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Second-Order Eikonal Approximation for Potential Scattering

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An eikonal approximation for potential scattering at high energies is derived to second order in $1/k$ for the Schrödinger equation. The procedure is based on the same assumptions and restrictions introduced by Glauber, except that the derivation is carried one order further. Whereas Glauber's solution provides for phase modulation of the incoming plane wave, the second-order eikonal approximation, in addition to amending the phase term, turns out also to include an amplitude modulation factor. This approximation, which is the result of maintaining the original calculation to higher order, differs in general from the perturbation correction by Wallace of Glauber's first-order eikonal approximation, although both reduce to the same correct solution for the special case of Coulomb scattering.

INTRODUCTION

The approach taken by Glauber[1] in deriving an eikonal approximation for the Schrödinger equation in potential scattering was followed[2] by a corresponding solution of the Dirac equation, and then applied to various nuclear models for electron scattering from heavy nuclei, where the Born approximation becomes inadequate. The resulting expression for the differential cross section agreed with that obtained also by Schiff[3] for use as a small-angle approximation, but was found to give accurate results at all scattering angles. Some attempt was made to seek to justify the validity of this approximation over the entire angular range, but in any case it was observed[2] that the diffraction peaks agreed extremely well in shape, although somewhat displaced in scattering angle with respect to those of phase-shift analysis. It was noted that merely changing the scale of linear dimensions shifted the curves into correct position, and that this high-energy approximation was then found to produce results without significant perceptible deviation from those of an exact calculation. (See Figs. 4 through 6 of Ref. 2.)

An analysis of high-energy approximations by Moore[4] made some interesting comparisons between a series expansion of the Glauber approximation and the terms of the infinite Born series. It was found that the first Born term is of course reproduced exactly (as Glauber had pointed out), but that if the subsequent Born terms are broken up into on-energy-shell contributions (real intermediate states) and off-energy-shell contributions (virtual intermediate states), then further correspondences can be made. Every term of the Born series may itself be expanded in successive powers of $1/p$, where p is the incident particle momentum. It was noted that the nth Born term is of leading order V^n/p^{n-1} for the on-energy-shell contribution, and V^n/p^n for the off-energy-shell contribution. (Moore uses V for the potential itself as well as its integral in the z direction, but since we are making order-of-magnitude comparisons, we shall not worry about this distinction.) The interesting thing is that the Glauber expansion reproduces exactly for every Born term the leading term of the order of V^n/p^{n-1} *at all scattering angles* for the on-energy-shell contribution, but completely fails to reproduce any of the off-energy-shell states of leading order V^n/p^n.

There are two inferences to be drawn from all this. One is that the success of the eikonal approximation is due to its inclusion of all the on-energy-shell states to leading order for all momentum transfers, i.e., all scattering angles, despite the fact that a small-angle approximation was implicit in the way it was originally derived. One can look for alternate derivations or arguments[2] to avoid making the small-angle approximation, but whether or not these are convincing does not alter the fact that the result correctly reproduces terms of the Born series at all momentum transfers, not just small ones. The second inference is that an important limitation of the Glauber approximation appears to be not so much an angular restriction as it is the absence of off-energy-shell contributions.

An extension of the Glauber approximation has recently been made by Wallace[5] in the form of a perturbation expansion of the T matrix, obtaining a set of ordered corrections to the Glauber formula. These were applied to the particular case of the Yukawa potential, where they were found to approach the results of phase-shift analysis.

The fact that Glauber's result can (at least in particular cases) be adjusted[2] (by a simple "change of scales") or perturbed[5] (in a T-matrix expansion) with such success suggests that it might be fruitful to pursue his original line of approach, but retaining now a higher order of accuracy from the very beginning. It turns out that it is indeed possible to solve the Schrödinger integral equation consistently to the next higher order in the expansion parameter, simply by carrying Glauber's procedure one step further.

