$$\left\{\int_{-\infty}^{\infty} d\nu \,\mathrm{Re}S_1^{\{ab\}}(\nu, q^2)\bigg|_{\mathrm{II}}\right\} \equiv \left[\int_{-\infty}^{\infty} d\nu\,\mathrm{Re}S_1^{\{ab\}}(\nu, q^2; q^- = 0)\bigg|_{\mathrm{II}}\right] - \lim_{q^- \to 0}\left[\int_{-\infty}^{\infty} d\nu\,\mathrm{Re}S_1^{\{ab\}}(\nu, q^2; q^-)\bigg|_{\mathrm{II}}\right].$$

If the Class-II contributions diverge, the sum rules to which they contribute are presumably invalid.

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Analytic Continuation of Reduced Pion-Nucleon Partial-Wave Amplitudes*

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It is shown that one can write the partial-wave amplitudes $h_l^+ = (2k^2/4\mu^2)^{-l} f_l^+$ and $h_{l+1}^- = (2k^2/4\mu^2)^{-l} f_{l+1}^-$ for meson-nucleon scattering as $h_l^\pm = h_{l,\mathrm{I}}^\pm + h_{l,\mathrm{II}}^\pm$, where $h_{l,\mathrm{I}}^\pm$ is analytic in the l plane except for fixed poles at negative integers, and $h_{l,\mathrm{II}}^\pm$ is analytic in the energy W complex plane except for cuts which do not disconnect the plane. Moreover, $h_{l,\mathrm{II}}^+(W) = -h_{l+1,\mathrm{II}}^-(-W)$. Consequences of this result are discussed, in particular its relevance for the theory of Regge poles and the problem of the uniqueness of Mandelstam amplitudes for meson-nucleon scattering.

The problem of determining uniqueness conditions for amplitudes satisfying Mandelstam's representation was thoroughly investigated by Martin.[1] One of his results is that the scattering amplitude is uniquely determined by its absorptive part in the elastic region of one channel. The original derivation was given for the scattering of two identical spinless particles. In a recent paper Cheung and Chen-Cheung[2] generalized this result to cover cases of scattering of particles with spin and isospin, such as pion-nucleon scattering. The argument for this generalization depends on the relation[3]

$$f_1(W,\cos\theta)=-f_2(-W,\cos\theta), \tag{1}$$

where f_1 and f_2 are the pion-nucleon scattering amplitudes related to the invariant amplitudes A and B by

$$f_1=\frac{E+M}{8\pi W}[A+(W-m)B],$$
$$f_2=-\frac{E-M}{8\pi W}[A-(W+m)B], \tag{2}$$

and $W=\sqrt{s}$ and θ are the total energy and scattering angle in the center-of-mass frame. Now, assuming that A and B satisfy Mandelstam's representation, one can show that, for $1-2(\mu^2/m^2)<\cos\theta<1$, $f_1(W,\cos\theta)$ and $-f_2(-W,\cos\theta)$ are analytic continuations of one another in the W complex plane.

Equation (1) has been stated as a relation for partial waves in the form[3]

$$f^-_{l+1}(-W)=-f^+_l(W), \tag{3}$$

but since the projection of partial waves requires integration in $\cos\theta$ over the interval $(-1, 1)$ one finds that (3) cannot be understood in the sense of analytic continuation. It is our purpose here to show that one can replace (3) by a modified partial-wave relation in which analytic continuation is possible connecting W in the physical region to $-W$. In addition these modified partial waves can be analytically continued as functions of l, and their W-dependent singularity structure is similar to that of the original partial waves (in the common domain of analyticity in W).

Consider the amplitudes[4]:

$$h^+_l(s)=(2k^2/4\mu^2)^{-l}f^+_l(s),$$
$$h^-_{l+1}(s)=(2k^2/4\mu^2)^{-l}f^-_{l+1}(s), \tag{4}$$

with the partial waves $f^\pm_l(s)$ being defined by the Froissart-Gribov projections:

$$f^+_{l,\sigma}(s)=\frac{1}{4k^2}\bigg\{\sigma\int_{(m+\mu)^2}^\infty du' f_{1u}(u',s)Q_l(x'_u)+\int_{4\mu^2}^\infty dt' f_{1t}(t',s)Q_l(x'_t)$$
$$-\sigma\int_{(m+\mu)^2}^\infty du' f_{2u}(u',s)Q_{l+1}(x'_u)+\int_{4\mu^2}^\infty dt' f_{2t}(t',s)Q_{l+1}(x'_t)\bigg\}, \tag{5}$$

