Lévy σ model, $\alpha = 1$.

Finally we should mention that the effects of symmetry breaking upon higher-order terms such as $\sigma^3, \sigma^2\Phi^2, \ldots$ have not yet been investigated. These higher-order terms contribute to $\pi N \to 3\pi N$ and other processes of high multiplicity.

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[1]M. G. Olsson and Leaf Turner, Phys. Rev. Letters 20, 1127 (1968).

[2]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[3]In the linear σ model $f = M_n/G$ and the unrenormalized value of $|g_A/g_V| = 1$. (M_n = nucleon mass.)

[4]S. Weinberg, Phys. Rev. 166, 1568 (1968).

[5]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, p. 101.

[6]An example where α has a singularity at $\mu_\pi^2 = 0$ is the Lagrangian whose nonderivative part is

$$\frac{\mu_\sigma^2 - 3\mu_\pi^2}{4}(\tilde{\sigma}^2 + \Phi^2) + \frac{\mu_\pi^2 - \mu_\sigma^2}{4f}\tilde{\sigma}(\tilde{\sigma}^2 + \Phi^2) + \left(\frac{3\mu_\pi^2 + \mu_\sigma^2}{4}\right) f\tilde{\sigma}.$$

In this model

$$\alpha = \tfrac{1}{2}(3 - \mu_\sigma^2/\mu_\pi^2),$$

and therefore

$$\lim_{\mu_\pi^2 \to 0} c_2(\Phi^2) = -\tfrac{1}{4}\mu_\sigma^2 \neq 0.$$

In the limit $\mu_\sigma \to \infty$ this model cannot yield the conventional nonlinear $\nu = 2$ amplitudes.

[7]Steven Weinberg, Phys. Rev. Letters 17, 616 (1966); 18, 188 (1967).

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$\pi\pi$ Scattering and the πN σ Term*

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The methods of an earlier work are modified so that the unitarity constraints on the $\pi\pi$ amplitude are better satisfied. The modification permits us to examine the extent to which $\pi\pi$ scattering affects the determination of the σ term in πN scattering.

Some renewal of interest has developed in the venerable problem of calculating $\pi\pi$ scattering from the general principles of analyticity, unitarity, and crossing symmetry. The impetus has come from current algebra. What the local-operator methods mean to an S-matrix approach is twofold. First, the Ward identities obtained from the current commutation relations provide equations relating matrix elements which are analytic in the invariant variables. As such they offer a vehicle for invoking unitarity. For a low-energy treatment this represents a distinct advantage over the use of partial-wave dispersion relations because only local analyticity needs to be employed. Secondly, the low-energy theorems[1] of current algebra are incorporated and effectively normalize the results of the analytic approach.

Schnitzer[2] has proposed methods for such a scheme, and an analysis of what can be predicted has been carried out.[3] The purpose of the present investigation is to show how a slight modification of what was done in Ref. 3 leads to considerable improvement on the extent to which unitarity is satisfied. This is achieved by making a minor alteration in the parametrization. Of course a more general treatment of the $\pi\pi$ problem admits other parametrizations so that the results we obtain are not unique beyond the context of the methods of Ref. 3. The same modification also allows us to inquire how $\pi\pi$ scattering affects the determination of the σ term in πN scattering at the Cheng-Dashen point.[4] The latter question has recently been raised by Schnitzer.[5]

In what follows we shall pursue the consequences of our alteration in the development of Ref. 3. We shall generally transmit only those equations which are necessary to indicate this modification. The notation and development is otherwise identical to Ref. 3. The new results for the low-energy $\pi\pi$ phase shifts will be given. Finally we shall discuss the question of the πN σ term.

The s- and p-wave amplitudes are given by the following hard-pion formulas:

$$T_{0T}=(H_Tf_T+\Psi_T)/F_\pi{}^2, \qquad T=0 \text{ and } 2$$

and (1)

$$T_{11}=(H_1F+\Psi_1)/F_\pi{}^2,$$

where $H_T=\Gamma_T(s-\alpha_T)$ and $H_1=\frac{1}{2}(1+\Gamma s)$. In (1), f_T and F are the form factors of the σ commutator and the vector current, respectively. They satisfy the following on-shell Ward identities:

$$f_T+l_T=H_T\Delta_T/F_\pi{}^2, \qquad T=0 \text{ and } 2$$

and (2)