THE INTEGRAL EQUATION

Following Glauber, we look for a solution of the Schrödinger equation,

$$\psi(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}} - \frac{2m}{4\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\psi(\vec{r}')d\vec{r}' , \tag{1}$$

by employing a trial function consisting of a plane wave multiplied by an unknown modulating function,

$$\psi(\vec{r}) = \Phi(\vec{r})e^{i\vec{k}_0\cdot\vec{r}} . \tag{2}$$

The result is an integral equation for the unknown function, which like Glauber we integrate by parts in spherical coordinates over $\mu = \cos\theta$, obtaining

$$\Phi(\vec{r}) = 1 + \frac{2m}{4\pi\hbar^2}\int_0^\infty dr''\int_0^{2\pi} d\phi \left(\frac{e^{ikr''(1-\mu)}}{ik} V(\vec{r}-\vec{r}'')\Phi(\vec{r}-\vec{r}'')\right)_{\mu=-1}^{\mu=+1}$$
$$- \frac{2m}{4\pi\hbar^2}\int_0^\infty dr''\int_0^{2\pi} d\phi\int_{-1}^{+1} d\mu \frac{e^{ikr''(1-\mu)}}{ik}\frac{\partial(V\Phi)}{\partial\mu} . \tag{3}$$

Equation (3) is obtained by substituting (2) into (1) and making the change of variables $\vec{r}'' = \vec{r} - \vec{r}'$ before integrating by parts. For $\mu = \pm 1$ the vector $\vec{r}''$ extends from the origin of the double-prime coordinate system in the $\pm z$ direction, corresponding to directions parallel and antiparallel, respectively, to the incoming momentum vector $\vec{k}_0$. Since for $\mu = \pm 1$ there is no azimuthal dependence in the first integrand of Eq. (3), integration over ϕ becomes trivial for this term.

Glauber's approximation was made by dropping both the second integral of Eq. (3) and the $\mu = -1$ contribution to the first integral, since (as we shall see explicitly) these are of higher order in $1/k$, and hence of $1/kd$, where d is a characteristic dimension of the scattering center. His resulting integral equation for the unknown modulating function, retaining only the $\mu = +1$ term, and converting from spherical to Cartesian coordinates and back to the single-prime variable, is

$$\Phi_G(x,y,z) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')\Phi_G(x,y,z')dz' . \tag{4}$$

Equation (4) has the exact solution

$$\Phi_G(x,y,z) = \exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')dz'\right) , \tag{5}$$

so that the wave function (2) becomes, when one is content with having satisfied integral equation (3) only to first order in $1/kd$,

$$\psi_G(x,y,z) = \exp\left(ikz - \frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')dz'\right) . \tag{6}$$

We now investigate the consequence of satisfying Eq. (3) to second order in $1/kd$. The $\mu = -1$ contribution to the first integral of (3) becomes

$$\frac{2m}{4\pi\hbar^2}\int_0^\infty dr''\int_0^{2\pi} d\phi \left[\frac{e^{ikr''(1-\mu)}}{ik} V(\vec{r}-\vec{r}'')\Phi(\vec{r}-\vec{r}'')\right]_{\mu=-1} = \frac{i}{\hbar v}\int_z^\infty e^{-2ik(z-z')} V(x,y,z')\Phi(x,y,z')dz' .$$

This may be integrated by parts over z', obtaining for the leading term

$$- \frac{1}{2\hbar k v} V(x,y,z)\Phi(x,y,z) , \tag{7}$$

which we note to be of second order in $1/k$, plus successively higher-order terms obtained from repeated integration by parts; these we can drop.

The last integral of Eq. (3) may likewise again be integrated by parts over μ, but this time we retain only the $\mu=+1$ term (of second order),

$$-\frac{2m}{4\pi\hbar^2}\int_0^\infty dr''\int_0^{2\pi}d\phi\int_{-1}^{+1}d\mu\,\frac{e^{ikr''(1-\mu)}}{ik}\,\frac{\partial(V\Phi)}{\partial\mu}=-\frac{2m}{4\pi\hbar^2}\int_0^\infty dr''\int_0^{2\pi}d\phi\left[\frac{1}{k^2r''}\,\frac{\partial(V\Phi)}{\partial\mu}\right]_{\mu=1},\tag{8}$$

plus terms of still higher order in $1/k$ which may be dropped.

