Interaction of Isovector Scalar Mesons with a Static Source*

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A self-consistent method is used to find an approximate ground state in the case of charged scalar and isovector scalar static source theories. In both cases there is a spectrum of stable excited states. In the case of a point source, the physical states are grouped into orthogonal uncoupled subspaces.

I. INTRODUCTION

In a recent paper,[1] a self-consistent method for treating interactions mediated by mesons was described. It is also possible to apply such a method to the self-interaction of a source that can emit and absorb mesons. In particular, the simplest such system is that in which the source is static.[2] This paper deals with the self-consistent approximation for the cases of charged and isovector mesons interacting with a static isospin-$\frac{1}{2}$ source. The meson field is taken to be scalar under space rotations. Thus the Hamiltonian to be considered is

$$H=\sum_\alpha \int \omega(k) a_\alpha^\dagger(k) a_\alpha(k) dk - \sum_\alpha \tau_\alpha \int [W^*(k) a_\alpha(k) + W(k) a_{-\alpha}^\dagger(k)] dk, \qquad (1)$$

where the sums run over $\alpha=0, \pm 1$ for the isovector case, $\alpha=\pm 1$ only for the charged case with

$$\tau_\pm \equiv (\tau_1 \pm i\tau_2)/\sqrt{2}\,, \quad \tau_0=\tau_3, \quad \omega(k)=(k^2+m^2)^{1/2}. \qquad (2)$$

The source function $W(k)$ is taken to have the form

$$W(k)=kv(k)\left(\frac{\gamma}{\pi\omega(k)}\right)^{1/2}, \qquad (3)$$

where γ replaces $g^2/4\pi$ and $v(k)$ is unity for a point source.

In Sec. II, the details of the self-consistent approximation are exhibited. Section III gives the spectra for the special case of a point source. In the charged case, the result is identical with that derived by Pais and Serber[3] for the case of strong coupling; moreover, the approximation indicates that the coupling is always strong in the charged case. In the isovector case, there is a critical value γ_c of the coupling constant. For $\gamma \leq \gamma_c$ the coupling is weak, and there are no stable excited states like those found in the charged case. For $\gamma > \gamma_c$ the spectrum is like that in the charged case.

In the case of a point source, there is an interesting orthogonality property that is like the one pointed out by Van Hove[4] for the case of a neutral scalar field interacting with a point source. In the present case, not only are the physical states orthogonal to the bare ones, but the physical states are grouped into orthogonal subspaces that are uncoupled from each other.

Section IV contains some remarks on methods of improving the approximate state vectors in the case of an extended source.

II. DETAILS

The eigenstates of the Hamiltonian (1) are complicated superpositions of states of the form

$$|i;k_1\alpha_1, k_2\alpha_2, \ldots, k_n\alpha_n\rangle = \prod_{l=1}^{n} a_{\alpha_l}^\dagger(k_l)\ \beta_i, \qquad (4)$$

where the β_i are the eigenspinors of τ_0. The eigenvalues of H are degenerate with degeneracy $2T+1$, where $T=\frac{1}{2}, \frac{3}{2}, \ldots$ is the isospin of the state. Here use will be made of the fact that a superposition of states belonging to different eigenvalues of T_3 and T,

$$T_3=\tfrac{1}{2}\tau_0+\int [a_+^\dagger(k)a_+(k) - a_-^\dagger(k)a_-(k)]dk = \tfrac{1}{2}\tau_0+\sum_\alpha \int \alpha a_\alpha^\dagger(k) a_\alpha(k) dk, \qquad (5)$$

can possibly have a simpler structure than an eigenstate of H and T_3. Since states with differing values of T_3 are to be superposed, the operator to be investigated is not H, but rather G:

$$G=H-\lambda T_3, \qquad (6)$$

where λ is a Lagrange multiplier.