$$f^-_{l+1,\sigma}(s)=\frac{1}{4k^2}\bigg\{-\sigma\int_{(m+\mu)^2}^\infty du' f_{1u}(u',s)Q_{l+1}(x'_u)+\int_{4\mu^2}^\infty dt' f_{1t}(t',s)Q_{l+1}(x'_t)$$
$$+\sigma\int_{(m+\mu)^2}^\infty du' f_{2u}(u',s)Q_l(x'_u)+\int_{4\mu^2}^\infty dt' f_{2t}(t',s)Q_l(x'_t)\bigg\}, \tag{6}$$

where f_{1u}, f_{1t} and f_{2u}, f_{2t} are the absorptive parts in the u and t channels of f_1 and f_2, respectively, $\sigma=\pm1$ is the signature, and

$$x'_u=-1+[u'+s-2(m^2+\mu^2)]/2k^2, \tag{7}$$

$$x'_t=1+t'/2k^2. \tag{8}$$

(From now on we shall drop the signature index.)

Now we separate the integrals over u' and t' into two parts corresponding to the intervals $((m+\mu)^2, u_0)_{\rm I}$, $(u_0,\infty)_{\rm II}$ for the u' integrals and $(4\mu^2, t_0)_{\rm I}$, $(t_0,\infty)_{\rm II}$ for the t' integrals. The values of t_0 and u_0 are chosen in such a way that the roots of the equations

$$t_0+2k^2(1-\cos\theta)=0, \tag{9}$$

$$u_0+s-2(m^2+\mu^2)-2k^2(1-\cos\theta)=0 \tag{10}$$

fall in the nonoverlapping intervals $(-\infty, s_1)$ and $(s_2, (m-\mu)^2)$, with $s_1<s_2\leq 0$, for $\cos\theta$ in the interval $(-1, 1)$. This condition is satisfied if we take

$$t_0>4m^2, \tag{11}$$

$$u_0>m^2+\mu^2+\tfrac{1}{2}t_0-\tfrac{1}{2}[(t_0-4m^2)(t_0-4\mu^2)]^{1/2}. \tag{12}$$

The second condition is verified for all values of $t_0>4m^2$ if we take $u_0>3m^2+\mu^2$.

According to this separation of the intervals of integration we shall write

$$f^+_l=f^+_{l,\rm I}+f^+_{l,\rm II},$$
$$f^-_l=f^-_{l,\rm I}+f^-_{l,\rm II}, \tag{13}$$

where $f^\pm_{l,\rm I}$ and $f^\pm_{l,\rm II}$ are Froissart-Gribov partial-wave projections from pairs of amplitudes $(f_{1,\rm I}, f_{2,\rm I})$ and $(f_{1,\rm II}, f_{2,\rm II})$, each pair satisfying a relation of the form (1). The indices I and II refer to the integration ranges of the variables u' and t' as indicated before. However, now, for the second pair, that relation is valid over the whole interval $-1\leq\cos\theta\leq 1$.

We shall also define amplitudes $h^\pm_{l,\rm I}$ and $h^\pm_{l,\rm II}$ in terms of $f^\pm_{l,\rm I}$ and $f^\pm_{l,\rm II}$ by relations of the form (4).

Since the u' and t' integrations in the definition of $h^\pm_{l,\rm I}$ run over finite intervals it follows that these functions are analytic in the whole complex l plane except for fixed poles at negative integers. On the other hand, it is shown in the Appendix that, for

$\mathrm{Re}\,l > N$, $h^{\pm}_{l,\mathrm{II}}$ are analytic functions of s with cuts on the real axis, and that, apart from the square-root kinematical branch point at $s=0$, there is an interval on the negative real axis where these functions are analytic and single-valued. (More precisely, it is possible to define paths encircling the origin, such that going around such paths twice the functions $h^{\pm}_{l,\mathrm{II}}$ return to their original values.) Therefore $h^{\pm}_{l,\mathrm{II}}$, for $\mathrm{Re}\,l > N$ are analytic functions of W in a cut plane. The cuts are on the real and imaginary axis but do not extend over the entire axis so that the whole plane remains connected. Therefore, taking into account that $f_{1,\mathrm{II}}(W, \cos\theta) = -f_{2,\mathrm{II}}(-W, \cos\theta)$ over the $\cos\theta$ interval $(-1, 1)$ and the definitions of $h^{\pm}_{l,\mathrm{II}}$ in terms of $f_{1,\mathrm{II}}$ and $f_{2,\mathrm{II}}$, it follows that $h^{+}_{l,\mathrm{II}}(W)$ and $-h^{-}_{l+1,\mathrm{II}}(-W)$ are analytic continuations of one another in the W plane.

We have thus proved the following:

(i) One can write the partial-wave amplitudes $h^{\pm}_{l}$ in the form

$$h^{\pm}_{l} = h^{\pm}_{l,\mathrm{I}} + h^{\pm}_{l,\mathrm{II}}, \tag{14}$$

where (a) the only singularities of $h^{\pm}_{l,\mathrm{I}}$ in the l plane are fixed poles at negative integers and an essential singularity at infinity, and (b) $h_{l,\mathrm{II}}$ is analytic in the whole complex W plane except for cuts on the real and imaginary axis which do not disconnect the plane.