One must be careful in integrating (8) over azimuth, since the integrand becomes multivalued in the limit $\mu=1$. This is best seen by writing it in terms of the θ variable. The expression then becomes

$$\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\int_0^\infty dr''\,\frac{1}{r''\sin\theta}\int_0^{2\pi}d\phi\,\frac{\partial(V\Phi)}{\partial\theta}\,.\tag{9}$$

For fixed magnitude r'' the integration over azimuth in this limit corresponds to evaluating the partial derivative with respect to θ as the tip of the position variable $\vec{r}''$ sweeps about the z'' axis in all the different azimuthal directions, and then letting θ go to zero. Unfortunately the non-Cartesian character of the coordinate system presents a problem, but one can avoid having to deal with a multivalued partial derivative in this limit by transforming to Cartesian derivatives, which for a smooth function become independent of azimuth at $\theta=0$. Then (9) becomes

$$\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\int_0^\infty dr''\frac{1}{r''\sin\theta}\int_0^{2\pi}d\phi\left[\frac{\partial(V\Phi)}{\partial x''}\,r''\cos\theta\cos\phi+\frac{\partial(V\Phi)}{\partial y''}\,r''\cos\theta\sin\phi-\frac{\partial(V\Phi)}{\partial z''}\,r''\sin\theta\right].\tag{10}$$

The last term of (10) is easiest to evaluate, becoming simply

$$\begin{aligned}-\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\int_0^\infty dr''\int_0^{2\pi}d\phi\,\frac{\partial(V\Phi)}{dz''}&=-\frac{1}{\hbar kv}\int_0^\infty dz''\left[\frac{\partial(V\Phi)}{\partial z''}\right]_{x''=y''=0}\\&=\frac{1}{\hbar kv}\int_{-\infty}^{z}\frac{\partial}{\partial z'}\,[V(x,y,z')\Phi(x,y,z')]dz'\\&=\frac{1}{\hbar kv}\,V(x,y,z)\Phi(x,y,z)\,.\end{aligned}\tag{11}$$

The two transverse-derivative terms of (10) are not quite so simple. What keeps them (and the third term as well, for that matter) from vanishing altogether in the limit $\theta\to0$ is the fact that the entire integral over azimuth is being divided by $\sin\theta$; otherwise the azimuthal integral of the term in brackets of (10) actually vanishes in this limit. As it is, however, there is an indeterminacy to be evaluated. Thus the first term of (10) becomes, again transforming the partial with respect to θ (this time due to L'Hospital's rule) into Cartesian partial derivatives,

$$\begin{aligned}&\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\frac{1}{\tan\theta}\int_0^\infty dr''\int_0^{2\pi}d\phi\,\frac{\partial(V\Phi)}{\partial x''}\,\cos\phi\\&\quad=\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\frac{1}{\sec^2\theta}\int_0^\infty dr''\int_0^{2\pi}d\phi\,\frac{\partial}{\partial\theta}\left[\frac{\partial(V\Phi)}{\partial x''}\right]\cos\phi\\&\quad=\frac{1}{2\pi\hbar kv}\lim_{\theta\to0}\int_0^\infty dr''\int_0^{2\pi}d\phi\left[\frac{\partial^2(V\Phi)}{\partial x''^2}\,r''\cos\theta\cos^2\phi\right.\\&\qquad\left.+\frac{\partial^2(V\Phi)}{\partial y''dx''}\,r''\cos\theta\sin\phi\cos\phi+\frac{\partial^2(V\Phi)}{\partial z''\partial x''}\,r''\sin\theta\cos\phi\right].\end{aligned}\tag{12}$$

As $\theta\to0$ the Cartesian partial derivatives approach their limit values on the z'' axis and may be taken outside the integral over ϕ, whereupon the coefficients of the mixed partial-derivative terms vanish in the azimuthal integration, leaving only the second partial with respect to x'',

$$\begin{aligned}\frac{1}{2\hbar kv}\lim_{\theta\to0}\int_0^\infty dr''r''\,\frac{\partial^2(V\Phi)}{\partial x''^2}&=\frac{1}{2\hbar kv}\int_0^\infty dz''z''\left[\frac{\partial^2(V\Phi)}{\partial x''^2}\right]_{x''=y''=0}\\&=\frac{1}{2\hbar kv}\int_{-\infty}^{z}dz'(z-z')\,\frac{\partial^2}{\partial x^2}\,V(x,y,z')\Phi(x,y,z')\,.\end{aligned}\tag{13}$$

