- has an opposite charge-conjugation parity and ψ_3 and ψ_4 for 1^- have no definite C parity. However, for large n, we find

$$\int (\psi_{3,4})^2 d^3r \Big/ \int \psi_1^2 d^3r = O(1/n)$$

so that the charge-conjugation property is restored for the asymptotic states.

For application of the wave function (19) to physical problems, we need to determine its normalization. We normalize by

$$(\Phi_P, j_\mu^+(0)\Phi_P) = -\left(\Phi_P, j_\mu^+(0)\int j_\lambda^-(x)d\sigma^\lambda(x)\Phi_P\right)$$
$$= 2P_\mu\,. \tag{23}$$

Here $j_\mu^\pm(x) = \bar{q}^{(\pm)}(x)\gamma_\mu q^{(\pm)}(x)$ are currents for the quark and the antiquark, respectively. $\int d\sigma$ is a spacelike surface integral. Using (2), we find that (23) is equal to

$$\int \mathrm{tr}[\bar{\chi}_P(\xi)\gamma_\mu\chi_P(\xi)\gamma_\lambda]d\sigma^\lambda(\xi) = 2P_\mu\,, \tag{24}$$

where

$$\bar{\chi} \equiv \gamma_0\chi^\dagger\gamma_0\,.$$

Introducing (19) into (24), we find

$$1 = 4A_n^2 d m_n^{-2}\int [\chi_{n,0}^2(r) + C_n^2(r)\chi_{n,0}'^2(r)]P\cdot d\sigma, \tag{25}$$

where d is unity for the Gell-Mann–Zweig model,[3] and three for three-triplet models, either the Han-Nambu[4] or the parastatistics quark[5] of rank 3. In the rest frame, the surface integral reduces to a space integral. Since $\chi_{n,0}$ is normalized, we have

$$A_n = \tfrac{1}{2}(m_n/d)^{1/2}(1+C_n)^{-1/2}\,, \tag{26}$$

where

$$C_n = \int d^3r \frac{\chi_{n,0}'(r)^2}{[M_0 + (m_n^2 a/r)^{1/2}]^2}\,. \tag{27}$$

As stated before, C_n decreases like $1/n$. For $n=1$, we have from (13) and (21),

$$C_1 = \tfrac{1}{2}\int_0^\infty d\rho \frac{\rho^2 e^{-\rho}}{(1+2\sqrt{\rho})^2} = 0.2\,. \tag{28}$$

Although our solutions are expected to be valid only for large n because of the approximations to neglect higher-order terms in $1/r$ and also because of the charge-conjugation problem as stated above, it is nevertheless extremely interesting to apply the ground-state wave functions to evaluate the pion decay constant f_π and ρ-γ coupling constant f_ρ. We should stress that we are here calculating these parameters for the unperturbed ground states, with perturbative interaction switched off, which breaks our SU_6-type degeneracy to bring down energy levels to the observed values. This means that we should use m_1, given by (21) as the π and ρ mass, instead of the actual masses. We assume, however, that the level spacing Δ is not perturbed appreciably, so that we may use the physical value $2m_\rho^2$ for it. Thus from (21), we have

$$\Delta = 2M_0/a = 2m_\rho^2\,, \tag{29}$$

which also determines

$$m_1^2 = 2m_\rho^2\,. \tag{30}$$

The fact that the ground-state energy m_1 is uniquely determined from the level spacing, with no space for an additive constant, is a unique feature of our model. We will see shortly that the value (30), together with the SU_6-type symmetry of our wave functions, leads to the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation,[6] which has never been derived in a proper sense.[7] We are of course assuming here that the values of f_π and f_ρ or at least the product $f_\pi f_\rho$ will not change appreciably by perturbation. The pion decay constant f_π is defined by

$$if_\pi P_\mu = (\Phi_0, j_{5\mu}^3(0)\Phi_P)$$
$$= \mathrm{tr}[\gamma_\mu\gamma_5(\tfrac{1}{2}\tau_3)\chi_P^{\pi^0}(0)]\,.$$

From (19), (13), and (26) we have

$$f_\pi = (4dA/\sqrt{2}\,m_1)\chi_{10}(0)$$
$$= \left(\frac{2dM_0^3}{\pi(1+C_1)m_1}\right)^{1/2}\,. \tag{31}$$

