Approximately Relativistic Lagrangians for Classical Interacting Point Particles*

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In classical special-relativistic dynamics, equations of motion for interacting point particles can be derived from Lorentz-invariant variational principles of the Fokker type. Similarly, approximately relativistic equations of motion for such particles, obtained by expansion of the exact equations in inverse powers of the speed of light, follow from variational principles involving approximately relativistic Lagrangians of the type first found by Darwin for electrodynamics. Here the general form of such Lagrangians is established by directly approximating Lorentz-invariant variational principles describing point particles interacting through two-body forces. Only interactions which possess a static Newtonian limit are considered; the interaction is not assumed to be symmetric in the particles' variables. The exact variational principle is assumed to depend at most on velocities and thus leads to at most acceleration-dependent forces. The same is found to be true of the approximate variational principles. The general approximate Lagrangian obtained is characterized by the absence of terms of order c^{-1} and by the possible presence, for each relativistic particle interaction, of three new functions of the Euclidean interparticle separation which may be independent of the static Newtonian potential. The form of the usual ten approximate conservation theorems is established using the invariance properties of the approximate Lagrangian. Two examples of approximate Lagrangians are evaluated explicitly. The first establishes the form of the approximate Lagrangians associated with relativistic particle interactions which allow definition of "adjunct fields," while the second establishes this form for a particular interaction connecting pairs of points on the world lines of particles with spacelike separation. Possible applications of the results in the classical and quantum domains are discussed.

I. INTRODUCTION

The Newtonian mechanics of point particles is based on the concept of instantaneous interaction at a distance. As a consequence of the success of this theory it served as a model for physical theories for two centuries. However, in time it was supplanted by field theories in which an agent, the field, is held responsible for transmitting forces between the particles. Concurrently the space-time concepts of Newtonian mechanics were superseded by those of the special theory of relativity. Thus there was a transition from a description of a physical system in terms of particles interacting at a distance to one involving fields acting only within an infinitesimal neighborhood (near action); this further involved a change from a system with a finite number of degrees of freedom to one with an infinite number. The transition from Newtonian point mechanics to field theory[1] was greatly accelerated by the enormous success of Maxwell's electrodynamics, which fitted the invariance requirements of special relativity as well as the available macroscopic experiments. Indeed, since there seemed to be no other way to describe radiation on the classical level, nor to develop a relativistic theory of interacting elementary particles on the quantum level, the belief that the field was the basis of all physical phenomena became quite firmly entrenched.

However, later developments such as the Wheeler-Feynman theory[2] showed that on the classical level it might be possible to describe radiation within the concepts of a theory of directly interacting particles. Furthermore, quantum field theory is not free from difficulties, and various authors have realized that a relativistic theory of direct particle interactions[3] may be of help in overcoming some of these difficulties. Thus, most attempts at formulating a classical relativistic dynamics of interacting particles are made with a view toward eventual quantization.

Three general approaches may be identified. One is to replace the Newtonian Galilei-invariant force laws by Lorentz-invariant ones, an approach taken by Poincaré[4] even before the special theory of relativity was created. This approach was later developed by Havas and Plebański.[5,3]

The second approach is through various canonical formalisms, initiated by Dirac.[6,7] One of the formalisms he suggested has the advantage of dealing with variables at a single time, as in Newtonian dynamics. Dirac himself was unable to demonstrate that his formalism was not empty. Bakamjian and Thomas[8] were able to show that it allowed a description of interacting particles provided that one renounced the concept of Lorentz-invariant world lines for these particles[9]; on the other hand, it was shown through various "no-interaction" theorems (first developed by Currie[10]) that maintaining this concept within the canonical formalism excluded the possibility of interactions.

The third approach is to base the equations of motion on Lorentz-invariant variational principles. Such variational principles depending on the four-dimensional separation of the particles and on the product of their four-velocities were first introduced by Fokker,[11] and developed by various authors.[12] A general dependence on other relativistic two-body invariants depending on the separation and velocities of the particles was introduced by Havas[13,14]; the equations of motion following from such generalized variational principles form a subclass of the equations considered in Ref. 5.

All such equations of motion depend on more than one time. Little is known about the general mathematical properties of, exact solutions to, or the quantization of such equations. Consequently, either very special examples are solved[15] or else a single-time approximation is made. This latter has the advantage that it is possible to develop the quantum theory corresponding to the resulting equations by standard methods.[16] Here the approximation approach is exploited on the classical level. A discussion of the corresponding quantum theory will be given elsewhere.[17]

The approximation most commonly employed in relativistic dynamics is an expansion[18] in powers of c^{-1}, the most famous examples of which are those by Darwin in electrodynamics[19] and by Einstein, Infeld, and Hoffmann (EIH) in the general theory of relativity.[20] The usual applications of c^{-2} equations have been to few-electron atoms, using the Darwin Hamiltonian, and to two- or three-body motions of celestial bodies, using the EIH Hamiltonian. However, many-body applications are also possible; investigations include a study of the qualitative aspects of the N-body problem for special cases of approximately relativistic Lagrangians,[21] and application of the Darwin Hamiltonian in magnetism[22] as well as in the study of relativistic effects in the statistical mechanics of charged particles.[23]

Recent work on approximately relativistic Lagrangians by Havas and Stachel[24] focused on the invariances of such Lagrangians (correct to order c^{-2}), and in particular on deriving the center-of-mass theorem from invariance considerations. Some of the questions raised by their results are answered here.

It was noted in HS that all the particular approximately relativistic Lagrangians considered in their paper fitted a general form, but that it had not been established whether all approximate Lagrangians following from exact (special or general) relativistic variational principles had to be of that form, i.e., whether all relativistic corrections were of the same type, differing only in the factor of one term.

This led us to search for the general form of the approximately relativistic Lagrangian (to order c^{-2}) appropriate to those Lorentz-invariant variational principles considered in H which have a static Newtonian limit [i.e., a potential $V_{ij}(r_{ij})$ describing the instantaneous interaction of particles i and j separated by the distance r_{ij}]; they are not restricted to be symmetric in the particle variables. The resulting form derived in Sec. III includes the form suggested by HS, which follows from those exact variational principles considered in H which are most closely related to the customary linear field theories, as shown in Sec. IV and discussed in detail in Sec. VI.[25]

Another question broached by their analysis is whether any approximately relativistic equation correct to order c^{-2} and derivable from a Lagrangian must lead to the particular functional form for the conserved total linear momentum associated with their generalized approximately relativistic Lagrangian (which allowed their expressions for the center-of-mass coordinate to be integrated and hence expressed in terms of the particle variables). Section V uses the invariance methods of HS on the approximate Lagrangian derived in Sec. III to find the usual ten associated conserved quantities. The resulting total linear momentum agrees with that of HS for approximate Lagrangians symmetric in the particle variables; for those not symmetric in these variables, it is still possible to obtain a center-of-mass coordinate expressed in terms of them.

Previous work on establishing the form of approximately relativistic Lagrangians[16,25] proceeded by starting from the Newtonian theory and then generalizing to order c^{-2} on the basis of invariance considerations. Such an approach is not unique.[26] The attack used here is to start from the Lorentz-invariant variational principle considered in H; the action principle is expanded and terms are retained to order c^{-2}. This method has the virtue of establishing the form of the approximately relativistic Lagrangian *determined by* the assumptions rather than, as in previous work, computing what *could be* consistent with those assumptions. While previous work only considered correction terms depending on the Newtonian potential $V_{ij}(r_{ij})$, we find that, in addition, three new functions of r_{ij} may appear, which can be independent of V_{ij}. An interesting by-product of the analysis is the proof that no terms of order c^{-1} can ever appear in the Lagrangian (and thus in the equations of motion) in any theory which is consistent with the assumptions made.

In Sec. II the notation used is introduced, and it is established that the approximate variational principle can be obtained by direct approximation from the exact one, rather than by integration of the approximate equations of motion. The form of the approximately relativistic Lagrangian is established in Sec. III. Section IV exhibits two special cases of approximate Lagrangians: the important case of field-theoretically related Lagrangians, and an example with a spacelike interaction. The approximate conservation theorems are established in Sec. V. The results obtained are discussed in Sec. VI.

II. LORENTZ-INVARIANT AND NEWTONIAN VARIATIONAL PRINCIPLES

We shall be concerned with a dynamical system consisting of N interacting point particles. In special relativity it is convenient to describe the motion of this system in a four-space with coordinates x^μ ($\mu = 0, 1, 2, 3$), where x^0 is the time coordinate. Repetition of a Greek index implies summation over this range. The metric of this space is

$$\eta_{\mu\nu} = 0 \quad \text{if } \mu \neq \nu, \qquad \eta_{00} = 1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -c^{-2}. \tag{1}$$

The world line of the jth particle, with coordinates z_j^μ, can be parametrized by the proper time τ_j, where

$$d\tau_j \equiv (\eta_{\mu\nu}\, dz_j^\mu\, dz_j^\nu)^{1/2}. \tag{2}$$

We define a four-velocity $v_j^\mu(\tau_j)$ and a four-acceleration $a_j^\mu(\tau_j)$,

$$v_j^\mu(\tau_j) \equiv \frac{dz_j^\mu}{d\tau_j}, \quad a_j^\mu(\tau_j) \equiv \frac{d^2 z_j^\mu}{d\tau_j{}^2}, \tag{3}$$

so that

$$v_j \equiv (v_j^\mu v_{j\mu})^{1/2} = 1, \quad v_j^\mu a_{j\mu} = 0. \tag{4}$$

On the other hand the space of Newtonian mechanics is the usual three-dimensional space with Euclidean geometry. In this space the path of the jth particle $\vec{r}_j(t)$ is parametrized by the absolute time t; the three-velocity $\vec{v}_j(t)$ and three-acceleration $\vec{a}_j(t)$ are

$$\vec{v}_j(t) \equiv \frac{d\vec{r}_j}{dt}, \quad \vec{a}_j(t) \equiv \frac{d^2\vec{r}_j}{dt^2}. \tag{5}$$

Actually, neither of these formulations is necessary. A "three-plus-one" description of special relativity is given in most relativity texts,[27] while four-dimensional formulations of Newtonian mechanics[28] are also possible. However, the three-dimensional form is chosen here for the Newtonian case since we are interested in approximately relativistic quantities, which are usually expressed in three-plus-one form.[16,24,25] The four-dimensional form is chosen for the relativistic case because the salient problem in relativistic dynamics is the formulation of Lorentz-invariant interactions; for these the four-dimensional form provides a natural and direct mode of expression, whereas an invariant three-plus-one formulation is quite cumbersome.

The physically important invariance group is the proper orthochronous subgroup of the full inhomogeneous Lorentz group. In the following, "Lorentz invariance" will always mean invariance under this subgroup.

Lorentz-invariant equations of motion for point particles interacting through two-body forces can be obtained from a variational principle

$$\delta I = 0, \quad I \equiv I_1 + I_2, \tag{6a}$$

where I_1 is defined by

$$I_1 \equiv -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_i\, d\tau_j\, \Lambda_{ij}(s_{ij}^\mu, v_i^\mu, v_j^\mu),$$

$$s_{ij}^\mu \equiv z_i^\mu(\tau_i) - z_j^\mu(\tau_j), \tag{6b}$$

and I_2 will be considered below. The g_i's are the "coupling constants" of the interaction, which are introduced in analogy to the constants used in electro- and mesodynamics. The form $g_i g_j \Lambda_{ij}$ implies that the interactions are characterized by only N coupling constants, but by $\frac{1}{2}N(N-1)$ possibly distinct functions Λ_{ij} (not necessarily symmetric in i and j). A further generalization could be introduced by considering different types of interactions between a given pair of particles, each similarly characterized. This would require a separate index for each such type and summation over all types; since it would not change any of the subsequent calculations, but only introduce a cumbersome notation, this will not be done explicitly. However, it should be understood that the final result could be generalized to any number of types of interactions as noted, and indeed use is made of this possibility in a particular example discussed in Sec. VI.

Each Λ_{ij} is assumed to be invariant under the infinitesimal transformations of the proper orthochronous Lorentz group and to depend only on the positions and velocities of the particles. Thus it can be a function only of the two-body invariants of the group, which will be given later. Since our focus is on approximations, Λ_{ij} is assumed to be infinitely differentiable. Particular examples [11,12,29] often can be written in terms of distributions [30] which have this desirable feature by definition.

The variations of the components of each world line z_j^μ are not independent, since Eq. (4) implies

$$v_j^\mu \delta v_{j\mu} = 0. \tag{7}$$

Thus the term δI_2 is needed to maintain these conditions. It is defined by

$$\delta I_2 \equiv -c^2 \sum_i \int_{-\infty}^{\infty} d\tau_i \, M_i(\tau_i) v_i^\mu \delta v_{i\mu} \,, \tag{6c}$$

where the $M_i(\tau_i)$'s are Lagrange multipliers, and a factor of c^2 has been introduced so that they have dimensions of mass. It is not necessary to define I_2 explicitly in this formulation. The minus signs in (6b) and (6c) are chosen so that (6) has the usual Newtonian limit.