The variational principle can be used to find the best approximate eigenstates of G within the subspace consisting of states that are simple products of a spinor χ describing the source and a factor $|g\rangle$ describing the meson field

$$\psi = |g\rangle\chi\ . \tag{7}$$

As in I, the simplest procedure is to consider $(\chi, G\chi)$

$$(\chi, G\chi) = \int \sum_\alpha \epsilon_\alpha(k) a_\alpha^\dagger(k) a_\alpha(k)\, dk - \tfrac{1}{2}\lambda\langle\tau_0\rangle - \sum_\alpha \int [W^*(k)\langle\tau_\alpha\rangle a_\alpha(k) + W(k)\langle\tau_{-\alpha}\rangle a_\alpha^\dagger(k)]\, dk\ . \tag{8}$$

where

$$\epsilon_\alpha(k) = \omega(k) - \alpha\lambda\ , \quad \langle\tau_i\rangle = (\chi, \tau_i\chi)\ , \tag{9}$$

and (3) gives

$$\langle\tau_{-\alpha}\rangle = \langle\tau_\alpha^\dagger\rangle = \langle\tau_\alpha\rangle^*\ . \tag{10}$$

The quantity $(\chi, G\chi)$ can be simplified by introducing the new creation and annihilation operators $b_\alpha^\dagger(k)$ and $b_\alpha(k)$:

$$b_\alpha(k) = a_\alpha(k) - d_\alpha(k)\ , \quad d_\alpha(k) = W(k)\langle\tau_\alpha\rangle^*/\epsilon_\alpha(k)\ . \tag{11}$$

Then

$$(\chi, G\chi) = \int \sum_\alpha \epsilon_\alpha(k) b_\alpha^\dagger(k) b_\alpha(k)\, dk + G'\ , \quad G' = -\tfrac{1}{2}\lambda\langle\tau_0\rangle - \sum_\alpha |\langle\tau_\alpha\rangle|^2 \int \frac{|W(k)|^2}{\epsilon_\alpha(k)}\, dk\ , \tag{12}$$

and it follows that the appropriate meson state $|g\rangle$ is the lowest meson eigenstate of $(\chi, G\chi)$, namely, the vacuum of the $b_\alpha(k)$

$$b_\alpha(k)|g\rangle = 0 \quad \text{all } \alpha, k\ . \tag{13}$$

G' is the approximate eigenvalue of G

$$G' = (\chi, \langle g|G|g\rangle\chi) = \langle g|(\chi, G\chi)|g\rangle\ . \tag{14}$$

The state factor χ is found by first substituting (11) into G with the result

$$\begin{aligned} G &= G_0 - G_{\text{int}}\ , \\ G_0 &= G' + G_{0m} + G_{0f}\ , \\ G_{0m} &= \sum_\alpha \int \epsilon_\alpha(k) b_\alpha^\dagger(k) b_\alpha(k)\, dk\ , \\ G_{0f} &= -\tfrac{1}{2}\lambda(\tau_0 - \langle\tau_0\rangle) - 2\sum_\alpha(\tau_\alpha - \langle\tau_\alpha\rangle)\langle\tau_\alpha\rangle^* I(\alpha\lambda)\ , \\ G_{\text{int}} &= -\sum_\alpha(\tau_\alpha - \langle\tau_\alpha\rangle) \times \int [W^*(k) b_\alpha(k) + W(k) b_{-\alpha}^\dagger(k)]\, dk\ , \end{aligned} \tag{15}$$

where

$$I(x) = \int \frac{\omega(k)|W(k)|^2 dk}{\omega^2(k) - x^2} = I(-x)\ . \tag{16}$$

Now χ is an eigenspinor of G_{0f}, which is of the form

$$G_{0f} = A - \vec{\tau}\cdot\vec{\mathrm{B}} = A - \tau_0 B_0 - \tau_+ B_- - \tau_- B_+\ , \quad B_0 = \tfrac{1}{2}\lambda + 2I(0)\langle\tau_0\rangle^*\ , \tag{17}$$

$$B_\pm = 2I(\lambda)\langle\tau_\mp\rangle^* = 2I(\lambda)\langle\tau_\pm\rangle\ . \tag{18}$$