(ii) The relation

$$h^{+}_{l,\mathrm{II}}(W) = -h^{-}_{l+1,\mathrm{II}}(-W) \tag{15}$$

holds for $\mathrm{Re}\,l > N$.

(iii) It follows from (i) that all the moving (W-dependent) singularities of $h^{\pm}_{l}$ coincide with those of $h^{\pm}_{l,\mathrm{II}}$, for W in the common domain of analyticity of these functions. We shall now give some applications of this result. First, it follows immediately from (ii) and (iii) that if $h^{+}_{l}(W)$ has a moving singularity, given by $j = l + \frac{1}{2} = \alpha^{+}(W)$, for instance a Regge pole, then $h^{-}_{l}(W)$ has a pole at $j = l - \frac{1}{2} = \alpha^{-}(W) = \alpha^{+}(-W)$. In particular, if the trajectories $\alpha^{\pm}(W)$ are analytic functions of s with a cut along the physical region on the positive real axis, then Regge trajectories for fermions with opposite parity and signature are degenerate, and one obtains parity doublets.[5]

Second, we apply our results to the problem of uniqueness discussed by Cheung and Chen-Cheung.[2] Let us consider two sets of partial-wave amplitudes $f^{\pm}_{l}$ and $f'^{\pm}_{l}$, and let us assume that for some interval in the elastic region of the s channel and $\mathrm{Re}\,l > 1$, we have

$$f^{+}_{l} - f'^{+}_{l} = 0, \tag{16}$$

$$f^{-}_{l+1} - f'^{-}_{l+1} = f^{-}_{l+1}(s+i\epsilon) + f^{-}_{l+1}(s-i\epsilon). \tag{17}$$

Then in this region we have

$$\Delta h^{+}_{l,\mathrm{I}} + \Delta h^{+}_{l,\mathrm{II}} = 0, \tag{18}$$

$$\Delta h^{-}_{l+1,\mathrm{I}} + \Delta h^{-}_{l+1,\mathrm{II}} = h^{-}_{l+1}(s+i\epsilon) + h^{-}_{l+1}(s-i\epsilon), \tag{19}$$

where $\Delta h^{\pm}_{l} = h^{\pm}_{l} - h'^{\pm}_{l}$.

In the first equation, since $\Delta h^{+}_{l,\mathrm{I}}$ considered as an analytic function of l has no s-dependent singularities, it follows that $h^{+}_{l,\mathrm{II}}$ has also only fixed singularities. However, $-\Delta h^{-}_{l+1,\mathrm{II}}(-W)$ is the analytic continuation of $\Delta h^{+}_{l,\mathrm{II}}(W)$ in the W plane, and therefore in the l plane it can only have the same singularities as $\Delta h^{+}_{l,\mathrm{II}}(W)$. Therefore the left-hand side of (19) has only fixed singularities in the l plane. But the right-hand side of (19) cannot have only fixed poles in the l plane at negative integers. In fact, unitarity requires the existence of moving cuts, and Mandelstam[6] has shown that they do indeed occur. Therefore the assumptions (16) and (17) lead to a contradiction. This is the desired result, required in Ref. 2 for the extension of Martin's uniqueness condition to the case of meson-nucleon scattering.

It is apparent that these results can be generalized to the case of particles with higher spins. The relations analogous to (3) for the general spin case were derived by Hara.[7]

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APPENDIX: ANALYTIC STRUCTURE OF $h^{\pm}_{l,\mathrm{II}}(s)$.

We shall investigate here the analytic structure of $h^{\pm}_{l,\mathrm{II}}$ in the s plane. We use for $f^{\pm}_{l,\mathrm{II}}$ the Froissart-Gribov representations (5) and (6), with the lower limits of integrations being replaced by u_0 and t_0 in the u' and t' integrations, respectively. We assume these representations to be valid in the domain $\mathrm{Re}\,l > N(s)$. According to usual Regge theory, $N(s) < 1$, at least in some interval of the elastic region in pion-nucleon scattering. The validity of the Mandelstam representation requires $N(s)$ to be bounded, $N(s) < N$. For general l the function $Q_l(x'_u)$ has the following branch points:

$x'_u = -1$,

$$u' + s - 2(m^2+\mu^2) = 0\,, \quad \Rightarrow s_1 = 2(m^2+\mu^2) - u'\,,$$
$$s = 0 \tag{A1}$$

$x'_u = 1$,

$$u' + s - 2(m^2+\mu^2) - 4k^2 = 0\,, \quad \Rightarrow s_2 = \frac{(m^2-\mu^2)^2}{u'}\,,$$
$$s = -\infty \tag{A2}$$