The "usual Newtonian limit" means a variational principle [31]

$$\delta I = 0, \quad I = \int_{-\infty}^{\infty} dt \, L[\vec{r}_i(t), \vec{v}_i(t)], \quad i = 1, \ldots, N$$

$$L \equiv T - V, \quad T \equiv \tfrac{1}{2} \sum_{i=1}^{N} m_i \vec{v}_i^{\,2}, \tag{8}$$

$$V \equiv \sum_{i<j}\sum g_i g_j V_{ij}(r_{ij}), \quad r_{ij} \equiv |\vec{r}_i(t) - \vec{r}_j(t)| \,.$$

The g_i's are constants characteristic of the interaction and are included for convenience in comparing the limits of the relativistic case to (8). The potential energy $g_i g_j V_{ij}(r_{ij})$ depends only on the instantaneous interparticle separation. Forces derivable from such potential energies are static and central. The variational principle (8) is invariant up to a total time derivative under the infinitesimal transformations of the Galilei group. However, it by no means represents the most general Galilei-invariant variational principle.[13] Furthermore, it does not even yield the most general Newtonian point mechanics, nor is it the only possible limit of Lorentz-invariant variational principles. However, here we restrict ourselves only to such interactions Λ_{ij} which lead to (8).

We note that, while the Newtonian variational principle (8) is the nonrelativistic limit of the Lorentz-invariant variational principle (6), there are substantial differences between them. The relativistic principle involves $4N$ coordinates and N parameters in a four-space, while the Newtonian one involves $3N$ coordinates and a single parameter in a three-space. The Newtonian principle leads to equations of motion which are written in terms of a single time and depend at most on accelerations, and thus describes a system of particles possessing $6N$ degrees of freedom. On the other hand, the number of degrees of freedom of particle systems described by relativistic equations of motion involving N independent parameters has not been established.[3] The two types of principles (6) and (8) will be contrasted with approximately relativistic ones in Sec. III.

Variation of I_1 for arbitrary $\delta z_i^\mu(\tau_i)$ which vanish at infinity yields

$$\delta I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_i d\tau_j \left(\frac{\partial \Lambda_{ij}}{\partial s_{ij}^\mu} \delta s_{ij}^\mu + \frac{\partial \Lambda_{ij}}{\partial v_i^\mu} \delta v_i^\mu + \frac{\partial \Lambda_{ij}}{\partial v_j^\mu} \delta v_j^\mu \right) . \tag{9}$$

Rearrangement of terms and integration by parts give

$$\delta I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_i d\tau_j \left\{ \left[\frac{\partial \Lambda_{ij}}{\partial s_{ij}^\mu} \delta z_i^\mu - \frac{d}{d\tau_i}\left(\frac{\partial \Lambda_{ij}}{\partial v_i^\mu} \right) \delta z_i^\mu \right] + \left[\frac{\partial \Lambda_{ij}}{\partial s_{ji}^\mu} \delta z_j^\mu - \frac{d}{d\tau_j}\left(\frac{\partial \Lambda_{ij}}{\partial v_j^\mu} \right) \delta z_j^\mu \right] \right\} . \tag{10}$$

It is convenient to define the Lagrangian derivative

$$\mathfrak{L}_{j\mu} \equiv \frac{\partial}{\partial z_j^\mu} - \frac{d}{d\tau_j}\left(\frac{\partial}{\partial v_j^\mu} \right); \tag{11}$$

then Eq. (10) can be rewritten as

$$\delta I_1 = -\sum_i g_i \int_{-\infty}^{\infty} d\tau_i \delta z_i^\mu \mathfrak{L}_{i\mu} \sum_{j>i} g_j \int_{-\infty}^{\infty} d\tau_j \Lambda_{ij} \Big] - \sum_i g_i \int_{-\infty}^{\infty} d\tau_i \delta z_i^\mu \mathfrak{L}_{i\mu} \Big[\sum_{j<i} g_j \int_{-\infty}^{\infty} d\tau_j \Lambda_{ji} \Big] , \tag{12}$$

where we have relabeled the particles in the second term. Equation (6c) is changed by an integration by parts into

$$\delta I_2 = +c^2\sum_i \int_{-\infty}^{\infty} d\tau_i \delta z_i^\mu \frac{d}{d\tau_i}[M_i(\tau_i)v_{i\mu}]. \tag{13}$$

We now define the "generalized potential"

$$V_i(z_i^\mu, v_i^\mu) \equiv \sum_{j>i} g_j \int_{-\infty}^{\infty} d\tau_j \Lambda_{ij} + \sum_{j<i} g_j \int_{-\infty}^{\infty} d\tau_j \Lambda_{ji} . \tag{14}$$

Then the equations of motion following from Eqs. (12) and (13) are

$$\frac{d}{d\tau_i}(M_i c^2 v_{i\mu}) = g_i \mathfrak{L}_{i\mu} V_i , \tag{15}$$

where the Lagrange multiplier M_i is as yet unknown. Its form can be determined by contracting (15) with v_i^μ and using (4), yielding

$$c^2 \frac{dM_i}{d\tau_i} = g_i\left[v_i^\alpha \frac{\partial V_i}{\partial z_i^\alpha} - v_i^\alpha \frac{d}{d\tau_i}\left(\frac{\partial V_i}{\partial v_i^\alpha}\right)\right] . \tag{16}$$

Using

$$\frac{d}{d\tau_i} V_i = \frac{\partial V_i}{\partial z_i^\alpha} v_i^\alpha + \frac{\partial V_i}{\partial v_i^\alpha} a_i^\alpha \tag{17}$$

in (16) gives

$$\frac{dM_i}{d\tau_i} = \frac{g_i}{c^2}\left[\frac{dV_i}{d\tau_i} - \frac{d}{d\tau_i}\left(v_i^\alpha \frac{\partial V_i}{\partial v_i^\alpha}\right)\right] , \tag{18}$$

from which we obtain

$$M_i = m_i + \frac{g_i}{c^2}\left(V_i - v_i^\alpha \frac{\partial V_i}{\partial v_i^\alpha}\right) , \tag{19}$$

where m_i is an arbitrary constant of integration. Inserting (19) into the equations of motion (15) gives

$$m_i a_i^\mu = \frac{g_i}{c^2}\left\{\frac{\partial V_i}{\partial z_i^\mu} - \frac{d}{d\tau_i}\left[\frac{\partial V_i}{\partial v_i^\mu} + v_i^\mu\left(V_i - v_i^\alpha \frac{\partial V_i}{\partial v_i^\alpha}\right)\right]\right\}. \tag{20}$$

Consequently, m_i can be interpreted as the inertial mass of the ith particle, and therefore should be taken as positive. The development to this point is essentially the same as that of H.

In calculating (20) from the variational principle (6) it was necessary to introduce and evaluate the Lagrange multipliers M_i because the variations δz_i^μ were restricted by conditions (7), which are a consequence of having chosen the proper times for parametrizing the world lines. The advantage of the proper time over any other parameter for the world line is its relationship to the particle's three-coordinates and the time through

$$d\tau_i = dt\left[1 - \frac{\vec{v}_i(t_i)\cdot\vec{v}_i(t_i)}{c^2}\right]^{1/2},$$
$$\vec{v}_i(t_i) \equiv \frac{d\vec{r}_i(t_i)}{dt_i} . \tag{21}$$

Equations (21) follow from (2) and from considering the particle's three spatial coordinates to be functions of its zeroth coordinate $z_i^0 = t_i$. Until some relation is established between them, t_i is independent of t_j. Thus (21) is convenient in helping to establish the connection between the Lorentz-invariant variational principle and its approximations.

In view of the complication of the Lagrange multiplier it might be thought that in order to calculate approximate Lagrangians it would be necessary first to approximate the exact equations of motion and then to integrate the approximate ones. However, it is possible to approximate the variational principle itself. This can be seen by using an alternative variational principle to obtain the exact equations of motion.

The alternative approach is to use an arbitrary parameter T_i (which may or may not be identical to τ_i), for which

$$\mathfrak{v}_i \equiv (\mathfrak{v}_i^\alpha \mathfrak{v}_{i\alpha})^{1/2}, \quad \mathfrak{v}_i^\alpha \equiv \frac{dz_i^\alpha(T_i)}{dT_i} , \tag{22}$$

where $\mathfrak{v}_i$ is not necessarily equal to unity. The variational principle (6) is constructed to be parameter-invariant by choosing

$$I_1 \equiv -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dT_i dT_j \lambda_{ij} \tag{23a}$$

and

$$I_2 \equiv -c^2\sum_j \int_{-\infty}^{\infty} dT_i M_i(T_i)(\mathfrak{v}_i^\alpha \mathfrak{v}_{i\alpha})^{1/2}, \tag{23b}$$

with

$$\lambda_{ij} \equiv \mathfrak{v}_i \mathfrak{v}_j \Lambda_{ij}\left(s_{ij}^\mu, \frac{\mathfrak{v}_i^\mu}{\mathfrak{v}_i}, \frac{\mathfrak{v}_j^\mu}{\mathfrak{v}_j}\right). \tag{23c}$$

Then the quantity M_i turns out to be m_i. This is seen by carrying out the variation (6a) using (23). The variation of (23a) is trivial since the same sequence of steps that lead to (12) is retraced. But instead of postulating Eq. (13) for δI_2, we now must vary Eq. (23b), yielding

$$\delta I_2 = -c^2\sum_i \int_{-\infty}^{\infty} dT_i M_i(T_i)(\mathfrak{v}_i^\alpha \mathfrak{v}_{i\alpha})^{-1/2}\mathfrak{v}_{i\mu}\delta\mathfrak{v}_i^\mu , \tag{24}$$

which, subjected to a procedure identical to that used on Eq. (6c), gives

$$\frac{d}{dT_i}\left[\frac{M_i c^2 \mathfrak{v}_{i\mu}}{(\mathfrak{v}_i^\alpha \mathfrak{v}_{i\alpha})^{1/2}}\right] = g_i \mathfrak{L}_{i\mu} V_i \tag{25}$$

instead of Eq. (15), where now

$$\mathfrak{L}_{i\mu} \equiv \frac{\partial}{\partial z_i^\mu} - \frac{d}{dT_i}\left(\frac{\partial}{\partial \mathfrak{v}_i^\mu}\right),$$

$$V_i[z_i^\mu(T_i),\ \mathfrak{v}_i^\mu(T_i)] \equiv \sum_{j<i} g_j \int_{-\infty}^{\infty} dT_j\,\lambda_{ji} + \sum_{j>i} g_j \int_{-\infty}^{\infty} dT_j\,\lambda_{ij}\,. \tag{26}$$

We now use Eqs. (26) and the derivatives of λ_{ij} or λ_{ji} following from Eq. (23c) (omitting the subscripts on the λ's and Λ's)

$$\frac{\partial\lambda}{\partial \mathfrak{v}_i^\mu} = \frac{\mathfrak{v}_j}{\mathfrak{v}_i}\mathfrak{v}_{i\mu}\Lambda + \mathfrak{v}_i\mathfrak{v}_j\frac{\partial\Lambda}{\partial(\mathfrak{v}_i^\rho/\mathfrak{v}_i)}[\delta_\mu^\rho \mathfrak{v}_i^{-1} - \mathfrak{v}_i^\rho \mathfrak{v}_{i\mu}\mathfrak{v}_i^{-3}] \tag{27}$$

in Eq. (25), and let $T_i \to \tau_i$, so that $\mathfrak{v}_i^\mu \to v_i^\mu$, $\mathfrak{v}_i \to v_i$ ($\to 1$ after the variation has been performed), and

$$\frac{\partial\lambda}{\partial \mathfrak{v}_i^\mu} \to \frac{\partial\Lambda}{\partial v_i^\mu} + v_{i\mu}\left(\Lambda - v_i^\alpha\frac{\partial\Lambda}{\partial v_i^\alpha}\right). \tag{28}$$

Then Eq. (25) becomes

$$\frac{d}{d\tau_i}(M_i v_i^\mu) = \frac{g_i}{c^2}\left\{\frac{\partial V_i}{\partial v_i^\mu} - \frac{d}{d\tau_i}\left[\frac{\partial V_i}{\partial v_i^\mu} + v_i^\mu\left(V_i - v_i^\alpha\frac{\partial V_i}{\partial v_i^\alpha}\right)\right]\right\}. \tag{29}$$

Since the right-hand side of Eq. (29) is identical to that of Eq. (20), if M_i were calculated from (29) it would result in m_i, i.e., a quantity independent of τ_i. Thus indeed Eq. (6a) with (23) gives the same equations of motion (20) as do Eqs. (6). This means that for the purpose of directly approximating the variational principle itself, the alternative variational principle (23) can be written with m_i replacing M_i, and the arbitrary parameters can be chosen to be the proper times. Thus, once the variational principle is approximated and put into the form of an integral over a single time, the approximate Lagrangian L can be read off the approximate variational principle

$$\delta I = 0,\quad I = \int_{-\infty}^{\infty} dt\, L[\vec{r}_i(t),\ \vec{v}_i(t)]\,,\ i = 1, \ldots, N. \tag{30}$$

The argument of Λ_{ij} in (6b) was assumed to be Lorentz-invariant. This invariance can be made manifest by choosing the argument of Λ_{ij} to depend only on the following independent invariants [13] formed from the four-dimensional separation s_{ij}^μ and the corresponding four-velocities v_i^μ and v_j^μ:

$$s_{ij}{}^2 \equiv \eta_{\mu\nu}s_{ij}^\mu s_{ij}^\nu,\quad \omega_{ij} \equiv v_i^\mu v_{j\mu}\,,$$
$$\kappa_i \equiv v_{i\mu}s_{ji}^\mu,\quad \kappa_j \equiv v_{j\mu}s_{ij}^\mu, \tag{31}$$

which are a complete set of independent polynomial invariants formed from the particle variables considered.