There is a similar term involving the second partial derivative with respect to y. These two terms, together with (7) and (11), represent the contributions of second order in $1/kd$ to the integral equation (3). Other terms are of still higher order. Thus Eq. (3) may now be written

$$\Phi(x,y,z)=1-\frac{i}{\hbar v}\int_{-\infty}^{z}V(x,y,z')\Phi(x,y,z')dz' + \frac{1}{2\hbar kv}V(x,y,z)\Phi(x,y,z)+\frac{1}{2\hbar kv}\int_{-\infty}^{z}(z-z')\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V(x,y,z')\,\Phi(x,y,z')dz'. \tag{14}$$

Equation (14) contains all contributions up to second order in $1/kd$, as compared with Glauber's Eq. (4), which included terms of first order only. It should be noted that no additional assumptions were made in deriving (14) beyond those introduced by Glauber to obtain (4); the process was simply carried one step further. There remains the problem of solving (14), to obtain a wave function that satisfies the Schrödinger integral equation to second order.

SOLUTION FOR SPHERICALLY SYMMETRIC POTENTIALS

The transverse second partial derivatives in the last term of Eq. (14) may be taken outside the integral, i.e., the differentiation may be commuted with the integration, provided only that the potential has continuous derivatives.[6] It is convenient also to transform the expression in terms of the impact parameter $b=(x^2+y^2)^{1/2}$. Equation (14) then takes the form for spherically symmetric potentials

$$\Phi(b,z)=1-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,z')\Phi(b,z')dz' + \frac{1}{2\hbar kv}V(b,z)\Phi(b,z) + \frac{1}{2\hbar kv}\left(\frac{1}{b}+\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b} \times\int_{-\infty}^{z}(z-z')\,V(b,z')\Phi(b,z')dz'. \tag{15}$$

If it were not for the z' in the integrand of the right-hand integral of (15), this integral would be of the same form as the first integral, and one could then simply look for a suitable function Φ which would make the integrand an exact differential. But the integral involving the added factor z' is to be contended with. However, Eq. (14) is correct only to second order in $1/k$; this means one is required only to find a solution which satisfies it to that order; in other words, one is free to add or subtract terms of third or higher order to the right-hand side of (15) if this facilitates the solution. We also make the tentative assumption at this point that the unknown function Φ will turn out to have the property that its derivative is of higher order in $1/k$ then Φ itself (as was true of Glauber's solution). This will of course have to be verified subsequently. Hence the term including the factor z' in the integrand may be written

$$-\frac{1}{2\hbar kv}\left(\frac{1}{b}+\frac{\partial}{\partial b}\right)\int_{-\infty}^{z}z'\,\frac{\partial V(b,z')}{\partial b}\,\Phi(b,z')dz',$$

dropping the term involving $\partial\Phi/\partial b$ which will be of third or higher order. But spherical symmetry of the potential means that

$$z'\,\frac{\partial V(b,z')}{\partial b}=b\,\frac{\partial V(b,z')}{\partial z'}.$$

As a result this term becomes an exact differential, since again we can neglect the derivative of Φ as of higher order. Thus

$$-\frac{1}{2\hbar kv}\left(\frac{1}{b}+\frac{\partial}{\partial b}\right)\int_{-\infty}^{z}b\,\frac{\partial}{\partial z'}(V\Phi)dz' = -\frac{1}{2\hbar kv}\left(2+b\,\frac{\partial}{\partial b}\right)V(b,z)\Phi(b,z). \tag{16}$$

Equation (15) therefore takes the form

$$\Phi(b,z)=1-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,z')\Phi(b,z')dz' - \frac{1}{2\hbar kv}\left(1+b\,\frac{\partial}{\partial b}\right)V(b,z)\Phi(b,z) + \frac{z}{2\hbar kvb}\left(1+b\,\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b} \times\int_{-\infty}^{z}V(b,z')\Phi(b,z')dz'. \tag{17}$$