The eigenvalues of G_{0f} are $A \pm |B|$. Let the angles in isospin space of the vector B be θ and φ, so that

$$B_0 = B\cos\theta\ , \quad B_\pm = \frac{1}{\sqrt{2}} B e^{\pm i\varphi}\sin\theta\ . \tag{19}$$

The eigenspinors of (17) are

$$\begin{pmatrix} \exp(-i\varphi/2) & \cos(\theta/2) \\ \exp(i\varphi/2) & \sin(\theta/2) \end{pmatrix}$$

and (20)

$$\begin{pmatrix} -\exp(-i\varphi/2) & \sin(\theta/2) \\ \exp(i\varphi/2) & \cos(\theta/2) \end{pmatrix}\ ,$$

with

$$\langle\tau_0\rangle = \cos\theta, \quad \langle\tau_\pm\rangle = \frac{1}{\sqrt{2}} e^{\pm i\varphi}\sin\theta$$

and (21)

$$\langle\tau_0\rangle = -\cos\theta, \quad \langle\tau_\pm\rangle = -\frac{1}{\sqrt{2}} e^{\pm i\varphi}\sin\theta\ ,$$

respectively. The second set with (18) and (19) gives $B = -2I(\lambda) < 0$ and is therefore inconsistent. The first set gives

$$B = I(\lambda)\ , \quad \chi = \begin{pmatrix} \exp(-i\varphi/2) & \cos(\theta/2) \\ \exp(i\varphi/2) & \sin(\theta/2) \end{pmatrix}\ . \tag{22}$$

In solving for $\cos\theta$, two cases arise. Let

$$c(\lambda) \equiv \frac{\lambda}{4J(\lambda)}\ , \quad J(\lambda) \equiv I(\lambda) - I(0)\ , \tag{23}$$

then

$$\cos\theta = c(\lambda)\ , \quad |c(\lambda)| < 1 \qquad = s = \pm 1\ , \quad |c(\lambda)| \geq 1\ , \tag{24}$$

and correspondingly

$$G' = -I(0) - J(\lambda) - \frac{\lambda^2}{16J(\lambda)}, \quad |c(\lambda)| < 1$$
$$= -I(0) - \tfrac{1}{2}\lambda s, \quad |c(\lambda)| \geq 1 . \tag{25}$$

The value of λ must be chosen to make the expectation value of T_3 equal to the desired value, which will be denoted T (not to be confused with total isospin)

$$T = \frac{\lambda}{8J(\lambda)} + 2\lambda\left(1 - \frac{\lambda^2}{16J^2(\lambda)}\right)K(\lambda), \quad |c(\lambda)| < 1$$
$$= \tfrac{1}{2}s, \quad |c(\lambda)| \geq 1 \tag{26}$$

$$K(\lambda) = \int \frac{\omega(k)|W(k)|^2}{[\omega^2(k) - \lambda^2]^2}\, dk . \tag{27}$$

Finally,

$$\langle H \rangle = G' + \lambda T$$
$$= -I(0) - J(\lambda) + \frac{\lambda^2}{16J(\lambda)}$$
$$+ 2\lambda^2\left(1 - \frac{\lambda^2}{16J^2(\lambda)}\right) K(\lambda), \quad |c(\lambda)| < 1$$
$$= -I(0), \quad |c(\lambda)| \geq 1 . \tag{28}$$