III. THE APPROXIMATELY RELATIVISTIC LAGRANGIAN

The meaning of Lorentz-invariant variational principles and their Newtonian limits was given in Sec. II. Here the meaning of approximately relativistic variational principles will be delineated and the approximately relativistic Lagrangian will be calculated.

As noted earlier, in taking the Newtonian limit of the Lorentz-invariant variational principle the equation changes from one involving $4N$ coordinates and N parameters in a four-dimensional space to one involving $3N$ coordinates and one parameter in a three-dimensional space. This limit corresponds to the limit $c^{-1} \to 0$, and can be obtained by putting the Lorentz-invariant variational principle into the form of an integral over a single time and then letting $c^{-1} \to 0$. Instead of going to this limit the variational principle will now be expanded in powers of c^{-1}, retaining terms only up to some designated power of c^{-1}, yielding an equation of the form (30); this will be called an approximately relativistic variational principle to that designated order, and its integrand an approximately relativistic Lagrangian to that same order. Thus the approximately relativistic principle involves an integration over a single time parameter, as does the Newtonian variational principle, but it retains some of the information of its relativistic progenitor, and hence lies somewhere between them.

Since we are concerned only with those variational principles which lead to the "usual" Newtonian limit of a static potential (8), this restriction will now be built directly into the Lorentz-invariant variational principle. (The alternative to this restriction will be discussed in detail in Sec. VI.) Unfortunately, not all the two-body invariants introduced in (31) [following H] possess a static limit. However, an equivalent set of invariants can be used which does have this property, as will be shown later. This set is

$$\sigma_{ij} \equiv c^2 s_{ij}{}^2 \equiv c^2(t_i - t_j)^2 - |\vec{r}_i(t_i) - \vec{r}_j(t_j)|^2 \equiv c^2 t_{ij}{}^2 - \vec{r}_{ij}{}^2(t_i, t_j),$$

$$\omega_{ij} \equiv v_i^\mu v_{j\mu} \equiv \gamma_i\gamma_j\left[1 - \frac{\vec{v}_i(t_i)\cdot\vec{v}_j(t_j)}{c^2}\right], \tag{32a}$$

$$\chi_{ij} \equiv c\kappa_i \equiv -\gamma_i[ct_{ij} - c^{-1}\vec{v}_i(t_i)\cdot\vec{r}_{ij}(t_i, t_j)],$$

$$\zeta_{ij} \equiv c\kappa_j \equiv +\gamma_j[ct_{ij} - c^{-1}\vec{v}_j(t_j)\cdot\vec{r}_{ij}(t_i, t_j)],$$

where

$$\gamma_i \equiv [1 - c^{-2}\vec{v}_i(t_i)\cdot\vec{v}_i(t_i)]^{-1/2}. \tag{32b}$$

It is not obvious at this point why the set (32) has a static Newtonian limit while the set (31) does not, since this cannot be seen until some connection is made between the two independent times t_i and t_j in the double integral of (6b) in a form suitable for taking the nonrelativistic limit, i.e., a form which would result in only one time. This will be done in the actual calculation and discussed in detail in Sec. VI.

In view of the change of argument for the integrand of the double integral it is appropriate to use a new notation

$$g_i g_j \Lambda_{ij}(s_{ij}^\mu, v_i^\mu, v_j^\mu) = c g_i g_j U_{ij}(\sigma_{ij}, \omega_{ij}, \chi_{ij}, \zeta_{ij}), \tag{33}$$

where the factor c is introduced for convenience so that $g_i g_j U_{ij}$ has dimensions of force and the nonrelativistic limit of the Lagrangian can be made to correspond easily to well-known examples of relativistic variational principles.[11,12] It is understood that U_{ij} does not involve c except through the invariants indicated. Then Eqs. (6a) with (23) become, putting $T_i = \tau_i$, and consequently $\mathfrak{v}_i = v_i$,

$$\delta I = 0, \quad I \equiv I_1 + I_2, \tag{34a}$$

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\, d\tau_i\, d\tau_j \times v_i v_j U_{ij}\left(\sigma_{ij}, \frac{\omega_{ij}}{v_i v_j}, \frac{\chi_{ij}}{v_i}, \frac{\zeta_{ij}}{v_j}\right), \tag{34b}$$

$$I_2 = -\sum_i \int_{-\infty}^{\infty} d\tau_i\, m_i c^2 (v_i^\mu v_{i\mu})^{1/2}. \tag{34c}$$

I_2 can be expanded easily by changing the variable of integration from τ_i to t_i using Eqs. (21) and then expanding in a Taylor series, to obtain

$$I_2 \approx -\sum_i \int_{-\infty}^{\infty} dt_i m_i c^2 \left[1 + \tfrac{1}{2}\left(-\frac{\vec{v}_i^{\,2}}{c^2}\right) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}\left(-\frac{\vec{v}_i^{\,2}}{c^2}\right)^2 + \cdots\right]. \tag{35}$$

Here t_i is an ordinary integration variable, rather than a path-dependent parameter as is τ_i. Thus the subscript is superfluous and can be deleted, yielding

$$I_2 \approx +\sum_i \int_{-\infty}^{\infty} dt \left[-m_i c^2 + \tfrac{1}{2} m_i \vec{v}_i^{\,2}(t) + \tfrac{1}{8} m_i \vec{v}_i^{\,2}(t) \frac{\vec{v}_i^{\,2}(t)}{c^2}\right]. \tag{36}$$

As noted before, the two integrations of I_1 must be connected in order to be able eventually to write all quantities as functions of a single time. This is accomplished by changing variable from τ_i to t_i and then to ζ_{ij}, which turns Eq. (34b) into

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_j \gamma_j^{-1} \int_{-\infty}^{\infty} c\gamma_i^{-1} U_{ij} \frac{d\zeta_{ij}}{d\zeta_{ij}/dt_i}, \tag{37a}$$

where

$$\frac{d\zeta_{ij}}{dt_i} = c\gamma_j\left[1 - \frac{\vec{v}_i(t_i)\cdot\vec{v}_j(t_j)}{c^2}\right] = c\gamma_i^{-1}\omega_{ij}. \tag{37b}$$

The variable change from t_i to ζ_{ij} is monotonic, since (37b) is manifestly positive definite. In fact, ζ_{ij} is the only invariant from the set (32) for which a change of variable from t_i to an invariant is monotonic across the entire range of integration. While calculating (37b), t_j, being independent, is unaffected by the t_i differentiation. In order to complete the change of variable, all t_i's must be replaced by the appropriate function of ζ_{ij} from

$$t_i = t_j + \frac{1}{c}\left[\zeta_{ij}\gamma_j^{-1} + \frac{\vec{v}_j \cdot \vec{r}_{ij}(t_i, t_j)}{c}\right], \tag{38}$$

which came from Eq. (32). If the Newtonian limit of Eq. (37) were the only form desired, it now could be obtained from Eq. (38) by going to the limit $c \to \infty$. However, since an expansion in powers of c^{-1} is needed here, the substitution (38) must be completed and followed by an expansion. Unfortunately, Eq. (38) is an implicit relation for t_i since it occurs on the right-hand side of the equation within an unknown function.

The standard method of handling implicit functions[32] of the form (38) is Lagrange expansion.[33] Since results are desired to order no higher than c^{-2} it is appropriate to make a Taylor expansion of the γ_j^{-1} in (38) to that order, resulting in

$$t_i \approx t_j + \frac{1}{c}\left[\zeta_{ij} + \frac{\vec{v}_j \cdot \vec{r}(t_i, t_j)}{c} - \tfrac{1}{2}\zeta_{ij}\frac{\vec{v}_j^{\,2}}{c^2}\right], \tag{39}$$

where the convention has been introduced that unless otherwise specified

$$\vec{r} \equiv \vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j, \quad r \equiv |\vec{r}|. \tag{40}$$

Then I_1 can be written in the form

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_j \gamma_j^{-1} \int_{-\infty}^{\infty} d\zeta_{ij} \left.\frac{U_{ij}(\sigma_{ij}, \omega_{ij}, \chi_{ij}, \zeta_{ij})}{\omega_{ij}}\right|_+, \tag{41}$$

where $|_+$ means that the expression is to be evaluated with t_i given by (39). In order to simplify the notation the subscripts i and j will be omitted from U_{ij} and the invariants [in analogy to Eq. (40)], and

$$U(\zeta)|_+ \equiv U(\sigma, \omega, \chi, \zeta)|_+ \tag{42}$$

is defined. Equations (32) show that both σ and χ contain t_i's, which must be replaced using (39), resulting in

$$\sigma|_+ \approx \left\{ c^2\left[\frac{1}{c}\left(\zeta + \frac{1}{c}\vec{v}_j\cdot\vec{r}(t_i,t_j) - \frac{\zeta}{2c^2}\vec{v}_j^{\,2}\right)\right]^2 - \vec{r}^{\,2}(t_i,t_j)\right\}\Big|_+ ,$$
$$\chi|_+ \approx \left\{ -\gamma_i\left[\left(\zeta + \frac{1}{c}\vec{v}_j\cdot\vec{r}(t_i,t_j) - \frac{\zeta}{2c^2}\vec{v}_j^{\,2}\right) - \frac{1}{c}\vec{v}_i(t_i)\cdot\vec{r}(t_i,t_j)\right]\right\}\Big|_+ . \tag{43a}$$

The quantity $\omega|_+$ can be expanded in a Taylor series as follows:

$$\omega|_+ \approx \left\{1 + \frac{1}{2c^2}[\vec{v}_i(t_i) - \vec{v}_j(t_j)]^2\right\}\Big|_+ . \tag{43b}$$

Thus Eqs. (43) allow (42) to be written explicitly as

$$U(\zeta)|_+ \approx U\left\{\left[\zeta^2 - \vec{r}^{\,2}(t_i,t_j) + \frac{2\zeta}{c}\vec{v}_j\cdot\vec{r}(t_i,t_j) + \frac{1}{c^2}[\vec{v}_j\cdot\vec{r}(t_i,t_j)]^2 - \frac{1}{c^2}\vec{v}_j^{\,2}\zeta^2\right],\right.$$
$$\left.\left[1 + \frac{1}{2c^2}[\vec{v}_i(t_i) - \vec{v}_j(t_j)]^2\right], \left[-\zeta + \frac{1}{c}[\vec{v}_i(t_i) - \vec{v}_j(t_j)]\cdot\vec{r}(t_i,t_j) + \frac{\zeta}{2c^2}[\vec{v}_j^{\,2} - \vec{v}_i^{\,2}(t_i)]\right], \zeta\right\}\Big|_+ . \tag{44}$$

Consequently (41) becomes

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_j \gamma_j^{-1}\int_{-\infty}^{\infty} d\zeta \left\{(\gamma_i\gamma_j)^{-1}\left[1 - \frac{\vec{v}_i(t_i)\cdot\vec{v}_j(t_j)}{c^2}\right]^{-1} U(\zeta)\right\}\Big|_+ . \tag{45}$$

A Lagrange expansion must still be made of Eq. (45) using[32]

$$f(t_i,t_j)|_+ = f(t_j,t_j) + \sum_{n=1}^{\infty}\frac{1}{c^n n!}\frac{\partial^{n-1}}{\partial t_i^{\,n-1}}\left\{\left[\zeta + \frac{1}{c}\vec{v}_j\cdot\vec{r}(t_i,t_j) - \frac{\zeta}{2c^2}\vec{v}_j^{\,2}\right]^n \frac{\partial f(t_i,t_j)}{\partial t_i}\right\}\Big|_{t_i=t_j} , \tag{46}$$

where $f(t_i, t_j)$ is any infinitely differentiable function of t_i. The factors to the left of $U(\zeta)$ in the curly brackets of (45) are the reciprocal of (43b). Therefore, the terms containing t_i's are already of order c^{-2}, and only the first term of (46) is needed for their Lagrange expansion. Thus they are independent of ζ, and (45) becomes

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_j \left\{1 - \frac{1}{2c^2}[\vec{v}_i(t_j) - \vec{v}_j(t_j)]^2 - \frac{1}{2c^2}\vec{v}_j^{\,2}(t_j)\right\}\int_{-\infty}^{\infty} d\zeta U(\zeta)\Big|_+ . \tag{47}$$