We look for a solution of (17) of the form

$$\Phi(b,z)=f(V(b,z)) \times\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))d\bar z\right), \tag{18}$$

where the function $f(V(b,z))$ is to be determined so as to satisfy Eq. (17) to second order. This will make it possible to evaluate the integrals on the right-hand side of (17) explicitly, since it will make the integrand an exact differential. The exponential in (18) is chosen to provide phase modulation of the incoming plane wave, just as in the case of Glauber's first-order solution. The function f should have the property of reducing to the Glauber result $f=1$ when terms of higher order than the first are dropped from the integral equation. Thus $f(V(b,z))$ will have the form of unity plus higher-order terms. This means it is of zeroth order, and its derivative is at least of first order.

When (18) is substituted in (17), and the integrals explicitly evaluated, we have

$$f(V(b,z))\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))\,d\bar z\right)$$
$$=\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))\,d\bar z\right)$$
$$-\frac{1}{2\hbar kv}\left(1+b\frac{\partial}{\partial b}\right)V(b,z)f(V(b,z))\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))d\bar z\right)$$
$$-\frac{z}{2kib}\left(1+b\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b}\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))\,d\bar z\right). \tag{19}$$

The differential operators operate on everything to their right in each term of Eq. (19). However, in the middle term the only derivative that need be retained is $\partial V(b,z)/\partial b$, since the derivative of Φ will contribute terms of third or higher order to the integral equation. In the case of the third term of (19) the first differentiation results in

$$-\frac{z}{2kib}\left(1+b\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b}\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))\,d\bar z\right)$$
$$=\frac{z}{2\hbar kvb}\left(1+b\frac{\partial}{\partial b}\right)\left[\left(\int_{-\infty}^{z}\frac{\partial V(b,\bar z)}{\partial b}f(V(b,\bar z))\,d\bar z\right)\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)f(V(b,\bar z))\,d\bar z\right)\right] \tag{20}$$

to second order, since the derivative of f contributes higher order. The second derivative operator in (20) again may be considered to operate effectively on only the potential function outside the exponential for the same reason, so that finally Eq. (19) may be written to second order, with the exponential factored out,

$$f(V(b,z))=1-\frac{1}{2\hbar kv}\left[\left(1+b\frac{\partial}{\partial b}\right)V(b,z)\right]f(V(b,z))+\frac{z}{2\hbar kvb}\int_{-\infty}^{z}\left[\left(1+b\frac{\partial}{\partial b}\right)\frac{\partial V(b,\bar z)}{\partial b}\right]f(V(b,\bar z))\,d\bar z\,. \tag{21}$$

Equation (21) is equivalent to the original integral equation, which can effectively be recovered (consistent to second order) merely by multiplying through by the exponential. The only difficulty in solving for f is that it appears inside the integral on the right-hand side. However, since the function f consists of unity plus higher-order terms, it may be introduced in both terms on the right-hand side of (21) and only the unity part retained. A better way to see this is to be reminded that it is consistent to add or subtract higher-order terms on the right-hand side of the original integral equation. We are therefore free to add on the right-hand side of (21) the terms

$$-\frac{1}{2\hbar kv}\left[\left(1+b\frac{\partial}{\partial b}\right)V(b,z)\right][1-f(V(b,z))]+\frac{z}{2\hbar kvb}\int_{-\infty}^{z}\left[\left(1+b\frac{\partial}{\partial b}\right)\frac{\partial V(b,\bar z)}{\partial b}\right][1-f(V(b,\bar z))]\,d\bar z\,, \tag{22}$$

since $[1-f]$ introduces only higher-order terms; this is precisely equivalent to adding corresponding expressions to the original integral equation. With this addition to (21), we obtain the expression for f,

$$f(V(b,z))=1-\frac{1}{2\hbar kv}\left(1+b\frac{\partial}{\partial b}\right)V(b,z)+\frac{z}{2\hbar kvb}\left(1+b\frac{\partial}{\partial b}\right)\int_{-\infty}^{z}\frac{\partial V(b,\bar z)}{\partial b}\,d\bar z\,. \tag{23}$$