$x'_u = \infty$,

$$k^2 = 0 \Rightarrow \begin{cases} s = (m-\mu)^2 \\ s = (m+\mu)^2 \end{cases}.$$

Therefore $Q_l(x_u')$ has the following branch cuts:

$$x_u' \in (-1, 1) \Rightarrow s \in \{(-\infty, s_1), (0, s_2)\},$$

$$x_u' \in (1, \infty) \Rightarrow s \in \{(s_2, (m-\mu)^2), ((m+\mu)^2, \infty)\}.$$

Similarly $Q_l(x_t')$ has the following branch points:

$x_u' = -1$

$$t' + 4k^2 = 0$$

$$\Rightarrow s_\pm = m^2 + \mu^2 - \tfrac{1}{2}t' \pm \tfrac{1}{2}[(t' - 4m^2)(t' - 4\mu^2)]^{1/2},$$

$$x_t' = 1 \qquad \text{(A3)}$$

$$k^2 = \infty \Rightarrow \begin{cases} s = 0 \\ s = \infty \end{cases},$$

$x_t' = \infty$

$$k^2 = 0 \Rightarrow \begin{cases} s = (m-\mu)^2 \\ s = (m+\mu)^2 \end{cases}.$$

Therefore $Q_l(x_t')$ has the following branch cuts:

$$x_t' \in (-1, 1) \Rightarrow s \in \{(-\infty, s_-), (s_+, 0)\},$$

$$x_t' \in (1, \infty) \Rightarrow s \in \{(0, (m-\mu)^2), ((m+\mu)^2, \infty)\}.$$

The factor $(2k^2/4\mu^2)^{-l}$ has branch cuts on the intervals $(0, (m-\mu^2)$ and $((m+\mu)^2, \infty)$. These branch cuts exactly cancel the branch cuts of Q_l on the interval corresponding to $x' \in (1, \infty)$. Therefore the functions $(2k^2/4\mu^2)^{-l} Q_l(x_{u,t}')$ have branch cuts in the intervals

$$s \in \{(-\infty, s_1), (-\infty, s_-), (s_+, s_2)\}.$$

Now in order to make analytic continuation around the point $s = 0$, it is necessary that the branch line (s_+, s_2) which contains the point $s = 0$ not intercept or overlap the other two branch lines. This condition is fulfilled if the u', t' integrations are restricted to the intervals

$$t' > t_0 > 4m^2,$$

$$u' > u_0 > (m^2 + \mu^2) + \tfrac{1}{2}t_0 - \tfrac{1}{2}[(t_0 - 4m^2)(t_0 - 4\mu^2)]^{1/2}.$$

The second condition results from the requirement that $s_1 < s_+$.

Now, in addition to the singularities of $Q_l(x')$, the integrands in (5) and (6) have singularities in the absorptive amplitudes $f_{1,2u}(u', s)$ and $f_{1,2t}(t', s)$. Apart from the kinematical square root branch point at $s = 0$ the singularities of these functions are

(i) A branch cut along the physical region $s \in ((m+\mu)^2, \infty)$.

(ii) A branch cut along the line

$$u' + t' - 2(m^2 + \mu^2) + s = 0, \qquad \text{(A4)}$$

where u' and t' are in the support of the third double spectral function $\rho_{ut}(u', t')$, and either u' or t' is restricted to the interval (u_0, ∞) or (t_0, ∞). For $u' > u_0$, since $t' > 0$, it is clear that the solution of (A4) is inside the interval $(-\infty, s_1(u_0))$. On the other hand, the equation for $s_-(t_0)$ is

$$t_0 + s_-(t_0) - 2(m^2 + \mu^2) + \frac{(m^2 - \mu^2)^2}{s_-(t_0)} = 0,$$

and since $s_-(t_0) < 0$ it follows that

$$t_0 + s_-(t_0) - 2(m^2 + \mu^2) > 0.$$

Therefore, since $u' > 0$, it follows that the solution of (A4) for $t' > t_0$ is within the interval $(-\infty, s_-(t_0))$.

We conclude that for $\mathrm{Re}\, l > N$ the functions $h_{l,\mathrm{II}}^{\pm}(s)$ are analytic in s with cuts on the real axis in the intervals

$$s \in \{(-\infty, s_{\max}), (s_+(t_0), s_2(u_0)), ((m+\mu)^2, \infty)\} \qquad \text{(A5)}$$

where $s_{\max}$ is the greater of $s_1(u_0)$ and $s_-(t_0)$. In addition they have a purely kinematical square root branch point at $s = 0$. For $\mathrm{Re}\, l < N$, the Froissart-Gribov projections still define partial waves analytic in s, but only inside the domain bounded by $\mathrm{Re}\, l = N(s)$.

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