Equation (44) shows that $U(\zeta)|_+$ depends *explicitly* on c^{-1} through $\sigma|_+$, $\omega|_+$, and $\chi|_+$, and *implicitly* on c^{-1} through (39). Consequently, $U(\zeta)|_+$ first will be expanded in a Taylor series in c^{-1} to second order, and then a Lagrange expansion made to the same order. The Taylor expansion of $U(\zeta)$ is

$$U(\zeta) \approx {}^0U(\zeta) + \frac{1}{c}\,{}^0[U'(\zeta)] + \frac{1}{2c^2}\,{}^0[U''(\zeta)] + \cdots ,$$
$$U'(\zeta) \equiv \frac{dU(\zeta)}{d(c^{-1})}, \quad U''(\zeta) \equiv \frac{d^2U(\zeta)}{d(c^{-1})^2} , \tag{48}$$

where the superscript zero represents *explicit* $c^{-1} \to 0$, so that

$${}^0U(\zeta) \equiv U[\zeta^2 - \vec{r}^{\,2}(t_i,t_j), 1, -\zeta, \zeta] \equiv U[{}^0\sigma, {}^0\omega, {}^0\chi, \zeta] . \tag{49}$$

The first derivative is

$$U' = U_\sigma\sigma' + U_\omega\omega' + U_\chi\chi', \tag{50a}$$

where

$$U_\sigma \equiv \frac{\partial U}{\partial\sigma}, \quad U_\omega \equiv \frac{\partial U}{\partial\omega}, \quad U_\chi \equiv \frac{\partial U}{\partial\chi} . \tag{50b}$$

The derivatives of the invariants in (50a) are given by

$$\sigma' = 2\zeta\vec{v}_j\cdot\vec{r} + \frac{2}{c}[(\vec{v}_j\cdot\vec{r})^2 - \vec{v}_j^{\,2}\zeta^2],$$

$$\omega' = \frac{1}{c}(\vec{v}_i - \vec{v}_j)^2, \tag{50c}$$

$$\chi' = (\vec{v}_i - \vec{v}_j)\cdot\vec{r} + \frac{\zeta}{c}(\vec{v}_j^{\,2} - \vec{v}_i^{\,2}),$$

using the values of σ, ω, and χ given in (44). Thus ${}^0[U'(\zeta)]$ is

$${}^0[U'(\zeta)] \equiv {}^0U' = {}^0U_\sigma 2\zeta\vec{v}_j\cdot\vec{r} + {}^0U_\chi(\vec{v}_i - \vec{v}_j)\cdot\vec{r}, \tag{51}$$

using (50). Similarly, the second derivative is

$$U''(\zeta) = U_\sigma\sigma'' + [U_{\sigma\sigma}\sigma' + U_{\sigma\omega}\omega' + U_{\sigma\chi}\chi']\sigma' + U_\omega\omega'' + [U_{\omega\sigma}\sigma' + U_{\omega\omega}\omega' + U_{\omega\chi}\chi']\omega'$$
$$+ U_\chi\chi'' + [U_{\chi\sigma}\sigma' + U_{\chi\omega}\omega' + U_{\chi\chi}\chi']\chi', \tag{52a}$$

where

$$\sigma'' = 2[(\vec{v}_j\cdot\vec{r})^2 - \vec{v}_j^{\,2}\zeta^2],$$
$$\omega'' = (\vec{v}_i - \vec{v}_j)^2, \tag{52b}$$
$$\chi'' = \zeta(\vec{v}_j^{\,2} - \vec{v}_i^{\,2}),$$

as a consequence of (50c). Then (50) and (52) give

$${}^0[U''(\zeta)] \equiv {}^0U'' = 2\,{}^0U_\sigma[(\vec{v}_j\cdot\vec{r})^2 - \vec{v}_j^{\,2}\zeta^2] + 4\zeta^2\,{}^0U_{\sigma\sigma}(\vec{v}_j\cdot\vec{r})^2 + 4\zeta\,{}^0U_{\sigma\chi}[\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r} - (\vec{v}_j\cdot\vec{r})^2]$$
$$+ {}^0U_\omega(\vec{v}_i - \vec{v}_j)^2 + \zeta\,{}^0U_\chi(\vec{v}_j^{\,2} - \vec{v}_i^{\,2}) + {}^0U_{\chi\chi}[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2. \tag{53}$$

Using Eqs. (48)–(53) in (47), we obtain

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_j \left[1 - \frac{1}{2c^2}(\vec{v}_i - \vec{v}_j)^2 - \frac{1}{2c^2}\vec{v}_j^{\,2}\right]$$
$$\times \int_{-\infty}^{\infty} d\zeta \Big\{ {}^0U(\zeta) + \frac{1}{c}[2\zeta\,{}^0U_\sigma\vec{v}_j\cdot\vec{r} + {}^0U_\chi(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]$$
$$+ \frac{1}{2c^2}\{2\,{}^0U_\sigma(\vec{v}_j\cdot\vec{r})^2 - 2\zeta^2\,{}^0U_\sigma\vec{v}_j^{\,2} + 4\zeta^2\,{}^0U_{\sigma\sigma}(\vec{v}_j\cdot\vec{r})^2 + 4\zeta\,{}^0U_{\sigma\chi}[\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r} + (\vec{v}_j\cdot\vec{r})^2]$$
$$+ {}^0U_\omega(\vec{v}_i - \vec{v}_j)^2 + \zeta\,{}^0U_\chi(\vec{v}_j^{\,2} - \vec{v}_i^{\,2}) + {}^0U_{\chi\chi}[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2\}\Big\}\Big|_+ . \tag{54}$$

Here, a Lagrange expansion of all of the terms which are already of order c^{-2} can be trivially made from (46) by the simple substitution of t_j for t_i. However, those of order c^0 and c^{-1} require more effort. The Lagrange expansion of the term of order c^0 is

$${}^0U(\zeta)|_+ \approx \overline{U}(\zeta, r) + \frac{1}{c}\left[\left(\zeta + \frac{1}{c}\vec{v}_j\cdot\vec{r}\right)\frac{\partial\,{}^0U(\zeta)}{\partial t_i}\right]\Big|_{t_i=t_j\equiv t} + \frac{1}{2c^2}\left\{\frac{\partial}{\partial t_i}\left[\zeta^2\frac{\partial\,{}^0U(\zeta)}{\partial t_i}\right]\right\}\Big|_{t_i=t_j\equiv t}, \tag{55}$$

where the notation $\overline{U}(\zeta, r)$ means precisely

$$\overline{U}(\zeta, r) \equiv {}^0U(\zeta)|_{t_i=t_j\equiv t} \equiv U[\zeta^2 - \vec{r}^{\,2}(t,t), 1, -\zeta, \zeta], \tag{56}$$

where ${}^0U(\zeta)$ was defined in (49). The derivatives necessary for (55) are

$$\frac{\partial\,{}^0U(\zeta)}{\partial t_i} = {}^0U_\sigma(-2r)\left(\frac{\vec{v}_i\cdot\vec{r}}{r}\right),$$
$$\frac{\partial^2\,{}^0U(\zeta)}{\partial t_i^2} = -2[{}^0U_\sigma(\vec{a}_i\cdot\vec{r} + \vec{v}_i^{\,2}) + \vec{v}_i\cdot\vec{r}(-2\vec{v}_i\cdot\vec{r})\,{}^0U_{\sigma\sigma}], \tag{57}$$

so that (55) becomes

$${}^0U(\zeta)|_+ \approx \overline{U}(\zeta, r) + \frac{1}{c}\left(\zeta + \frac{1}{c}\vec{v}_j\cdot\vec{r}\right)(-2\overline{U}_\sigma\vec{v}_i\cdot\vec{r}) + \frac{1}{2c^2}\zeta^2(-2)[\overline{U}_\sigma(\vec{a}_i\cdot\vec{r} + \vec{v}_i^{\,2}) - 2(\vec{v}_i\cdot\vec{r})^2\overline{U}_{\sigma\sigma}]. \tag{58}$$

In (58) quantities which are coefficients of, e.g., $\bar{U}_\sigma$ should also have bars over them to indicate that $t_i = t_j \equiv t$. But since every term contains a U with a bar, and in order to prevent the notation from becoming unnecessarily cumbersome, it is to be understood that henceforth all quantities which are coefficients of a $\bar{U}$ are evaluated at $t_i = t_j \equiv t$. In a similar manner the Lagrange expansion of the terms of order c^{-1} is

$$\frac{1}{c}{}^0U'|_+ = \frac{1}{c}[2\zeta\vec{v}_j\cdot\vec{r}\,{}^0U_\sigma + (\vec{v}_i-\vec{v}_j)\cdot\vec{r}\,{}^0U_\chi]|_+$$

$$\approx \frac{1}{c}2\zeta\vec{v}_j\cdot\vec{r}\bar{U}_\sigma + \frac{1}{c}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}\bar{U}_\chi$$

$$+\frac{\zeta}{2c^2}\{4\zeta\vec{v}_i\cdot\vec{v}_j\bar{U}_\sigma - 8\zeta\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\bar{U}_{\sigma\sigma} - 4[(\vec{v}_i\cdot\vec{r})^2 - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\bar{U}_{\chi\sigma} + 2(\vec{a}_i\cdot\vec{r} + \vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j)\bar{U}_\chi\}. \quad (59)$$

Using (58) and (59) in (54) and simplifying, we obtain

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \Big\{\bar{U}(\zeta, r) + \frac{1}{c}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}(\bar{U}_\chi - 2\zeta\bar{U}_\sigma)$$

$$+\frac{1}{2c^2}\{(\vec{v}_i-\vec{v}_j)^2(\bar{U}_\omega - \bar{U} + \zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) - \vec{v}_j^{\,2}\bar{U} + (\vec{v}_j\cdot\vec{r})^2 2\bar{U}_\sigma - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,4\bar{U}_\sigma$$

$$+[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2(\bar{U}_{\chi\chi} - 4\zeta\bar{U}_{\chi\sigma} + 4\zeta^2\bar{U}_{\sigma\sigma}) + \vec{a}_i\cdot\vec{r}(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma)\}\Big\}. \quad (60)$$

The c^{-1}-order terms constitute a total time derivative, since

$$\frac{r}{c}(\bar{U}_\chi - 2\zeta\bar{U}_\sigma)\frac{1}{r}(\vec{v}_i-\vec{v}_j)\cdot\vec{r} = \frac{r}{c}(\bar{U}_\chi - 2\zeta\bar{U}_\sigma)\frac{dr}{dt} = \frac{d}{dt}\left\{\int^{r(t)} dr\left[\frac{r}{c}(\bar{U}_\chi - 2\zeta\bar{U}_\sigma)\right]\right\}, \quad (61)$$

and thus can be omitted from the variational principle.

The acceleration dependence of Eq. (60) can be removed by an integration by parts,

$$\int_{-\infty}^{\infty} dt\,\vec{a}_i\cdot\vec{r}(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) = -\int_{-\infty}^{\infty} dt\,\vec{v}_i\cdot\frac{d}{dt}[\vec{r}(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma)]$$

$$= \int_{-\infty}^{\infty} dt\left\{(-\vec{v}_i^{\,2} + \vec{v}_i\cdot\vec{v}_j)(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) + [-(\vec{v}_i\cdot\vec{r})^2 + \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{d}{dr}(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma)\right\}, \quad (62)$$

where it is assumed that U and its derivatives vanish sufficiently rapidly at infinity that the integrated term can be omitted; a similar assumption will be made for the integration (70) below. Inserting this into (60) gives

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \Big\{\bar{U}(\zeta, r) - \frac{1}{2c^2}\Big[\vec{v}_i\cdot\vec{v}_j\bar{U}(\zeta, r) - \vec{v}_i\cdot\vec{v}_j\bar{U}(\zeta, r)$$

$$+\vec{v}_j^{\,2}\bar{U}(\zeta, r) + (\vec{v}_i-\vec{v}_j)^2[\bar{U}(\zeta, r) - \bar{U}_\omega - \zeta\bar{U}_\chi + 2\zeta^2\bar{U}_\sigma]$$

$$+[\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j](2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) + 2[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma)_\sigma$$

$$+[(\vec{v}_i\cdot\vec{r})^2 - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{d}{dr}(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma)$$

$$+[4\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r} - 2(\vec{v}_j\cdot\vec{r})^2]\bar{U}_\sigma - [(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2\bar{U}_{\chi\chi}\Big]\Big\}. \quad (63)$$

Using the relation

$$\frac{d\bar{U}(\zeta, r)}{dr} = \bar{U}_\sigma\frac{d^0\sigma}{dr} = -2r\bar{U}_\sigma, \quad (64)$$

where $\bar{U}(\zeta, r)$ was defined by (56), and rearranging terms turns (63) into

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \bar{U}(\zeta, r)$$
$$+\frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\zeta \Big\{ \vec{v}_i\cdot\vec{v}_j \bar{U}(\zeta,r) - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\frac{1}{r}\frac{d\bar{U}(\zeta,r)}{dr}$$
$$+(\vec{v}_i-\vec{v}_j)^2[\bar{U}(\zeta,r)-\bar{U}_\omega-\zeta\bar{U}_\chi+2\zeta^2\bar{U}_\sigma]$$
$$+(\vec{v}_j^{\,2}-\vec{v}_i\cdot\vec{v}_j)\bar{U}(\zeta,r)+(\vec{v}_i^{\,2}-\vec{v}_i\cdot\vec{v}_j)(2\zeta\bar{U}_\chi-2\zeta^2\bar{U}_\sigma)$$
$$+[2\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}-(\vec{v}_j\cdot\vec{r})^2-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{d}{dr}(2\zeta\bar{U}_\chi-2\zeta^2\bar{U}_\sigma)$$
$$+[(\vec{v}_j\cdot\vec{r})^2-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{d\bar{U}(\zeta,r)}{dr}-[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2\bar{U}_{\chi\chi}\Big\}\,. \quad (65)$$

Comparing (36) and (65) to the Newtonian variational principle (8) makes obvious the identification

$$V_{ij}(r) \equiv \int_{-\infty}^{\infty} d\zeta\, \bar{U}(\zeta, r)\,, \quad (66)$$

so that $g_i g_j V_{ij}(r)$ is the Newtonian two-particle energy.