This is the solution to be substituted in Eq. (18) to obtain the modulating function for spherically symmetric potentials,

$$\Phi(b,z)=\left\{1-\frac{1}{2\hbar kv}\left[\left(1+b\frac{\partial}{\partial b}\right)V(b,z)-\frac{z}{b}\left(1+b\frac{\partial}{\partial b}\right)\int_{-\infty}^{z}\frac{\partial V(b,\bar z)}{\partial b}\,d\bar z\right]\right\}$$
$$\times\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar z)\left\{1-\frac{1}{2\hbar kv}\left[\left(1+b\frac{\partial}{\partial b}\right)V(b,\bar z)-\frac{\bar z}{b}\left(1+b\frac{\partial}{\partial b}\right)\int_{-\infty}^{\bar z}\frac{\partial V(b,\bar z)}{\partial b}\,d\bar z\right]\right\}d\bar z\right). \tag{24}$$

It can be readily verified by direct substitution that (24) satisfies integral equation (15) for spherically symmetric potentials, provided only than one is permitted to add on the right-hand side of (15) additional expressions such as (22) (multiplied by the exponential factor) and terms involving derivatives of Φ, all of which are of higher order. Since Eq. (14) was obtained by truncating expressions of higher order than the second from the exact integral equation, there is in principle no inconsistency in adding such terms. Glauber's solution (5) is an exponential which satisfies the integral equation to first order in $1/kd$; Eq. (24) is a corresponding exponential expression that satisfies it to second order. The exponential is retained to all orders, but the function f is significant only to second order.

When the modulating function (24) is introduced into Eq. (2), the resulting wave function consists of a plane wave modulated by the presence of the scattering center not only in phase (as was the Glauber solution) but in amplitude as well. This wave function may be substituted in the expression for the scattering amplitude,

$$f(\theta)=-\frac{2m}{4\pi\hbar^2}\int e^{i\vec{q}\cdot(\vec{b}+\hat{n}z)}V(b,z)f(V(b,z))\times\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}V(b,\bar{z})f(V(b,\bar{z}))d\bar{z}\right)dz\,d^2b\ , \tag{25}$$

where θ is the scattering angle, $\vec{q}=\vec{k}_0-\vec{k}_f$ is the momentum-transfer vector of magnitude $q=2k\sin\frac{1}{2}\theta$, $\hat{n}$ is the unit vector in the z direction, and $d^2b=b\,db\,d\phi$.

At this point one either takes advantage of the small-angle approximation or alternately invokes the argument of restoration of time-reversal invariance[2] in the scattering amplitude. We shall not here dwell upon these arguments, since they are precisely the same as were invoked previously.[1,2] The net result in any case is that the momentum-transfer vector $\vec{q}$ is taken to be perpendicular to the z direction, so that the integral over z in Eq. (25) becomes that of an exact differential. With the integral representation of the Bessel function, the result takes the same form as Glauber's,

$$f(\theta)=\frac{k}{i}\int_0^\infty J_0(qb)(e^{i\chi(b)}-1)\,b\,db\ , \tag{26}$$

where, however, now

$$\chi(b)=-\frac{1}{\hbar v}\int_{-\infty}^{\infty}V(b,z)\left\{1-\frac{1}{2\hbar kv}\left[\left(1+b\frac{\partial}{\partial b}\right)V(b,z)-\frac{z}{b}\left(1+b\frac{\partial}{\partial b}\right)\int_{-\infty}^{z}\frac{\partial V(b,\bar{z})}{\partial b}d\bar{z}\right]\right\}dz\ . \tag{27}$$

This is to be compared with Glauber's first-order function,[1]

$$\chi_G(b)=-\frac{1}{\hbar v}\int_{-\infty}^{\infty}V(b,z)dz\ . \tag{28}$$

It is significant that when the exponential of Eq. (26) is expanded in powers of V, the function (27) results in the appearance now of terms of order V^n/p^n in the scattering amplitude, as well as V^n/p^{n-1}, whereas the first-order solution (28) produces only terms of the latter type (although in one case it is the integral of a power of V and in the other case a power of the integral). Now Moore[4] has identified terms of leading order V^n/p^{n-1} as contributions to the scattering amplitude from real intermediate states on the energy shell, whereas the terms of order V^n/p^n not found in Glauber's solution correspond to virtual intermediate states off the energy shell, and this applies at all scattering angles. Since we have retained one order higher in $1/p$ and Glauber has already included all on-energy-shell contributions to leading order, it is to be expected that off-energy-shell contributions are now to be represented in the expansion of Eq. (26).