Of course, the above procedure can also be applied to the case of charged mesons interacting with a static source. The Hamiltonian for the charged case is the same as for the isovector case, except that there is no interaction with the neutral meson field. The results in that case are

$$c_c(\lambda) = \frac{\lambda}{4I(\lambda)},$$
$$\cos\theta_c = \frac{\lambda}{4I(\lambda)}, \quad |c_c(\lambda)| < 1$$
$$= s = \pm 1, \quad |c_c(\lambda)| \geq 1$$
$$T = \frac{\lambda}{8I(\lambda)} + 2\lambda\left(1 - \frac{\lambda^2}{16I^2(\lambda)}\right)K(\lambda), \quad |c_c(\lambda)| < 1$$
$$= \tfrac{1}{2}s, \quad |c_c(\lambda)| \geq 1 \tag{29}$$
$$\langle H \rangle = -I(0) - J(\lambda) + \frac{\lambda^2}{16I(\lambda)}$$
$$+ 2\lambda^2\left(1 - \frac{\lambda^2}{16I^2(\lambda)}\right) K(\lambda), \quad |c_c(\lambda)| < 1$$
$$= 0, \quad |c_c(\lambda)| \geq 1 .$$

As noted in I, there is a static meson field surrounding the source in this self-consistent approximation. In the case that $|c(\lambda)| \geq 1$, the static field consists entirely of neutral mesons (or is absent in the charged-meson case); the static neutral fields in $|g_{1/2}\rangle$ and $|g_{-1/2}\rangle$ are different. When $|c(\lambda)| < 1$, both neutral and charged static meson fields are present around the source. The neutral static field falls off at large distances like e^{-mr}/r, while the charged static field falls off like $e^{-m'r}/r$ with $m' = (m^2 - \lambda^2)^{1/2}$.

III. POINT SOURCE

For the case of a point source, the integrals $J(\lambda)$ and $K(\lambda)$ converge, and $I(\lambda)$ is divergent. As will be seen, $I(0)$ only appears as a source self-energy term and therefore causes no difficulties. The values of J and K are

$$J(\lambda) = \frac{\gamma m}{2}[1 - (1 - x^2)^{1/2}],$$
$$K(\lambda) = \frac{\gamma}{4m(1 - x^2)^{1/2}}, \tag{30}$$
$$x \equiv \frac{\lambda}{m} .$$

In the case of charged scalar theory, it follows that $|c_c(\lambda)| < 1$ always, so that

$$T = \frac{\gamma}{2}\frac{x}{(1 - x^2)^{1/2}},$$
$$x^2 = \frac{T^2}{T^2 + (\gamma/2)^2} \tag{31}$$

and, after some algebra,

$$\langle H \rangle = -I(0) + m\{[T^2 + (\gamma/2)^2]^{1/2} - \gamma/2\}. \tag{32}$$

This is the same result as that obtained by Pais and Serber.[3] As noted in Ref. 3, the states obtained for $T = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are all stable in this approximation, that is,

$$\langle H \rangle_T < \langle H \rangle_{T-1} + m, \quad T \geq \tfrac{3}{2}. \tag{33}$$

The result of Ref. 3 was restricted to the case of strong coupling. However, it is clear from the foregoing that in the case of a charged scalar field interacting with a *point* source, any coupling constant greater than zero gives strong coupling in the self-consistent approximation.

The isovector field is more varied. Here the solution of the equation does have two regions, one for weak coupling and one for strong coupling. The critical value of γ is γ_c with

$$\gamma_c = \tfrac{1}{2} . \tag{34}$$

For $\gamma \leq \gamma_c$, it follows that $\cos\theta = s = \pm 1$ and

$$\lambda = 0,$$
$$T_3' = \pm\tfrac{1}{2}, \quad \gamma \leq \gamma_c \tag{35}$$
$$\langle H \rangle = -I(0) .$$

Only for $\gamma > \gamma_c$ is there a spectrum of states; all values of $T \geq \frac{1}{2}$ are

$$x^2 = \frac{T^2}{T^2 + (\gamma/2 - 1/8\gamma)^2} ,$$

$$\langle H \rangle = -I(0) \quad (36)$$

$$+ m\{[T^2 + (\gamma/2 - 1/8\gamma)^2]^{1/2} - (\gamma/2 + 1/8\gamma)\},$$

$$T = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots, \quad \gamma > \gamma_c .$$

Again it is easily seen that

$$\langle H \rangle_T < \langle H \rangle_{T-1} + m, \quad T \geq \tfrac{3}{2} \quad (37)$$

so that all of the states listed are stable in this approximation.