In order to relate terms in (65), we now prove the identity

$$\int_{-\infty}^{\infty} d\zeta(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) \equiv V_{ij}(r) + W_{ij}(r)\,, \quad (67a)$$

where

$$W_{ij}(r) \equiv \int_{-\infty}^{\infty} d\zeta\, \zeta(\bar{U}_\chi + \bar{U}_\zeta)\,. \quad (67b)$$

Using the "chain rule" on $\zeta d\bar{U}/d\zeta$ yields

$$\zeta\frac{d\bar{U}}{d\zeta} = \zeta[2\zeta\bar{U}_\sigma + 0 - \bar{U}_\chi + \bar{U}_\zeta]\,. \quad (68)$$

Rearranging and integrating, we obtain

$$\int_{-\infty}^{\infty} d\zeta(2\zeta\bar{U}_\chi - 2\zeta^2\bar{U}_\sigma) = W_{ij}(r) - \int_{-\infty}^{\infty} d\zeta\,\zeta\frac{d\bar{U}}{d\zeta}\,. \quad (69)$$

But when the last term of (69) is integrated by parts, there results

$$-\int_{-\infty}^{\infty} d\zeta\,\zeta\frac{d\bar{U}}{d\zeta} = \int_{-\infty}^{\infty} d\zeta\,\bar{U} = V_{ij}(r)\,, \quad (70)$$

as long as $\zeta\bar{U}(\zeta, r)$ goes to zero fast enough at infinity. Using (70) in (69) proves (67a). Using (66) and (67) in (65) yields

$$I_1 \approx -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt\, V_{ij}(r)$$
$$+\frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \Big\{\vec{v}_i\cdot\vec{v}_j V_{ij}(r) - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\frac{1}{r}\frac{dV_{ij}}{dr}$$
$$+(\vec{v}_i-\vec{v}_j)^2\int_{-\infty}^{\infty} d\zeta[\bar{U}-\bar{U}_\omega+\zeta\bar{U}_\chi-(2\zeta\bar{U}_\chi-2\zeta^2\bar{U}_\sigma)]$$
$$+(\vec{v}_j^{\,2}-\vec{v}_i\cdot\vec{v}_j)V_{ij}(r)+(\vec{v}_i^{\,2}-\vec{v}_i\cdot\vec{v}_j)[V_{ij}(r)+W_{ij}(r)]$$
$$+[(\vec{v}_j\cdot\vec{r})^2-\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{dV_{ij}}{dr}$$
$$-[(\vec{v}_j\cdot\vec{r})^2-(\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r})]\frac{1}{r}\frac{d}{dr}(V_{ij}+W_{ij})-[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}]^2\int_{-\infty}^{\infty} d\zeta\bar{U}_{\chi\chi}\Big\}\,. \quad (71)$$

The coefficient of $(\vec{v}_i - \vec{v}_j)^2$ in (71) can be rearranged, using (66) and (67), as

$$\bar{U} - \bar{U}_\omega + \zeta\bar{U}_\chi - (\bar{U} + \zeta\bar{U}_\chi + \zeta\bar{U}_\zeta) = -\bar{U}_\omega - \zeta\bar{U}_\zeta \,. \tag{72}$$

Using (72) in (71) and combining terms appropriately gives

$$\begin{aligned} I_1 \approx & -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt\, V_{ij}(r) \\ & + \frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt \Big\{ \vec{v}_i\cdot\vec{v}_j V_{ij}(r) - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\,\frac{1}{r}\frac{dV_{ij}}{dr} + (\vec{v}_i - \vec{v}_j)^2 \Big[V_{ij}(r) + \int_{-\infty}^{\infty} d\zeta\,(-\bar{U}_\omega - \zeta\bar{U}_\zeta)\Big] \\ & - [(\vec{v}_i - \vec{v}_j)\cdot\vec{r}]^2 \int_{-\infty}^{\infty} d\zeta\,\bar{U}_{\chi\chi} + (\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j) W_{ij}(r) - [(\vec{v}_j\cdot\vec{r})^2 - \vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}]\frac{1}{r}\frac{dW_{ij}(r)}{dr}\Big\}. \end{aligned} \tag{73}$$

We define

$$X_{ij}(r_{ij}) \equiv -\int_{-\infty}^{\infty} d\zeta\,(\bar{U}_\omega + \zeta\bar{U}_\zeta) \tag{74a}$$

and

$$Y_{ij}(r_{ij}) \equiv -\int_{-\infty}^{\infty} d\zeta\,\bar{U}_{\chi\chi} \,. \tag{74b}$$

By comparing (36) and (73) to (30), the approximately relativistic Lagrangian L can be identified as

$$L = L_2 - V + I_{\rm PN} \,, \tag{75a}$$

where (following the notation of HS)

$$L_2 = -c^2 \sum_i m_i \left(1 - \frac{1}{2}\frac{\vec{v}_i^{\,2}}{c^2} - \frac{1}{8}\frac{\vec{v}_i^{\,4}}{c^4}\right), \tag{75b}$$

V is the Newtonian potential energy as given in (8), and the "post-Newtonian" interaction $I_{\rm PN}$ is given by

$$\begin{aligned} I_{\rm PN} = \frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \Big\{ & \vec{v}_i\cdot\vec{v}_j V_{ij}(r_{ij}) - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}} + (\vec{v}_i - \vec{v}_j)^2 [V_{ij}(r_{ij}) + X_{ij}(r_{ij})] \\ & + [(\vec{v}_i - \vec{v}_j)\cdot\vec{r}_{ij}]^2 Y_{ij}(r_{ij}) + (\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j) W_{ij}(r_{ij}) - [(\vec{v}_j\cdot\vec{r}_{ij})^2 - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}]\frac{1}{r_{ij}}\frac{dW_{ij}(r_{ij})}{dr_{ij}}\Big\} \,, \end{aligned} \tag{75c}$$

where for clarity the ij subscripts have been returned to r. The last two terms of (75c) reflect, in order c^{-2}, whatever asymmetry in χ_{ij} and ζ_{ij} exists in $U_{ij}(\sigma_{ij}, \omega_{ij}, \chi_{ij}, \zeta_{ij})$.

The term $\sum_i(-m_i c^2)$ in (75a) is constant and thus irrelevant for the equations of motion following from the variational principle (30). It thus could be omitted (and indeed must be before we can go to the Newtonian limit). However, we prefer to retain it both for ease of comparison with HS and to obtain the correct approximation to the relativistic energies of the individual particles as well as to the relativistic Hamiltonian in Sec. V; indeed, for the canonical formalism retention of this term is essential.[26]

As noted in Sec. II, there may be more than one type of interaction in the interaction term I_1 of (6). This would lead to appropriate summations over the various types of interactions within V as well as $I_{\rm PN}$ in Eq. (75).

IV. SPECIAL CASES OF APPROXIMATELY RELATIVISTIC LAGRANGIANS

The form of the approximately relativistic Lagrangian appropriate to Lorentz-invariant variational principles which allow definition of "adjunct fields"[11-13] can be obtained as a special case of the general approximate Lagrangian just derived. Such special cases are of interest, since, as noted before, both classical and quantum field theory have been emphasized during the last half-century.

Furthermore, for the purpose of comparison, a special case with a spacelike interaction will be calculated using the form (75). This case is useful here since the form of the interaction is sufficiently simple so that the derivation can be done alternatively by integrating the variational principle (34) exactly and then expanding in powers of c^{-1} (rather than expanding under the integral as in Sec. III). The results agree as required.

The special case of theories possessing adjunct fields can be obtained from the principles (6) or (23) provided a generalized potential V_i can be defined which can be separated into a sum of terms which are products of two factors, one of which depends on particle i only through its coordinates z_i^μ; then this factor, with the dependence on z_i^μ replaced by x^μ, can be considered as an adjunct potential defined at all points in space and determined by sources, as customary in field theory. However, unlike customary field theory, there is no *single* field, but the field adjunct to particle i involves as sources all particles other than i; on the other hand, this is precisely the property needed to avoid infinities for point sources. Furthermore, for macroscopic applications, i.e., for sources consisting of a very large number of particles, the difference between the various adjunct fields is negligible.

The desired separation is possible for all i and j provided that U_{ij} consists of a sum of terms each of which contains ω_{ij} only in the form $\omega_{ij}^{l_{ij}}$, and κ_i and κ_j (or equivalently χ_{ij} and ζ_{ij}) in the forms $\kappa_i^{m_{ij}}$ and $\kappa_j^{m_{ji}}$ (or $\chi_{ij}^{m_{ij}}$ and $\zeta_{ij}^{m_{ji}}$), where l_{ij} (= l_{ji}), m_{ij}, and m_{ji} are nonnegative integers. If all particles are to behave as if they were field sources which are similar except for the strength of their coupling (described by the g_i's), we must have $m_{ij} = m_{ji}$, and these numbers as well as the l_{ij} should be independent of i and j; thus the subscripts are no longer needed. (For simplicity, we exclude the possibility of sources which differ in some respects such as their multipole structure.) We have then

$$V_i^{(lm)}(z_i^\rho, v_i^\rho) = v_{i\alpha} v_{i\beta} \cdots v_{i\lambda} v_{i\alpha'} v_{i\beta'} \cdots v_{i\mu} \times \phi_i^{\alpha\beta\cdots\lambda\alpha'\beta'\cdots\mu}(z_i^\rho),$$

$$\phi_i^{\alpha\beta\cdots\lambda\alpha'\beta'\cdots\mu} = \sum_{j<i} g_j \int_{-\infty}^{\infty} d\tau_j v_j^\alpha v_j^\beta \cdots v_j^\lambda s_{ji}^{\alpha'} s_{ji}^{\beta'} \cdots s_{ji}^\mu \kappa_j^m \phi_{ji}^{(lm)}(s_{ij}) + \sum_{j>i} g_j \int_{-\infty}^{\infty} d\tau_j v_j^\alpha v_j^\beta \cdots v_j^\lambda s_{ji}^{\alpha'} s_{ji}^{\beta'} \cdots s_{ji}^\mu \kappa_j^m \phi_{ij}^{(lm)}(s_{ij}), \tag{76}$$

where $\phi_{ij}^{\alpha\beta\cdots\lambda\alpha'\beta'\cdots\mu}$ is of rank $l+m$. These equations are of the appropriate form (14) used before, except that for purposes of defining a "field" quantity a common term depending only on $v_{i\rho}$ was factored out from U_{ij} and U_{ji}; clearly this has no effect on the calculations of Secs. II and III.

The adjunct potential $\phi_i^{\alpha\beta\cdots\lambda\alpha'\beta'\cdots\mu}(x^\rho)$ is obtained from $\phi_i^{\alpha\beta\cdots\lambda\alpha'\beta'\cdots\mu}(z_i^\rho)$ by replacement of z_i^ρ by x^ρ everywhere in the integrands, as discussed above. Its essential property which justifies considering it as a field quantity is precisely its being a function of the x^ρ alone, as well as its implicit dependence on "sources." The question of the existence of suitable (partial differential or other) equations determining it directly from the source distribution is secondary, and of no direct concern for a theory which considers a variational principle of the type (6) as basic. Nevertheless, it is of importance for the interpretation of the theory to be able to establish a connection with known field equations in some cases. In particular, this is possible if $m=0$ and if the $\phi_{ij}^{(l0)}$, considered as functions G_j of $x^\rho - z_j^\rho$ rather than s_{ij}^ρ, are Green functions of some linear partial differential equation, i.e., if

$$\mathcal{L} G_j[\eta_{\mu\nu}(x^\mu - z_j^\mu)(x^\nu - z_j^\nu)] = 4\pi\delta^4(x^\rho - z_j^\rho), \tag{77}$$

where $\mathcal{L}$ is a linear differential operator and δ^4 is a fourfold product of Dirac δ functions. Then

$$\mathcal{L}\phi_i^{\alpha\beta\cdots\lambda}(x^\rho) = 4\pi \sum_{j\neq i} g_j \int_{-\infty}^{\infty} d\tau_j v_j^\alpha v_j^\beta \cdots v_j^\lambda \delta^4[x^\rho - z_j^\rho(\tau_j)] \equiv 4\pi j_i^{\alpha\beta\cdots\lambda}, \tag{78}$$

where $j_i^{\alpha\beta\cdots\lambda}$ is the source density of the adjunct field of the ith particle. Special cases of theories of direct particle interaction which allow such association with known field theories are Fokker's principle of electrodynamics[11,2] ($l=1$, $G_j=\delta$), and the principles of scalar or vector mesodynamics[12] ($l=0$ or 1). However, more complicated associations with field equations, even for $m\neq 0$, are also possible.[34]