A comparison of Eq. (27) with Wallace's[5] first correction to the Glauber approximation shows that introduction of Glauber's first-order result into a T-matrix perturbation expansion is not in general equivalent to having carried the original calculation to higher order, although, as we shall see, it does indeed give the same answer in particular cases.

It should be remembered that the solution (27) is restricted to spherically symmetric potentials, and presupposes also that the potential has continuous derivatives.[6]

COULOMB POTENTIAL

Since the Glauber approximation provides the correct solution of the Schrödinger equation for the Coulomb potential, we should like the second-order correction in (27) to vanish for this case; otherwise there would be the unpleasant task of explaining why a more accurate calculation should give a poorer result. (Actually, since the infinity at the origin violates Glauber's basic restriction on the behavior of the potential, there is really no *a priori* reason to expect it to work so well; and in fact it does not give the correct result for the Dirac equation.[2,7])

There are two terms to be evaluated in the expression in brackets of Eq. (27). The first term contributes an integral of the form

$$\int_{-\infty}^{\infty} V(b,z)\left(1+b\frac{\partial}{\partial b}\right) V(b,z)dz = \left(1+\tfrac{1}{2}b\frac{\partial}{\partial b}\right)\int_{-\infty}^{\infty}[V(b,z)]^2 dz\,.$$

Aside from constant factors this becomes for the Coulomb potential

$$\left(1+\tfrac{1}{2}b\frac{\partial}{\partial b}\right)\int_0^{\infty}\frac{dz}{b^2+z^2}=\frac{\pi}{4b}\,.$$

The second term requires evaluation of

$$-\frac{z}{b}\left(1+b\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b}\int_{-\infty}^{z}\frac{d\bar{z}}{b^2+\bar{z}^2}=\frac{z^2}{b}\frac{\partial V(b,z)}{\partial b}$$

when

$$V(b,z)=\frac{1}{(b^2+z^2)^{1/2}}\,.$$

Hence the second term becomes

$$\int_0^{\infty} V(b,z)\frac{z^2}{b}\frac{\partial V(b,z)}{\partial b}dz=\frac{1}{2b}\frac{\partial}{\partial b}\int_0^{\infty} z^2[V(b,z)]^2 dz = -\frac{\pi}{4b}$$

for this potential. Since the two terms are seen to cancel in this case, the second-order correction in Eq. (27) vanishes for the Coulomb potential, leaving Glauber's correct first-order result of Eq. (28).

ONE-DIMENSIONAL LIMIT

It is interesting also to consider the solution of the one-dimensional Schrödinger equation to second order. However, the result in (27) may not be used here directly, since it was derived for spherically symmetric potentials only. We therefore return instead to the integral equation (14). This becomes for the one-dimensional problem

$$\Phi(z)=1-\frac{i}{\hbar v}\int_{-\infty}^{z} V(z')\Phi(z')dz'+\frac{1}{2\hbar k v}V(z)\Phi(z)\,. \tag{29}$$

Although an exact solution of Eq. (29) may be found in the form of (18) correct to all orders, the integral equation itself is significant only to second order. Hence the function $f(V(z))$ which satisfies this equation should be represented only to second order. It should be noted that (both here and in the three-dimensional problem) from the viewpoint of mathematical consistency alone the exponential in (18) is not required to all orders. But knowledge of the physics of the problem suggests a solution in terms of an explicit exponential factor providing phase modulation of the incoming plane wave. Precisely the same consideration led to the choice of an exponential solution in the Glauber approximation, despite the fact that the integral equation itself was retained only to first order.