Each value of T has its corresponding meson vacuum $|g_T\rangle$. The value of $d_\alpha(k)$ in Eq. (11) depends on T; so does $b_\alpha(k)$. These will be written $d_\alpha(k, T)$ and $b_\alpha(k, T)$. Clearly

$$\begin{aligned} & b_\alpha(k, T) = a_\alpha(k) - d_\alpha(k, T), \\ & b_\alpha(k, T)|g_T\rangle = 0, \\ & d_\alpha(k, T) = W(k)\langle\tau_\alpha\rangle_T^*/\epsilon_\alpha(k, T), \\ & \epsilon_\alpha(k, T) = \omega(k) - \alpha\lambda(T), \\ & \langle\tau_0\rangle_T = \cos\theta_T, \\ & \langle\tau_\pm\rangle_T = \mp \frac{1}{\sqrt{2}} e^{\pm i\varphi_T} \sin\theta_T . \end{aligned} \quad (38)$$

It follows by standard techniques that

$$2 \ln\langle g_T|g_{T'}\rangle = -\sum_\alpha \int |d_\alpha(k, T) - d_\alpha(k, T')|^2 dk . \quad (39)$$

Consider first $\langle g_{1/2}|g_{-1/2}\rangle$. It is convenient to set

$$\begin{aligned} & \varphi_{-1/2} = \pi + \varphi_{1/2} , \\ & \theta_{-1/2} = \pi - \theta_{1/2} \end{aligned} \quad (40)$$

so that

$$\begin{aligned} & (\chi_{1/2}, \chi_{-1/2}) = 0, \\ & \langle\tau_\alpha\rangle_{-1/2} = -\langle\tau_\alpha\rangle_{1/2} . \end{aligned} \quad (41)$$

Then

$$\ln\langle g_{1/2}|g_{-1/2}\rangle = -2 \int |W(k)|^2 \left(\frac{1}{\omega^2(k)} \cos^2\theta + \frac{\omega^2(k)}{[\omega^2(k) - \lambda^2]^2} \sin^2\theta\right) dk$$

$$= -\infty \quad (42)$$

so that

$$\langle g_{1/2}|g_{-1/2}\rangle = 0. \quad (43)$$

Similarly, it also follows for strong coupling that

$$\langle g_T|g_{T'}\rangle = \delta_{TT'} . \quad (44)$$

Thus, for each value of T there is a set of states $|T; k_1\alpha_1, k_2\alpha_2, \ldots\rangle$ defined by

$$|T; k_1\alpha_1, k_2\alpha_2, \ldots\rangle = b^\dagger_{\alpha_1}(k_1, T) b^\dagger_{\alpha_2}(k_2, T) \cdots |g_T\rangle \chi_T . \quad (45)$$

The sets (45) for different values of T are in orthogonal subspaces. There is no interaction that connects the different subspaces. This situation has some resemblance to what happens in the coupling of a neutral scalar field to a static point source; in that case all the physical states are orthogonal to all the bare states.[4] In the case treated here it is also true that all the physical states are orthogonal to all the bare states. However, here there is also no coupling between the states $|\tfrac{1}{2}; k\alpha\rangle$ and the states $|-\tfrac{1}{2}; k'\alpha'\rangle$.

IV. EXTENDED SOURCE

Consider now the situation when the source function $v(k)$ is chosen to make the integral in (42) finite. It is convenient to set

$$Z = \langle g_{1/2}|g_{-1/2}\rangle . \quad (46)$$

In order to improve the wave function, it is necessary to add components orthogonal to $|\tfrac{1}{2}\rangle$. There are two sets of states that can be used. For one set, take the set $|\tfrac{1}{2}; k_1\alpha_1, k_2\alpha_2, \ldots\rangle$ defined in (45); then for the other set, it is possible to take either the set A consisting of the states $|-\tfrac{1}{2}; k_1\alpha_1, k_2\alpha_2, \ldots\rangle$ or the set B of states $|\tilde{\tfrac{1}{2}}; k_1\alpha_1, k_2\alpha_2, \ldots\rangle$ defined by

$$|\tilde{\tfrac{1}{2}}; k_1\alpha_1, k_2\alpha_2, \ldots\rangle = b^\dagger_{\alpha_1}(k_1, \tfrac{1}{2}) b^\dagger_{\alpha_2}(k_2, \tfrac{1}{2}) \cdots |g_{1/2}\rangle \chi_{-1/2} . \quad (47)$$