Thus, for interactions corresponding to the generalized potential (76) which can be related to a field theory, the exact variational principle (23) takes the form

$$I_1^{(lm)} = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_i d\tau_j v_i^{1-m} v_j^{1-m} \times \omega_{ij}^l \chi_{ij}^m \zeta_{ij}^m \phi_{ij}^{(lm)}(\sigma_{ij}), \tag{79a}$$

$$I_2 = -\sum_i \int_{-\infty}^{\infty} d\tau_i m_i c^2 v_i. \tag{79b}$$

In order to calculate the approximate Lagrangian appropriate to (79) we need

$$\begin{aligned} &\overline{U}^{(lm)} = (-\zeta)^m \zeta^m \phi^{(lm)}(\zeta^2 - \vec{r}^2), \\ &\overline{U}_\omega^{(lm)} = l\overline{U}^{(lm)}, \quad \zeta\overline{U}_\zeta^{(lm)} = m\overline{U}^{(lm)}, \\ &\zeta\overline{U}_\chi^{(lm)} = -m\overline{U}^{(lm)}, \\ &\overline{U}_{\chi\chi}^{(lm)} = m(m-1)(-\zeta)^{m-2}\zeta^m\phi^{(lm)}(\zeta^2 - \vec{r}^2), \end{aligned} \tag{80}$$

which can be used in (66), (67), and (74) to yield

$$V_{ij}^{(lm)}(r_{ij}) = \int_{-\infty}^{\infty} d\zeta\,(-\zeta)^m \zeta^m \phi_{ij}^{(lm)}(\zeta^2 - \vec{r}_{ij}^{\,2}),$$

$$W_{ij}^{(lm)}(r_{ij}) = 0, \quad X_{ij}^{(lm)}(r_{ij}) = -(l+m)V_{ij}^{(lm)}, \quad \text{(81a)}$$

$$Y_{ij}^{(lm)}(r_{ij}) = \frac{m(1-m)}{2m-1}\frac{1}{r_{ij}}\frac{dV_{ij}^{(lm)}(r_{ij})}{dr_{ij}}.$$

The relation between $Y_{ij}^{(lm)}$ and $V_{ij}^{(lm)}$ follows most easily by direct differentiation of $V_{ij}^{(lm)}$ and comparison with the definition (74b). Inserting Eq. (81a) into Eq. (75c) we obtain

$$I_{\rm PN} \equiv I^{(lm)} = \frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \Big\{[(1-l-m)(\vec{v}_i - \vec{v}_j)^2 + \vec{v}_i\cdot\vec{v}_j]V_{ij}^{(lm)}(r_{ij}) - \Big(\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij} - \frac{m(1-m)}{2m-1}[(\vec{v}_i - \vec{v}_j)\cdot\vec{r}_{ij}]^2\Big)\frac{1}{r_{ij}}\frac{dV_{ij}^{(lm)}}{dr_{ij}}\Big\}. \tag{81b}$$

For the particular case $m=0$, which may allow field equations of the form (78), we have (omitting the superfluous index m)

$$I^{(l)} = \frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \Big\{[(1-l)(\vec{v}_i - \vec{v}_j)^2 + \vec{v}_i\cdot\vec{v}_j]V_{ij}^{(l)}(r_{ij}) - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}^{(l)}}{dr_{ij}}\Big\}. \tag{81c}$$

The relation of these results to the general form of the interaction suggested by HS is discussed in Sec. VI.

An example of a spacelike interaction follows from a U_{ij} of the form

$$U_{ij} = \omega_{ij}^{l}\,\delta(\chi_{ij}) f_{ij}[(-\sigma_{ij})^{1/2}], \tag{82}$$

where f_{ij} is an arbitrary analytic function of its argument. Its spacelike quality is due to $\delta(\chi_{ij})$, since it follows from Eq. (32) that χ_{ij} vanishes whenever

$$t_{ij} = \frac{1}{c^2}\,\vec{v}_i(t_i)\cdot\vec{r}_{ij}(t_i, t_j). \tag{83}$$

This should be contrasted with the case of electrodynamics, where the interaction involves $\delta(\sigma_{ij})$, and thus

$$t_{ij} = \pm\frac{1}{c}\,r_{ij}(t_i, t_j) \tag{84}$$

whenever σ_{ij} vanishes.

In order to evaluate Eq. (75c) for the interaction (82), we need

$$\begin{aligned} &\overline{U} = \delta(-\zeta) f_{ij}[(\vec{r}^{\,2} - \zeta^2)^{1/2}], \\ &\overline{U}_\omega = -l\delta(-\zeta) f_{ij}[(\vec{r}^{\,2} - \zeta^2)^{1/2}], \quad \overline{U}_\zeta = 0, \\ &\zeta\overline{U}_\chi = -\zeta\frac{d\delta(\zeta)}{d\zeta} f_{ij}[(\vec{r}^{\,2} - \zeta^2)^{1/2}], \qquad \overline{U}_{\chi\chi} = \frac{d^2\delta(-\zeta)}{d(-\zeta)^2} f_{ij}[(\vec{r}^{\,2} - \zeta^2)^{1/2}], \end{aligned} \tag{85}$$

which can be used in (66), (67), and (74) to yield

$$\begin{aligned} &V_{ij}(r_{ij}) = f_{ij}(r_{ij}), \quad X_{ij}(r_{ij}) = -l\,V_{ij}(r_{ij}), \\ &Y_{ij}(r_{ij}) = \frac{1}{r_{ij}}\frac{dV_{ij}(r_{ij})}{dr_{ij}}, \quad W_{ij}(r_{ij}) = V_{ij}(r_{ij}). \end{aligned} \tag{86a}$$

Inserting this in Eq. (75c), we obtain

$$I_{\rm PN} = \frac{1}{2c^2}\sum_{i<j}\sum g_i g_j \Big\{[(1-l)(\vec{v}_i - \vec{v}_j)^2 + \vec{v}_i^{\,2}]V_{ij} + [(\vec{v}_i\cdot\vec{r}_{ij})^2 - 2\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}]\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\Big\}. \tag{86b}$$

In general, it is not necessary that W_{ij} (or X_{ij} or Y_{ij}) depend on V_{ij}; it just happens to be so for the particular interaction kernel (82).

This kernel was chosen as an example of a spacelike interaction because it is sufficiently simple to allow an exact integration to be performed. The exact expression then can be expanded to obtain the same Lagrangian (75) with (86b).

Using (82) in (34b) gives

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\,d\tau_i\,d\tau_j\,\delta(\chi_{ij}) f_{ij}[(-\sigma_{ij})^{1/2}]\omega_{ij}^{l}\,, \tag{87}$$

which, by virtue of Eq. (32) and the properties of the δ function, can be written

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_i\,d\tau_j\,\delta(\kappa_i) f_{ij}[(-\sigma_{ij})^{1/2}]\omega_{ij}^{l}\,. \tag{88}$$

A change of integration variable from τ_j to κ_i involves

$$\frac{d\kappa_i}{d\tau_j} = \frac{d}{d\tau_j}(v_i^{\alpha} s_{ji\alpha}) = \omega_{ij}\,, \tag{89}$$

which changes Eq. (88) to

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} d\tau_i \int_{-\infty}^{\infty} d\kappa_i\,\delta(\kappa_i)\{f_{ij}[(-\sigma_{ij})^{1/2}]\omega_{ij}^{l-1}\}\big|_{t_j=t_i+\kappa_i\gamma_i^{-1}-\vec{v}_i\cdot\vec{r}_{ij}/c^2}\,. \tag{90}$$

The κ_i integration can be done immediately, resulting in

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} d\tau_i\{f_{ij}[(-\sigma_{ij})^{1/2}]\omega_{ij}^{l-1}\}\big|_{-}\,, \tag{91}$$

where $|_-$ now means evaluated with t_j given by

$$t_j = t_i - \frac{1}{c^2}\vec{v}_i(t_i)\cdot\vec{r}_{ij}(t_i, t_j)\,, \tag{92}$$

which is the same as (83).

In Eq. (91) the quantity in curly brackets can be expanded in a Taylor series as

$$\left\{f_{ij}\left[r\left(1-\frac{(\vec{v}_i\cdot\vec{r})^2}{\vec{r}^{\,2}c^2}\right)^{1/2}\right]\omega_{ij}^{l-1}\right\}\bigg|_{-} \approx \left\{\left[f_{ij}(r) - \frac{1}{2c^2}(\vec{v}_i\cdot\vec{r})^2\frac{1}{r}f'_{ij}(r)\right]\left[1+\frac{l-1}{2c^2}(\vec{v}_i-\vec{v}_j)^2\right]\right\}\bigg|_{-}\,, \tag{93}$$

using the convention (40). A Lagrange expansion can then be made, resulting in

$$\left[f_{ij}(r) - \frac{\vec{v}_i\cdot\vec{r}}{c^2}f'_{ij}(r)(-1)\frac{\vec{v}_j\cdot\vec{r}}{r} - \frac{1}{2c^2}(\vec{v}_i\cdot\vec{r})^2\frac{f'_{ij}(r)}{r} + \frac{l-1}{2c^2}(\vec{v}_i-\vec{v}_j)^2 f_{ij}(r)\right]\bigg|_{t_j=t_i}\,. \tag{94}$$

Using this in Eq. (91) yields

$$I_1 = -\sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt_i\gamma_i^{-1}\left\{f_{ij}(r) + 2\vec{v}_i\cdot\vec{r}\,\vec{v}_j\cdot\vec{r}\frac{f'_{ij}(r)}{2rc^2} - (\vec{v}_i\cdot\vec{r})^2\frac{f'_{ij}(r)}{2rc^2} + \frac{l-1}{2c^2}(\vec{v}_i-\vec{v}_j)^2 f_{ij}(r)\right\}\,, \tag{95}$$

which can be written, expanding γ_i^{-1} and omitting the subscript i from the integration variable, as

$$I_1 = \sum_{i<j}\sum g_i g_j \int_{-\infty}^{\infty} dt\left\{-f_{ij}(r) + \frac{1}{2c^2}\{(1-l)(\vec{v}_i-\vec{v}_j)^2 f_{ij}(r) + \vec{v}_i^{\,2} f_{ij}(r) + [(\vec{v}_i\cdot\vec{r}_{ij})^2 - 2\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}]\frac{1}{r}f'_{ij}(r)\}\right\}\,. \tag{96}$$

This is indeed the required integral of $-V + l_{\rm PN}$ of Eqs. (75), as given by Eqs. (86).

V. THE APPROXIMATE CONSERVATION THEOREMS

The exact variational principles (6) or (23) are invariant under the full ten-parameter Lorentz group and thus, according to Noether's theorem,[35] imply ten exact conservation laws. Using the method of Dettman and Schild,[36] these conservation laws were determined in H.[37]

Since the approximate Lagrangian (75) is clearly invariant under a group consisting of the time and space translations and rotations, Noether's theorem similarly establishes the form of the conservation of energy, linear momentum, and angular momentum, respectively. These forms are[24]

$$\frac{dE}{dt}=0, \quad E\equiv\sum_i \vec{p}_i\cdot\vec{v}_i-L, \quad \vec{p}_i\equiv\frac{\partial L}{\partial\vec{v}_i}\,, \tag{97a}$$

$$\frac{d\vec{P}}{dt}=0, \quad \vec{P}\equiv\sum_i \vec{p}_i\,, \tag{97b}$$

$$\frac{d\vec{J}}{dt}=0, \quad \vec{J}\equiv\sum_i \vec{r}_i\times\vec{p}_i\,. \tag{97c}$$

Equations (97) apply equally to Newtonian and approximately relativistic Lagrangians. However, the latter are not exactly invariant under either the usual Galilei transformations or the Lorentz transformations relating different inertial frames. Such transformations generate, via Noether's theorem, the center-of-mass theorem for Galilei- and Lorentz-invariant particle systems.