$$F-F_\Gamma=H_1\Delta_V/F_\pi{}^2,$$

where $F_\Gamma=1-C_V/2F_\pi{}^2$; Δ_T and Δ_V are the propagators. The Ψ's are as given in Eqs. (16) of Ref. 3. The constraint of elastic unitarity is imposed by requiring (i) that each Ψ vanish in a neighborhood above threshold, and (ii) that

$$|1+Z_T|^2=1 \tag{3}$$

where

$$Z_T=\Psi_T(H_Tf_T)^{-1}, \qquad T=0 \text{ and } 2$$

and

$$Z_1=\Psi_1(H_1F)^{-1}.$$

Equation (3) is to be satisfied over $4m_\pi{}^2\leq s\leq 16m_\pi{}^2$ to the maximum accuracy by varying the parameters which remain after condition (i) has been imposed. In Ref. 3 we were able to make each $\Psi=0$ through $O(\nu)$, where $\nu=s-4m_\pi{}^2$, and to satisfy (3) to within a departure which reached about 20% at $s=16m_\pi{}^2$.

In Ref. 3 we used

$$\alpha_T=2m_\pi{}^2+\Gamma_T{}^{-1} \tag{4}$$

as a consequence of adopting a form for the $AA\sigma$ vertex which was smoothest in the momenta [see Eq. (A17) of Ref. 3]. We also, somewhat arbitrarily, set[6]

$$\Gamma=0$$

so that

$$F_\Gamma=0,$$

i.e.,

$$C_V=2F_\pi{}^2. \tag{5}$$

In the present investigation we shall take Γ_T and α_T to be independent parameters. With these free to be varied along with Γ, H_T and H_1 become the most general linear coupling polynomials. It was noted in Ref. 3 that condition (4) effectively constrained our $\pi\pi$ construction not to modify the σ-term contribution to πN scattering at the Cheng-Dashen point. Once (4) is relinquished, this question can be broached; we shall turn to this below. When we give up (4) and (5) we gain three additional free parameters. This permits us to impose condition (i) above as

$$\text{each } \Psi=0 \text{ through } O(\nu^2). \tag{6}$$

It is apparent that when this is done we shall satisfy (3) to an improved level of accuracy.

As in Ref. 3 we again introduce slope parameters f_T' and F' to describe the t dependence of $f_T(t)$ and $F(t)$ for small negative t [see Eqs. (29) of Ref. 3]. An immediate consequence of (6) is that (5) must be satisfied. Thus, expression (5), which was input in Ref. 3, is obtained as a result of applying unitarity in the present work. The remaining eight constraints implied by (6) are used to eliminate all but two of the remaining 10 parameters: α_0, α_2, l_2, f_0', f_2', F', Γ_0, Γ_2, η_1, and η_2. The relations among them are as follows:

$$\begin{aligned}
l_2&=\alpha_2-2m_\pi{}^2-\tfrac{2}{15}C[3\alpha_2m_\pi{}^2(1-Y)\\
&\qquad+(\alpha_0-2m_\pi{}^2)(5\alpha_0+7\alpha_2-12m_\pi{}^2)],\\
\Gamma_2&=\frac{1}{l_2}[1-\tfrac{2}{5}C(2\alpha_0-3m_\pi{}^2-m_\pi{}^2Y)],\\
\Gamma_0&=-\frac{2}{l_0}[1+C(\alpha_0-2m_\pi{}^2+2m_\pi{}^2Y)],\\
F'&=\frac{C}{2}\,\frac{\alpha_0-\alpha_2}{\alpha_2-2m_\pi{}^2},\\
\eta_1&=-\tfrac{2}{15}C(13-53Y),\\
\eta_2&=\tfrac{1}{5}C(3-23Y),\\
f_0'&=C/\Gamma_0,\\
f_2'&=2YC/\Gamma_2,
\end{aligned} \tag{7}$$

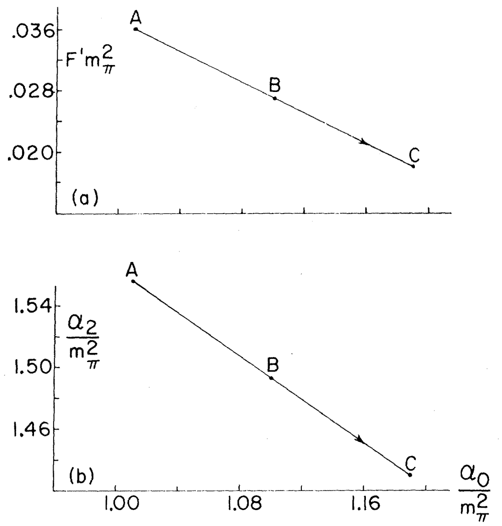

FIG. 1. Results of the parameter search to fit Eq. (3): (a) (α_0, F'); (b) (α_0, α_2). Improvement is indicated by the direction of the arrow.