Owing to the orthogonality relation (41), either of these two latter sets is orthogonal to the first set. The state $|\tilde{\tfrac{1}{2}}\rangle$ is the "upper state" of Ref. 3. The set A has the advantage of being closer to the set of physical states, while the set B has expectation values of G that start at $G' + 4I(\lambda)$, which is large. On the other hand, it is quite simple to develop a perturbation theory using set B, while set A seems less amenable to such a treatment. Moreover, set B gives a second order value for the source self energy that is much lower (greater in magnitude) than that obtained with set A. However, the self-energy of the source is not a physically interesting quantity; its accuracy is not a criterion by which to choose between sets A and B. The point of view taken here is that the more physical nature of set A is of overriding importance, and improvements in the zero-order state vector will be sought by using set A.

It is possible to find G and G_0 such that

$$G = G_0 + V \quad (48)$$

and all states $|\tfrac{1}{2}; k_i\alpha_i\rangle$ and $|-\tfrac{1}{2}; k_i\alpha_i\rangle$ are eigenstates of G_0; the choice is

$$G = H - \sum_{i=\pm 1/2} \lambda(i)\chi_i(\chi_i, T_3\chi_i)\chi_i^\dagger ,$$
$$G_0 = \sum_{i=\pm 1/2} \chi_i(i|G|i)\chi_i^\dagger . \qquad (49)$$

Then

$$V = \sum_{i=\pm 1/2} \chi_i(\chi_i|G|\chi_{-i})\chi_{-i}^\dagger ,$$
$$(\chi_i|G|\chi_{-i}) = (\chi_i|H|\chi_{-i}),$$
$$= -\sum_\alpha (i|\tau_\alpha|-i) \times \int [W^*(k)b_\alpha(k, -i) + W(k)b_{-\alpha}^\dagger(k, i)]\,dk , \qquad (50)$$

where the extra term vanishes because

$$W^*(k)d_\alpha(k, -i) + W(k)d_{-\alpha}^*(k, i) = 0 . \qquad (51)$$

The difficulty in constructing a perturbation theory arises because $V|\frac{1}{2}\rangle$ has components with states $|-\frac{1}{2}; k_1\alpha_1, \ldots, k_n\alpha_n\rangle$ with all possible numbers n of $-\frac{1}{2}$ mesons (except $n=0$). Thus, it seems more reasonable to use approximations that keep all terms up to N mesons, where N is small. These approximations are well known.[2] The only difference in the present case is that V connects states with meson numbers that differ by any number. Of course, it also follows that there is a factor Z in each matrix element of V, so that the coupling constant $\gamma^{1/2}$ is renormalized by a factor Z. The constant Z is also related to the source wave-function renormalization constant Z.

$$Z_1 = Z|\cos(\theta/2)|$$

and in weak coupling the two are equal.

V. SUMMARY

The self-consistent technique is simpler than previous methods[3] for approximate solution of the charged scalar theory. It can also be applied to isovector scalar fixed source theory with the interesting result that there is a critical value γ_c of the coupling constant. For coupling weaker than the critical value the self-consistent field is entirely neutral.

The orthogonality relations that arise in the theory applied to a point source are a sharper version of the ones originally noted by Van Hove[4] for the case of a neutral scalar field. In the case of an extended source, it seems likely that methods based on a limited number of mesons are most promising for improving the wave function and computing scattering matrix elements.

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