For the value of m_1 as given by (30), and $M_0 = 300$ MeV, for instance, we obtain

$$f_\pi = 110\sqrt{d}\ \mathrm{MeV} \tag{32}$$

to be compared with the experimental value of 94 MeV. In view of the crude nature of our evaluation, both values $d=3, 1$ are permissible. Similarly, $\rho-\gamma$ coupling constant is defined by

$$\epsilon_\mu m_\rho^2 f_\rho^{-1} = (\Phi_0, j_\mu^3(0)\Phi_P)$$
$$= \mathrm{tr}[\gamma_\mu(\tfrac{1}{2}\tau_3)\chi_P^{\rho^0}(0)]\,.$$

Here on the left-hand side the actual ρ mass must be used as it is the definition of f_ρ. Again from (19), (13), and (26) we have

$$f_\rho^{-1} = (4dA/\sqrt{2}\,m_\rho^2)\chi_{10}(0)$$
$$= \frac{1}{m_\rho^2}\left(\frac{2dm_1M_0^3}{\pi(1+C_1)}\right)^{1/2}\,. \tag{33}$$

With $m_1 = \sqrt{2}\,m_\rho$ and $M_0 = 300$ MeV we obtain

$$f_\rho = 5/\sqrt{d}\,.$$

Experimentally $f_\rho \sim 5.6$ and the value has the right order of magnitude. Now taking the ratio of (31) and (33) we have, from (30),

$$f_\pi f_\rho = m_\rho^2/m_1 = m_\rho/\sqrt{2}\,, \tag{34}$$

which is the KSRF relation. This ratio is independent of the value of $\chi_{10}(0)$, but the value of m_1 as given by (30) is crucial in deriving the KSRF relation. This is one positive result of our model.

III. VECTOR-MESON PRODUCTION BY VIRTUAL PHOTONS

As an application of the vector wave function derived in Sec. II, we shall consider vector-meson production by $e^+ + e^-$ collisions. In our resonance model, hadron production by a virtual photon γ^* would proceed via the conversion of γ^* into a vector meson of equal mass, and its subsequent decay into other hadrons. Therefore, the rate of $e^+ + e^- \to$ all hadrons is equal to the rate of $e^+ + e^- \to$ vector mesons. One might expect that the ratio

$$\lim_{s\to\infty} \frac{\sigma(e^+ + e^- \to \text{vector mesons})}{\sigma(e^+ + e^- \to \mu^+ + \mu^-)} = R \tag{35}$$

is still given by conventional prediction[8] $R = \sum Q_i^2$ (Q_i = charges of quarks making up the vector meson), since the effect of binding a quark and an antiquark into a vector meson would decrease as the squared energy s increases. It turns out that this is not the case in the present model. This is in sharp contrast to the oscillator potential model, where the statement is essentially true. In order to evaluate the ratio R, we calculate the decay width of a virtual photon γ^* of mass $\sqrt{s}$. For $\mu^+ + \mu^-$ decay we have

$$\Gamma(\gamma^* \to \mu^+ + \mu^-) = \sqrt{s}/12\pi \tag{36}$$

for spatially polarized γ^*.

For $\gamma^* \to$ vector mesons, we have

$$\Gamma(\gamma^* \to \text{vector mesons}) = 2\pi \sum_n \frac{\delta(\sqrt{s} - m_n)}{2m_n} \frac{1}{2\sqrt{s}} \times |\langle 0|\epsilon\cdot j^{\text{em}}(0)|n\rangle|^2 . \tag{37}$$

Here we do not have to consider $L=2$ vector mesons, whose wave function vanishes at $r=0$. The summation $\sum_n$ involves the third and eighth members (ρ' and ω_8') of the vector octet of mass $\sqrt{s}$. From (19), (13), and (26) we have

$$\langle 0|j_\mu^{\text{em}}(0)|\rho'\rangle = \text{tr}\,[\gamma_\mu(\lambda_3/2)\chi_P^{\rho'}(0)] = 4(1/\sqrt{2})dA_n\chi_{n,0}(0)\epsilon_\mu' , \tag{38a}$$

$$\langle 0|j_\mu^{\text{em}}(0)|\omega_8'\rangle = \text{tr}[\gamma_\mu(\lambda_8/2\sqrt{3})\chi_P^{\omega_8'}(0)] = 4(1/\sqrt{6})dA_n\chi_{n,0}(0)\epsilon_\mu' . \tag{38b}$$

Also

$$\sum_n \delta(\sqrt{s} - m_n)/2m_n = \sum_n \delta(s - m_n^2) = \Delta^{-1} , \tag{39}$$

where Δ is the level spacing in m^2. Introducing Eqs. (38) and (39) into (37), we obtain

$$\Gamma(\gamma^* \to \text{vector mesons}) = (\tfrac{2}{3}d)4M_0^3\Delta^{-1} . \tag{40}$$