But, as noted in Sec. I, such a Lagrangian is invariant under another three-parameter set of infinitesimal transformations which lead to a center-of-mass theorem,[24] given by

$$\frac{d\vec{G}}{dt}=0, \quad \vec{G}\equiv\frac{E}{c^2}\vec{R}-\vec{P}t, \tag{97d}$$

where $\vec{R}$ could be calculated from

$$\vec{R}(t)=\frac{c^2}{E}\int^t d\bar{t}\;\vec{P}[\vec{r}_i(\bar{t})] \tag{98a}$$

or

$$\vec{R}(t)=c^2\int^t d\bar{t}\;\frac{\vec{P}[\vec{r}_i(\bar{t})]}{E[\vec{r}_i(\bar{t})]}\,, \tag{98b}$$

along with the equations of motion which follow from the approximate variational principle. Equations (97) and (98) will be used to generate the forms of the approximately conserved quantities associated with (75). These forms agree with those approximated directly from the Lorentz-invariant conserved quantities.[38]

The canonical momentum of the ith particle is

$$\begin{aligned}\vec{p}_i\equiv\frac{\partial L}{\partial\vec{v}_i}&=m_i\vec{v}_i+\frac{1}{c^2}(\tfrac{1}{2}m_i\vec{v}_i^{\,2})\vec{v}_i\\ &+\frac{1}{2c^2}\sum_{j<i}g_ig_j\bigg\{\vec{v}_jV_{ji}-(\vec{v}_j\cdot\vec{r}_{ji})\vec{r}_{ji}\frac{1}{r_{ji}}\frac{dV_{ji}}{dr_{ji}}+2(\vec{v}_j-\vec{v}_i)(-1)(V_{ji}+X_{ji})\\ &\qquad+2[(\vec{v}_j-\vec{v}_i)\cdot\vec{r}_{ji}](-\vec{r}_{ji})Y_{ji}-\vec{v}_jW_{ji}-[2(\vec{v}_i\cdot\vec{r}_{ji})\vec{r}_{ji}-(\vec{v}_j\cdot\vec{r}_{ji})\vec{r}_{ji}]\frac{1}{r_{ji}}\frac{dW_{ji}}{dr_{ji}}\bigg\}\\ &+\frac{1}{2c^2}\sum_{j>i}g_jg_i\bigg\{\vec{v}_jV_{ij}-(\vec{v}_j\cdot\vec{r}_{ij})\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}+2(\vec{v}_i-\vec{v}_j)(V_{ij}+X_{ij})+2[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}]\vec{r}_{ij}Y_{ij}\\ &\qquad+(2\vec{v}_i-\vec{v}_j)W_{ij}-(-\vec{v}_j\cdot\vec{r}_{ij})\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\bigg\}\,.\end{aligned} \tag{99}$$

Using this in (97b) and simplifying the results gives

$$\begin{aligned}\vec{P}\equiv\sum_i\vec{p}_i=\sum_i\left(m_i+\tfrac{1}{2}m_i\frac{\vec{v}_i^{\,2}}{c^2}\right)\vec{v}_i&+\frac{1}{2c^2}\sum_{i<j}\sum g_ig_j\left[(\vec{v}_i+\vec{v}_j)V_{ij}-\vec{r}_{ij}(\vec{v}_i+\vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\right]\\ &+\frac{1}{2c^2}\sum_{i<j}\sum g_ig_j\left[(\vec{v}_i-\vec{v}_j)W_{ij}+\vec{r}_{ij}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\right],\end{aligned} \tag{100}$$

and consequently $\vec{P}$ does not depend on the X_{ij} or Y_{ij}. Equation (66) shows that V_{ij} depends on i and j only through its argument r_{ij}, which is inherently symmetric in i and j. Thus with the usual definition

$$V_{ji}\equiv V_{ij}, \tag{101}$$

Eq. (100) can be rewritten as

$$\vec{P}\equiv\vec{P}_V+\vec{P}_W, \tag{102a}$$

where

$$\vec{P}_V \equiv \sum_i \left(m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2}\right)\vec{v}_i + \frac{1}{2c^2}\sum\sum_{i\neq j} g_i g_j \left[\vec{v}_j V_{ij} - \vec{r}_{ij}(\vec{v}_j\cdot\vec{r}_{ij})\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\right] \tag{102b}$$

is the total linear momentum given by HS and

$$\vec{P}_W \equiv \frac{1}{2c^2}\sum\sum_{i<j} g_i g_j \left[(\vec{v}_i - \vec{v}_j)W_{ij} + \vec{r}_{ij}(\vec{v}_i - \vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\right]. \tag{102c}$$

The total energy can be calculated by using Eq. (99) in (97a), yielding

$$\begin{aligned} E = &\sum_i \left(m_i \vec{v}_i^{\,2} + \frac{1}{2c^2} m_i \vec{v}_i^{\,4}\right) \\ &+ \frac{1}{2c^2}\sum\sum_{i<j} g_i g_j \Big\{2\vec{v}_i\cdot\vec{v}_j V_{ij} - 2\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}} + 2(\vec{v}_i - \vec{v}_j)^2(V_{ij}+X_{ij}) + 2[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}]^2 Y_{ij} \\ &\qquad + 2(\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j)W_{ij} - 2[(\vec{v}_j\cdot\vec{r}_{ij})^2 - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}]\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\Big\} - L. \end{aligned} \tag{103}$$

Substituting Eq. (75) and combining terms yields finally

$$\begin{aligned} E = &\sum_i \left(m_i c^2 + \tfrac{1}{2} m_i \vec{v}_i^{\,2} + \tfrac{3}{8} m_i \frac{\vec{v}_i^{\,4}}{c^2}\right) \\ &+ \sum\sum_{i<j} g_i g_j V_{ij}(r_{ij}) + \frac{1}{2c^2}\sum\sum_{i<j} g_i g_j \Big\{\vec{v}_i\cdot\vec{v}_j V_{ij} - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}} + (\vec{v}_i - \vec{v}_j)^2(V_{ij}+X_{ij}) \\ &\qquad + [(\vec{v}_i - \vec{v}_j)\cdot\vec{r}_{ij}]^2 Y_{ij} + (\vec{v}_i^{\,2} - \vec{v}_i\cdot\vec{v}_j)W_{ij} \\ &\qquad - [(\vec{v}_j\cdot\vec{r}_{ij})^2 - \vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}]\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\Big\}. \end{aligned} \tag{104}$$

This can be rewritten in terms of the canonical momenta (99) to yield the Hamiltonian

$$\begin{aligned} H = &\sum_i \left(m_i c^2 + \frac{\vec{p}_i^{\,2}}{2m_i} - \frac{1}{8}\frac{\vec{p}_i^{\,4}}{c^2 m_i^3}\right) + \sum\sum_{i<j} g_i g_j V_{ij}(r_{ij}) \\ &- \frac{1}{2c^2}\sum\sum_{i<j} g_i g_j \Big\{\frac{\vec{p}_i\cdot\vec{p}_j}{m_i m_j} V_{ij} - \frac{\vec{p}_i\cdot\vec{r}_{ij}\vec{p}_j\cdot\vec{r}_{ij}}{m_i m_j}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}} \\ &\qquad + \left(\frac{\vec{p}_i}{m_i} - \frac{\vec{p}_j}{m_j}\right)^2 (V_{ij}+X_{ij}) + \left[\left(\frac{\vec{p}_i}{m_i} - \frac{\vec{p}_j}{m_j}\right)\cdot\vec{r}_{ij}\right]^2 Y_{ij} \\ &\qquad + \left(\frac{\vec{p}_i^{\,2}}{m_i^2} - \frac{\vec{p}_i\cdot\vec{p}_j}{m_i m_j}\right)W_{ij} - \left[\left(\frac{\vec{p}_j}{m_j}\cdot\vec{r}_{ij}\right)^2 - \frac{\vec{p}_i\cdot\vec{r}_{ij}\vec{p}_j\cdot\vec{r}_{ij}}{m_i m_j}\right]\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\Big\}. \end{aligned} \tag{105}$$

Using Eq. (99) in (97c) and rearranging we obtain the total angular momentum

$$\begin{aligned} \vec{J} = &\sum_i \left(m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2}\right)\vec{r}_i\times\vec{v}_i \\ &+ \frac{1}{2c^2}\sum\sum_{i<j} g_i g_j \Big\{(\vec{r}_i\times\vec{v}_j + \vec{r}_j\times\vec{v}_i)V_{ij} + \vec{r}_i\times\vec{r}_j(\vec{v}_i+\vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}} \\ &\qquad + 2\vec{r}_{ij}\times(\vec{v}_i - \vec{v}_j)(V_{ij}+X_{ij}) + [\vec{r}_i\times(\vec{v}_i - \vec{v}_j) + \vec{r}_{ij}\times\vec{v}_i]W_{ij} \\ &\qquad - \vec{r}_i\times\vec{r}_j(\vec{v}_i - \vec{v}_j)\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dW_{ij}}{dr_{ij}}\Big\}, \end{aligned} \tag{106}$$

which does not depend on Y_{ij}.

As indicated by HS, either one of Eqs. (98) could be used to calculate $\vec{R}$ in terms of the particle variables. However, HS found that for all of their examples the form (98a) was sufficient. This is true here also.

Their procedure consisted of "guessing" $(E/c^2)\vec{R}$ to be

$$\sum_i \left[m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2} + \frac{1}{2c^2} \sum_{j\neq i} g_i g_j V_{ij}\right]\vec{r}_i \,, \tag{107}$$

and then showing that its total time derivative is the total linear momentum [as demanded by (98a) and (97d)], making use of the Newtonian approximation to the equations of motion.

The form (102) of the linear momentum shows that here this procedure is not sufficient, since the "asymmetry potential" W_{ij} does not occur in Newtonian order. However, the additional momentum $\vec{P}_W$ given by Eq. (102c) is the total time derivative of

$$\frac{1}{2c^2} \sum_{i<j}\sum g_i g_j W_{ij} \vec{r}_{ij} \,. \tag{108}$$

Since HS showed that $\vec{P}_V$, given by Eq. (102b), is the time derivative of (107), in our case the appropriate form for $(E/c^2)\vec{R}$ is the sum of (107) and (108).

Equation (107) has the form of a sum over contributions directed along the N position vectors $\vec{r}_i$. Since a definition of W_{ji} for $j > i$ has not yet been given, it is possible to put Eq. (108) into a similar form by defining

$$W_{ji} \equiv -W_{ij}\,, \quad i<j \tag{109}$$

for then

$$\sum_{i<j}\sum g_i g_j W_{ij} (\vec{r}_i - \vec{r}_j) \equiv \sum_{i\neq j}\sum g_i g_j W_{ij} \vec{r}_i \,. \tag{110}$$

Using this in Eq. (108) and adding (107) we obtain

$$\frac{E}{c^2}\vec{R} = \sum_i \left[m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2} + \frac{1}{2c^2} \sum_{j\neq i} g_i g_j (V_{ij} + W_{ij})\right]\vec{r}_i \,. \tag{111}$$

Thus the conserved center-of-mass quantity $\vec{G}$ is given by Eq. (97d) with (111) and (102); explicitly it equals

$$\begin{aligned}\vec{G} = {} & \sum_i \left[m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2} + \frac{1}{2c^2} \sum_{j\neq i} g_i g_j (V_{ij} + W_{ij})\right]\vec{r}_i \\ & - t\Bigg\{\sum_i \left[m_i + \tfrac{1}{2} m_i \frac{\vec{v}_i^{\,2}}{c^2}\right]\vec{v}_i + \frac{1}{2c^2}\sum_{i\neq j}\sum g_i g_j \left[\vec{v}_j V_{ij} - \vec{r}_{ij}(\vec{v}_j \cdot \vec{r}_{ij}) \frac{1}{r_{ij}} \frac{dV_{ij}}{dr_{ij}}\right] \\ & + \frac{1}{2c^2} \sum_{i<j}\sum g_i g_j \left[(\vec{v}_i - \vec{v}_j) W_{ij} + \vec{r}_{ij}(\vec{v}_i - \vec{v}_j)\cdot \vec{r}_{ij} \frac{1}{r_{ij}} \frac{dW_{ij}}{dr_{ij}}\right]\Bigg\}. \end{aligned} \tag{112}$$

Thus it does not depend on the X_{ij} or Y_{ij}.

VI. DISCUSSION

The main result of this paper is Eq. (75), which is the approximately relativistic Lagrangian associated with Lorentz-invariant variational principles depending at most on the velocities of point particles interacting through two-body forces. The interaction is not necessarily symmetric in the particles' variables and is constructed to have a Newtonian limit. In previous work it has been taken for granted that the relativistic corrections would involve only the static Newtonian potential. However, the Lagrangian (75) includes the possibility of three new functions of r_{ij} for each relativistic particle interaction; none of these is necessarily related to the Newtonian potential. Of these three, $W_{ij}(r_{ij})$ may be the most interesting, since it shows for the first time that effects of a nonsymmetric Lorentz-invariant interaction[13] can be evident in order c^{-2}. While the result (75) and the functions appearing therein refer to a single type of interaction for a pair of particles, generalization to an arbitrary number of such types is immediate, as noted in Secs. II and III.

It is noteworthy that there are no nontrivial contributions of order c^{-1} in the Lagrangian (75), and thus none can appear in the equations of motion; such terms either cancel out by time symmetry, or, for nonsymmetric interactions, add up to a total time derivative (61). The absence of any optical effects of order c^{-1} (which were expected as a consequence of the supposed absolute motion of the earth) was of some historical importance before the development of the theory of relativity.[27] Our results show that there is a similar absence of mechanical effects of that order (which because of their velocity dependence might be similarly misconstrued), given a Lorentz-invariant dynamics as a starting point (analogous to the Lorentz-invariant optical theory available to the prerelativists).

The expansion used is designed to give equations which are functions of a single Newtonian time. Furthermore, since Eq. (75) will lead to at most acceleration-dependent forces, the number of degrees of freedom of a particle system described by this approximately relativistic Lagrangian is the same as for its Newtonian limit, i.e., $6N$. Since, as was noted in Sec. II, the number of degrees of freedom of the exactly relativistic case is unknown, it is likewise unknown whether for any given type of relativistic interaction a possible set of non-Newtonian solutions is being lost by virtue of the approximation procedure.[39] On the other hand, the restriction to Newtonian-type equations allows both the integration of the classical equations of motion by the usual techniques, and their quantization by standard methods,[17] thereby leading to possible quantitative and qualitative predictions which might be compared to experiments. In quantum theory, an independent study of possible effects of order c^{-2} in nuclear physics is being carried out by Coester.[40] In classical theory, a possible application is in the realm of gravitation. There has been renewed interest in alternative relativistic theories of gravitation, and the various effects of order c^{-2} predicted by them, especially in celestial mechanics.[41] The variational principle (6) allows a new class of theories of gravitation, with new post-Newtonian effects based on the approximate Lagrangian (75), which are currently being investigated.[42]

Our results also imply an immediate generalization of the results of Ref. 22. This paper showed that an apparent paradox in magnetism[43] can be resolved on the microscopic level "for any field theory (classical or quantum) described by a local, Lorentz-invariant Lagrangian." The authors' detailed calculations were based only on the validity, to order c^{-2}, of the laws of conservation of momentum and energy and of the center-of-mass theorem.[44] Since all of these are valid for the general approximate Lagrangian (75) (as discussed in Sec. V), the restriction to local field theories is not necessary, and the paradox is also resolved on the microscopic level, to order c^{-2}, for all interactions described by the exact variational principle (6) or the resulting approximate Lagrangian (75).