The solution of Eq. (29) is therefore

$$\Phi(z)=\left(1+\frac{V(z)}{4E}\right)\exp\left[-\frac{ik}{2E}\int_{-\infty}^{z} V(\bar{z})\left(1+\frac{V(\bar{z})}{4E}\right)d\bar{z}\right] \tag{30}$$

in terms of the particle's incident energy E.

Now in the solution of the one-dimensional Schrödinger equation the local wave number for a plane wave modulated by the presence of a potential $V(z)$ is

$$k(z)=\left(\frac{2m}{\hbar^2}[E-V(z)]\right)^{1/2}=k\left(1-\frac{V}{2E}-\frac{V^2}{8E^2}\cdots\right). \tag{31}$$

The eikonal approximation of Glauber corresponds to retention of only the first-order term $V/2E$ of this expansion. This is readily seen by substituting the expansion in the expression for the phase change,

$$\int_{-\infty}^{z}[k(\bar{z})-k]d\bar{z}=-\frac{k}{2E}\int_{-\infty}^{z} V(\bar{z})\left(1+\frac{V(\bar{z})}{4E}+\cdots\right)d\bar{z}\,. \tag{32}$$

The first term of (32) is identified to be the Glauber phase correction.

It has been suggested[8] that a considerable improvement in the eikonal approximation of Glauber might result if one could replace just the leading term of (32) by the correct expression for $[k(\bar{z})-k]$, namely, the entire expansion. It may be seen from Eq. (30) that the second-order eikonal approximation corresponds precisely to the inclusion of the second term of this expansion. Of course in three dimensions the problem is more complicated, and the very concept of a local wave number becomes questionable, but this term is in fact buried in the second-order correction of Eq. (27).

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Anomaly of the Axial-Vector Current in One Space and One Time Dimension*

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We use Wilson's theory of broken scale invariance to study the anomaly of axial-vector current in a world with one space and one time dimension. It is shown that in the Schwinger and Thirring models, Wilson's approach and the perturbative approach yield similar results for the anomaly of partial conservation of axial-vector current.

I. INTRODUCTION

In recent years the problem of the PCAC (partially conserved axial-vector current) anomaly in the presence of electromagnetism has been studied extensively in the framework of renormalized perturbation theory.[1] It was found that the anomaly is related to the breakdown of the naive Ward identity caused by the presence of a triangle graph in the renormalized perturbation theory. The anomaly also leads to many low-energy theorems for the electromagnetic decays of neutral pseudoscalar mesons[2] and other electromagnetic processes.[3]

Another approach to the problem of the PCAC anomaly was proposed by Wilson.[4] He applies his formulation of broken scale invariance and operator-product expansion to this problem. He shows qualitatively that the anomaly is related to the short-distance behavior of the product of currents.

Recently Crewther,[5] following the suggestion of Wilson, has proved that in fact the anomaly can be explained by the short-distance behavior of the product of currents. He also relates the anomaly constant to other physical constants in high-energy electroproduction and electron-positron annihilation processes.

So far, the anomaly has been studied either entirely in the framework of renormalized perturbation theory or in the framework of Wilson's theory of broken scale invariance.[6] However, the connection between these two different approaches has not been examined.[7] In the perturbation theory one can treat the anomaly successfully. Yet it is not at all clear whether Wilson's theory of broken scale invariance can be applied. On the other hand, in Wilson's approach although we have interesting results relating the anomaly to other physical quantities, we do not know how to calculate these quantities in strong interaction. It is therefore very desirable to find models in which both the perturbation theory and Wilson's theory of broken scale invariance can be applied.

It is the purpose of this note to study the PCAC anomaly in some solvable models. The models we discuss are the Schwinger model[8] and the Thirring model.[9] Both are field-theoretic models in one space and one time dimension. These models have been very useful to provide testing grounds for theoretical ideas. Although they are very special models, nevertheless, any general feature of quantum field theory should remain true. We will show that in these models the anomaly is related to the short-distance behavior of the product of two currents, and the results so obtained for the anomaly are the same as those obtained by perturbation theory.

In Sec. II we discuss the anomaly in one space and one time dimension using Wilson's theory of broken scale invariance and operator-product expansion. The PCAC anomaly will also be related