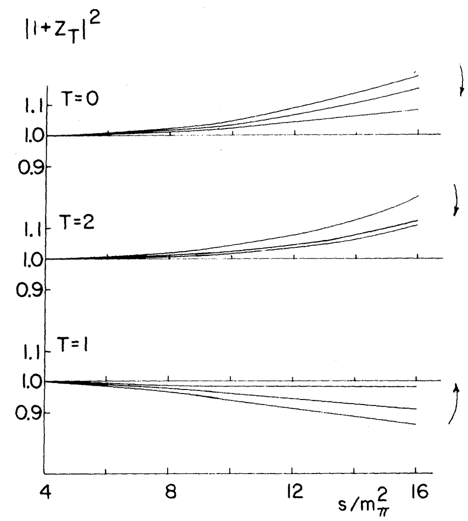

FIG. 2. $|1+Z_T|^2$ vs s for determinations A, B, and C.

where

$$Y=\frac{1}{5}\,\frac{\alpha_0-2m_\pi^{\,2}}{\alpha_2-2m_\pi^{\,2}}$$

and

$$C=\frac{5}{2}\,\frac{\alpha_2-2m_\pi^{\,2}}{\alpha_2-\alpha_0}\,\frac{4\alpha_0+5\alpha_2-12m_\pi^{\,2}}{5\alpha_2(\alpha_0-m_\pi^{\,2})-6m_\pi^{\,2}(\alpha_0-2m_\pi^{\,2})}\,.$$

Recall that $2l_0-5l_2=6m_\pi^{\,2}$.

Two of the parameters remain free to be varied in order to optimize the fit to the unitarity relation (3). It is most convenient if we choose these two to be α_0 and F'. The results of the search over this two-parameter space is shown in Fig. 1. The path indicated in Fig. 1(a) starts at the point A: $(\alpha_0/m_\pi^{\,2}, F'm_\pi^{\,2})=(1.01, 0.036)$, where the fit to (3) begins to improve upon the results of Ref. 3. The path terminates at the point C: (1.19, 0.018), where the fit ceases to improve. The intermediate point, B: (1.10, 0.027), is also indicated; its relevance is that $\alpha_0=1.10\,m_\pi^{\,2}$ resulted from the search conducted in Ref. 3. The corresponding path in the (α_0, α_2) space is shown in Fig. 1(b). Figure 2 shows the extent to which Eq. (3) is satisfied as we proceed along the path to increased improvement. The 20% departures obtained in Ref. 3 are reduced to 10% or better in this analysis. The arrow appearing in this figure and in the succeeding ones is correlated with the direction of the path in Fig. 1. We shall present results for each of the three points A, B, and C cited above.

In Fig. 3 we have plotted the s- and p-wave phase shifts which correspond to the determinations A, B, and C. The low-energy results exhibit a noteworthy stability to variation of the parameters. We can conclude that the method yields a set of low-energy phase shifts whose determination is virtually unique. Phenomenological-phase-shift results from pion-production data do not exist below 500 MeV; that which is available[7] has been indicated on the figures. The phase shift, δ_{00}, is almost certain to be strongly influenced in the 500–1000-MeV region by the opening of the $K\overline{K}$ channel.[8] Our calculation ignores this effect so that no conclusion should be drawn from Fig. 3(a) about the occurrence of a σ resonance. A ρ resonance is clearly in evidence but a decisive statement about its predicted position should also await a calculation with more firm higher-energy validity. Observe that the ρ of the real world appears in Fig. 3(b) near determination A.