Combining with (36) we have

$$\frac{\sigma(e^+ + e^- \to \text{vector mesons})}{\sigma(e^+ + e^- \to \mu^+ + \mu^-)} = (\tfrac{2}{3}d)\frac{48\pi M_0^3}{\Delta}\frac{1}{\sqrt{s}} . \tag{41}$$

Here $\frac{2}{3}d$ is just $\sum Q_i^2$ for the Gell-Mann-Zweig model ($d=1$) and rank-3 parastatistics for quarks ($d=3$). In the Han-Nambu model, as long as we exclude charmed vector mesons, there is no distinction from the case of the parastatistics. The remaining factor was expected to give unity. Taking $\Delta = 2m_\rho^2$, $M_0 = 300$ MeV, we have

$$48\pi M_0^3\Delta^{-1}s^{-1/2} = (3.3 \text{ GeV})s^{-1/2} .$$

The $s^{-1/2}$ decrease of the ratio R may be due to the singularity of the potential at $r=0$, and provide a decisive test of our model.

It is interesting to evaluate the ratio for the case of the three-dimensional oscillator potential. We replace (13) by the nonrelativistic oscillator wave function for reduced mass $\frac{1}{2}M$.

$$\chi_{n,0}(r) = \frac{1}{(2\pi)^{1/2}}(\tfrac{1}{2}M\omega)^{3/4}\left(\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right)^{1/2} \times e^{-M\omega r^2/4}L_{n-1}^{1/2}(\tfrac{1}{2}M\omega r^2) . \tag{42}$$

The nonrelativistic energy levels are related to the mass spectrum by

$$m_n^2 = (2M+E)^2 \sim 4M\omega(2n - \tfrac{1}{2}) + 4M^2 . \tag{43}$$

The level spacing Δ is given by $\Delta = 8M\omega$.

We extrapolate (43) to large n and find

$$\chi_{n0}(0) = \frac{\sqrt{2}}{\pi}(\tfrac{1}{2}M\omega)^{1/2}(\tfrac{1}{2}M\omega n)^{1/4} \sim \frac{1}{4\sqrt{2}\,\pi}\Delta^{1/2}m_n^{1/2} .$$

Using this value in Eqs. (38), we find

$$\Gamma(\gamma^* \to \text{vector mesons}) = (\tfrac{2}{3}d)\frac{\sqrt{s}}{8\pi} , \tag{44}$$

which gives

$$R = \tfrac{3}{2}(\sum Q_i^2) . \tag{45}$$

This enhancement factor of $\frac{3}{2}$ is quite interesting. Böhm, Joos, and Krammer[9] have found a similar factor $\frac{9}{4}$ using a relativistic oscillator-potential model. However, there is some reason to believe that this factor $\frac{3}{2}$ is fictitious, arising from the not completely relativistic treatment of spin in the vector-meson production.

IV. SUPPLEMENTARY REMARKS

In order to avoid the charge-conjugation difficulty of Eq. (8), we may take Bethe-Salpeter-type equations in the following way. The charge-conjugate relation to (8) is

$$\chi(\xi)(-\not{P}v + i\not{\partial} - M_0) = 0\,, \tag{46}$$

where we used (22). Combining (8) and (46), and symmetrizing, we obtain

$$(i\not{\partial} - M_0)\chi(i\overleftarrow{\not{\partial}} - M_0) = -v^2\not{P}\chi\not{P} - \tfrac{1}{2}iv'(\not{P}\chi\not{\xi}r^{-1} - r^{-1}\not{\xi}\chi\not{P})\,, \tag{47}$$

which is manifestly charge-conjugation-invariant, so that its solution will automatically satisfy the charge-conjugation condition. Similarly we may also consider the equation

$$(\tfrac{1}{2}\not{P} + i\not{\partial} - M_0)\chi(-\tfrac{1}{2}\not{P} + i\overleftarrow{\not{\partial}} - M_0) = -\not{P}\chi\not{P}(\tfrac{1}{2} - v)^2 - \tfrac{1}{2}iv'(\not{P}\chi\not{\xi}r^{-1} - r^{-1}\not{\xi}\chi\not{P})\,. \tag{48}$$

There is no guarantee, however, that the solutions of (47) and (48) are the same, because the charge-conjugate sets of Eqs. (8) and (46) are not compatible to start with, as we saw in Sec. II. Only for a large n or l, we may expect the solutions of all these equations to converge. We have not yet solved either of the charge-conjugate Eqs. (47) and (48).

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