In Table I we have listed the values determined for all the parameters by imposing the constraint of elastic unitarity. The last four columns list very stable values for the items of interest in low-energy pion scattering. The quantities $\overline{a}_{0T}$ and $\overline{a}_{11}$ are related to the scattering lengths a_{lT}:

$$a_{0T}=\frac{m_\pi\overline{a}_{0T}}{32\pi F_\pi^{\,2}}\quad\text{and}\quad a_{11}=\frac{\overline{a}_{11}}{24\pi m_\pi F_\pi^{\,2}}\,.$$

The tabulated quantities are to be compared with the Weinberg values[1] for $\overline{a}_{00}$, $\overline{a}_{02}$, and $\overline{a}_{11}$; viz., 7, −2, and 1. It has already been noted in Ref. 3 that we can attribute the agreement we get to the relatively small value obtained for the parameter

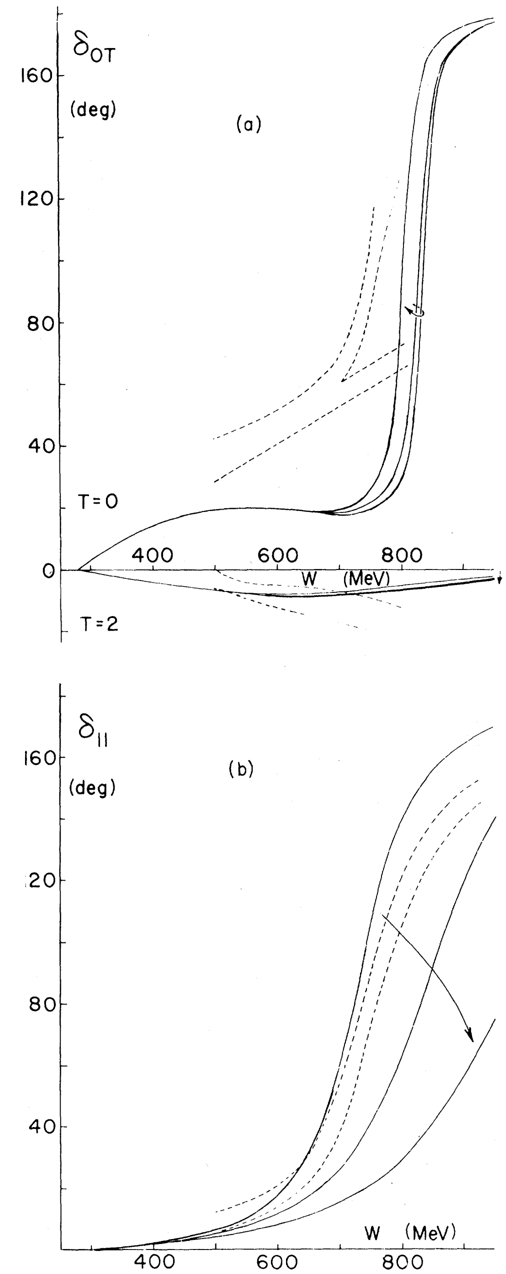

FIG. 3. The phase shifts: (a) s waves; (b) p wave. The phenomenological data of Ref. 7 are indicated.

l_2: Because $l_2 \simeq -\frac{1}{2}m_\pi^2$ (and so $l_0 \simeq \frac{7}{4}m_\pi^2$) we can conclude that the σ commutator is dominantly isoscalar. The tabulation in the last column pertains to the $\pi N\ \sigma$ term, to which we now address ourselves.

The nucleon-to-nucleon matrix element of the isoscalar part of the σ commutator is of great significance because, through it, πN data can be brought to bear on the question of chiral symmetry breaking. Cheng and Dashen[4] have proposed a means of isolating the σ term; Brown, Pardee, and Peccei[9] have confirmed their method in an analysis which keeps the pions on the mass shell. Schnitzer[5] has indicated how the Cheng-Dashen result should be modified by t-channel $\pi\pi$ scattering. The Ward identity expansion of the isospin-even part of the on-shell πN amplitude has the form (cf. Ref. 5)

$$M^{(+)} = F_\pi^2[f_0(t) + l_0]\Delta_0^{-1}(t)F_N(t) + \cdots ,$$

where the three dots denote terms which vanish when we approach the nucleon pole along the line $t = 2m_\pi^2$ (the Cheng-Dashen point) and $F_N(t)$ is the nucleon matrix element of the σ commutator. The desired modification factor can be extracted from our analysis; it is just

$$F_\pi^2[f_0(t) + l_0]\Delta_0^{-1}(t)$$

(in our notation), evaluated at $t = 2m_\pi^2$. The value of this factor, tabulated in the last column of Table I, is very stable and departs very little from -1, the value which corresponds to the Cheng-Dashen analysis.

We can understand this last result by means of some estimates based on Eqs. (7). Our $\pi\pi$ analysis can be characterized by the result that unitarity demands that the slope parameters F' and f_T' be small. Thus we have $C \simeq 0$ so that $l_2 \simeq \alpha_2 - 2m_\pi^2$ and $4\alpha_0 + 5\alpha_2 \simeq 12m_\pi^2$. As a result $l_0 \simeq 2(2m_\pi^2 - \alpha_0)$, thus $\Gamma_0 \simeq -2/l_0 \simeq (\alpha_0 - 2m_\pi^2)^{-1}$. Therefore, $\Gamma_0(2m_\pi^2 - \alpha_0) \simeq -1$, independently of α_0 and Γ_0. Our conclusion, based both on the estimates and on the detailed calculations, is that there is no appreciable modification of the Cheng-Dashen result due to t-channel $\pi\pi$ scattering. We note that

TABLE I. Parameters determined by imposing unitarity.

	α_0 (m_π^2)	F' (m_π^{-2})	α_2 (m_π^2)	l_2 (m_π^2)	l_0 (m_π^2)	Γ_0 (m_π^{-2})	Γ_2 (m_π^{-2})	f_0'	f_2'	η_1 (m_π^{-2})	η_2 (m_π^{-2})	$\bar{a}_{00}$	$\bar{a}_{02}$	$\bar{a}_{11}$	$\Gamma_0(2m_\pi^2-\alpha_0)$
A	1.01	0.036	1.557	−0.432	1.919	−1.036	−2.390	−0.056	−0.022	0.083	−0.085	6.85	−2.18	1.18	−1.026
B	1.10	0.027	1.493	−0.500	1.749	−1.128	−2.063	−0.062	−0.024	0.054	−0.072	6.67	−2.24	1.13	−1.016
C	1.19	0.018	1.430	−0.568	1.579	−1.240	−1.814	−0.069	−0.027	0.024	−0.060	6.51	−2.29	1.09	−1.005

Schnitzer has used essentially equivalent estimates and finds the correction factor to be −3. We can see no possibility for such a result in our work.[10]

Finally we may use our calculations to estimate $F_N(0)$. If we assume that

$$F_N(0)=\frac{f_0(0)}{f_0(2m_\pi^2)}F_N(2m_\pi^2),$$

then we get

$$F_N(0)=0.9F_N(2m_\pi^2).$$

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Interpretation of a $T=2$ σ Commutator*

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We analyze the circumstance in which the σ commutator, σ^{jk}, has isospin $T=2$ as well as $T=0$ components. We assign these along with $\partial_\mu A_\mu^k$ to a mixture of $(\frac{1}{2},\frac{1}{2})$ and $(1,1)$ representations, and we use a recent theoretical result to make the assignment quantitative. We add to the Gell-Mann–Oakes–Renner Hamiltonian in order to account for the $T=2$ effect. The modification we present as an example is $(8,8)$ transforming, constructed to provide the $(1,1)$ structure in $\partial_\mu A_\mu^k$ and σ^{jk}.

An operator of central importance in the study of chiral symmetry breaking is the σ commutator, defined by[1]

$$\sigma^{jk}=i[Q_5^j,D^k]\ . \tag{1}$$

We use $D^k=\partial_\mu A_\mu^k$. The Q_5^j together with Q^j denote the generators of SU(2)×SU(2) transformations. Isospin invariance implies that σ^{jk} is symmetric in jk so that the σ commutator has isospin $T=0$ and $T=2$ components[2]:

$$\sigma^{jk}=\delta_{jk}\sigma^0+\xi_{Jjk}\sigma^J\ ,\qquad \sigma^0=\tfrac{1}{3}\sigma^{jj}\ , \tag{2}$$

and

$$\sigma^J=\xi_{Jjk}\sigma^{jk}\ \ (J=1\ \text{to}\ 5).$$

Expectations are that the σ^J are small.[3] The only widely adopted model for the breaking of SU(3) ×SU(3) symmetry[4,5] is one in which the σ^J do not occur. That there is no *a priori* reason for the σ^J to be absent does not mean that the GOR (Gell-Mann, Oakes, and Renner) model should be abandoned. Instead we believe that accommodation of σ^J is a compelling reason for adding to it appropriately. The purpose of this paper is to offer a model which permits a quantitative interpretation of the $T=2$ components in the σ commutator.

A quantitative measure of the extent to which σ^J occurs in (2) is not readily available experimentally. As an alternative we shall refer to a measure which has recently been determined theoretically. To be precise we define

$$\sigma^{ijk}=-i[Q_5^i,\sigma^{jk}] \tag{3}$$

so that we may introduce a parameter l_2 appearing in