It should be noted that the types of interaction allowed by the exact relativistic variational principle (6) include not only interactions between particles with null or timelike separation, which are familiar from electrodynamics and mesodynamics, but also spacelike ones. No conceptual difficulties are introduced by such interactions as long as a *closed* system of particles is considered,[45] a restriction inherent also in the Newtonian mechanics of particle systems. Section IV gives a particular example of Eq. (75) applied to such a spacelike interaction. Also, this example is sufficiently simple to allow obtaining the exact Lagrangian by direct integration from the variational principle; expansion of this Lagrangian serves as a consistency check on Eq. (75) for this example. It also has the virtue of not being symmetric in the particles' variables, and thus is an example of a case in which the W_{ij} "potential" appears. On the other hand, this particular example is so simple that the W_{ij} "potential" happens to equal the Newtonian potential. This is not necessary, however.

Section IV also includes a very general form of relativistic interaction (79a) which allows definition of adjunct fields, leading to the approximate interaction terms (81b) or, for the special case closest to the familiar linear field theories, to (81c). As mentioned in the Introduction, it was noted in HS that all approximately relativistic Lagrangians discussed there could be put into the form (75a), with [Eqs. (67) and (68) of HS][46]

$$V=\tfrac{1}{2}\sum_{i\neq j}\sum V_{ij}(r_{ij})+V_2(r_{ij}) \tag{113a}$$

and

$$I=\frac{1}{4c^2}\sum_{i\neq j}\sum g_i g_j\left\{[A(\vec{v}_i-\vec{v}_j)^2+\vec{v}_i\cdot\vec{v}_j]V_{ij}-\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\right\}, \tag{113b}$$

where A is an integer. The term V_2 in (113a) appears only in the approximate Lagrangians following from general relativity, and thus does not concern us here. The question was raised by HS whether all approximate interactions must be of the form (113b). Comparison of (113b) with (81b) and (81c) shows that all interactions (81c) following from the field-related interactions (79a) with $m=0$ [which include those interactions which allow the field equations (78)] are indeed of this form, with

$$A=1-l. \tag{114a}$$

Furthermore, the more general field-related interactions (79a) also lead to this form, provided $m=1$; then Eq. (81b) shows that

$$A=-l. \tag{114b}$$

Thus in both cases A is an integer.

These identifications hold for the special case of a single type of interaction. However, as noted in Secs. II and III, the variational principle (6) may contain different types of interactions, leading to Eqs. (75) and, in the case under consideration, Eqs. (81), with appropriate summations. We now consider interactions which are such that they all have the same coupling constants g_i and imply the *same* functional form of the Newtonian potential $V_{ij}^{(lm)}$ independent of the values of l and m, such that

$$V_{ij}^{(lm)}=a_{ij}^{(lm)}V_{ij}(r_{ij}), \quad \sum_l\sum_m a_{ij}^{(lm)}=1 \tag{115}$$

where V_{ij} is the total Newtonian two-body potential, and the $a_{ij}^{(lm)}$ are constants. Then Eq. (81b), summed over l and m, can be written

$$I_{\rm PN}=\frac{1}{2c^2}\sum_{i<j}\sum g_ig_j\left\{[A(\vec{v}_i-\vec{v}_j)^2+\vec{v}_i\cdot\vec{v}_j]V_{ij}(r_{ij})-[\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}-B((\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij})^2]\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\right\}, \tag{116a}$$

where

$$A\equiv\sum_l\sum_m(1-l-m)a_{ij}^{(lm)} \tag{116b}$$

and

$$B\equiv\sum_l\sum_m\frac{m(1-m)}{2m-1}a_{ij}^{(lm)}, \tag{116c}$$

which are not necessarily integers. Eq. (116a) is precisely of the form of the most general interaction constructed by Breit,[47] which is characterized by two constants a and b, related to the ones introduced above by

$$a=1-2A, \quad b=1+2B. \tag{116d}$$

Breit's result, which was based on considerations of invariance and simplicity alone but not on any particular model of a relativistic interaction, thus is equivalent to a combination (115) of field-related interactions. The form (113b) of HS, which was gleaned from the form of approximately relativistic Lagrangians following from particular field theories, follows from Eq. (116a) for all combinations of interactions (115) for which B vanishes.[48]

Other interactions may allow forms of $I_{\rm PN}$ including functions other than V_{ij}; but even if all four functions appearing in Eq. (75) can be reduced to a single function and its derivative, as is the case for the interaction (82), the resulting expression is not necessarily of the form (116a), the expression (86b) following from (82) being a case in point.

It might be thought that this difference is a consequence of the non-field-theoretical character of the interaction of our example. However, taking instead of the form (82) the symmetric relativistic interaction

$$U_{ij}=\tfrac{1}{2}\omega_{ij}^{l}[\delta(\chi_{ij})+\delta(\zeta_{ij})]f_{ij}[(-\sigma_{ij})^{1/2}], \tag{117a}$$

we obtain from the calculations of Sec. IV instead of the nonsymmetric post-Newtonian interaction (86b) the symmetric one

$$I_{\rm PN}=\frac{1}{2c^2}\sum_{i<j}\sum g_ig_j\left\{[(\tfrac{3}{2}-l)(\vec{v}_i-\vec{v}_j)^2+\vec{v}_i\cdot\vec{v}_j]V_{ij}-\{\vec{v}_i\cdot\vec{r}_{ij}\vec{v}_j\cdot\vec{r}_{ij}-\tfrac{1}{2}[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}_{ij}]^2\}\frac{1}{r_{ij}}\frac{dV_{ij}}{dr_{ij}}\right\}, \tag{117b}$$

which is again of the form (116a), with $A=\frac{3}{2}-l$ and $B=\frac{1}{2}$. Indeed, this same $I_{\rm PN}$ can be obtained from interactions of the form (115), with a single (arbitrary) integer value of l; a possible set of coefficients is given by

$$a_{ij}^{(l0)}=\frac{1-n^2}{2n(1-n)}, \quad a_{ij}^{(l1)}=\frac{-n^2}{2n(1-n)}, \quad a_{ij}^{(ln)}=\frac{2n-1}{2n(1-n)}, \quad a_{ij}^{(lm)}=0 \text{ if } m\neq 0,1,n, \tag{118}$$

where n is a fixed, but arbitrary integer (≥ 2). Thus two different relativistic interactions, one of which cannot be related to a field theory, can lead to the same approximately relativistic Lagrangian to the order considered here (and, incidentally, infinitely many different field-theory related interactions can lead to the same approximate Lagrangian).

The difference between expressions (86b) and (116a) should likewise not be blamed on the spacelike character of the separation in the interaction (82). The field-related interactions (79a), even with $m=0$ [leading to Eq. (81c)] and with field equations of the form (78), do not necessarily imply separations which are not spacelike; indeed, field equations with Green functions implying spacelike separation have recently been studied extensively in connection with the tachyon problem. Thus, to order c^{-2}, the knowledge of the form of the approximate Lagrangian alone is not sufficient to allow us to decide whether the fully relativistic interaction involves timelike, spacelike, or null separations between the particles.[49]

The approximate Lagrangian (75) implies ten approximate conservation theorems, which are found from invariance considerations in Sec. V; they can also be obtained by directly approximating the exact Lorentz-invariant conservation theorems.[38] Each of the approximate theorems contains a contribution from interactions which are not symmetric in the particles. Just as in the field-related cases considered in HS, it is possible to find a center-of-mass theorem (112), with a center-of-mass coordinate defined in terms of individual contributions proportional to the position vectors of the particles [as shown in Eq. (111)].

It is worthwhile to note what has not been considered. The assumption of a static Newtonian limit is traditional, but by no means necessary. A Newtonian theory with interactions depending on velocities and their derivatives is not at all new,[50] and it is easy to devise a relativistic interaction which would reduce to such a theory in the nonrelativistic limit, as shown below. However, preliminary exploratory calculations of such cases showed them to be acceleration-dependent both in order c^{-2} and in order c^{-1}. Since only Lagrangians depending at worst on velocities were desired in both the exact and approximate realms, the nonstatic cases are not considered here in detail.

As noted in Sec. III, it is the particular choice (32) of the set of invariants which is ultimately responsible for the appearance of a static Newtonian limit. This is most clearly seen from Eq. (44), whose leading term depends only on ζ_{ij} [which is an integration variable in I_1, given by Eq. (45)], and $r(t_i, t_j)$, where in lowest order $t_i = t_j$, according to Eq. (38). Thus in lowest order I_1 involves only the static Newtonian potential (66). If we had desired to obtain a velocity-dependent Newtonian limit, we could, e.g., have chosen a set

$$\begin{aligned} &\sigma_{ij}, \quad \zeta_{ij}, \quad \Omega_{ij} \equiv c^2(\omega_{ij}-1) \equiv c^2\left\{\gamma_i\gamma_j\left[1-\frac{1}{c^2}\vec{v}_i(t_i)\cdot\vec{v}_j(t_j)\right]-1\right\}, \\ &\xi_{ij} \equiv c^2(\kappa_i+\kappa_j) \equiv c^2 t_{ij}(\gamma_i-\gamma_j)+[\gamma_j\vec{v}_j(t_j)-\gamma_i\vec{v}_i(t_i)]\cdot\vec{r}_{ij}(t_i,t_j) \end{aligned} \tag{119}$$

instead of the set (32). Then we could proceed with the same change of integration variable as in Sec. III, but would be led to

$$\begin{aligned} U(\zeta)|_+ \approx U\Bigg\{&\left[\zeta^2-\vec{r}^2(t_i,t_j)+2\frac{\zeta}{c}\vec{v}_j\cdot\vec{r}(t_i,t_j)+\frac{1}{c^2}[\vec{v}_j\cdot\vec{r}(t_i,t_j)]^2-\frac{\vec{v}_j^{\,2}\zeta^2}{c^2}\right], \quad \zeta, \\ &\left[\tfrac{1}{2}(\vec{v}_i-\vec{v}_j)^2+\frac{1}{8c^2}[(\vec{v}_i-\vec{v}_j)^4-4(\vec{v}_i\cdot\vec{v}_j)^2+2(\vec{v}_i^{\,4}+\vec{v}_j^{\,4})]\right], \left[(\vec{v}_i-\vec{v}_j)\cdot\vec{r}-\frac{\zeta}{2c}(\vec{v}_i^{\,2}-\vec{v}_j^{\,2})+\frac{1}{2c^2}\vec{v}_i^{\,2}(\vec{v}_i-\vec{v}_j)\cdot\vec{r}\right]\Bigg\}\Bigg|_+ \end{aligned} \tag{120}$$

instead of Eq. (44). The lowest-order terms of this equation involve $[\vec{v}_i(t)-\vec{v}_j(t)]^2$ and $[\vec{v}_i(t)-\vec{v}_j(t)]\cdot\vec{r}_{ij}(t)$, in addition to r_{ij} and ζ_{ij}, and thus the Newtonian limit of I_1 is not static.

While the sets (32) and (119) are not the only possible ones, the set (32) is the only one which gives a static Newtonian limit, and the set (119) contains the most general velocity-dependent Newtonian limit, encompassing the limits of sets such as $(\sigma_{ij}, \omega_{ij}, \xi_{ij}, \zeta_{ij})$ and $(\sigma_{ij}, \Omega_{ij}, \chi_{ij}, \zeta_{ij})$ as special cases. However, for the terms of order c^{-1} or higher the set (119) is not all-encompassing, but different functional forms are obtained from the different sets mentioned, as can be seen from the Taylor and Lagrange expansions of the individual invariants.

In the above, "Newtonian limit" means $c\to\infty$ as well as $t_i = t_j$; the latter condition was required here, but is not necessary for Galilei invariance, and thus more general Galilei-invariant theories are possible, as discussed in H.

The exact variational principle is postulated to be exactly Lorentz-invariant. This assumption is not necessary. It would be sufficient to allow it to be invariant up to a divergence in order to apply Noether's theorem and retain the advantage of particle systems having ten exactly conserved quantities. The Lagrangian of ordinary Newtonian mechanics is invariant only up to a divergence (i.e., in this case a total time derivative) in the nonrelativistic limit. Consequently, there is no reason for excluding such a possibility in the Lorentz case. However, this possibility has not been explored in the literature.

The Lorentz invariants (31) or (32) are not the only two-body invariants of the Lorentz group, but they are the only independent quantities which are polynomials in the positions and velocities of the particles. However, the Lorentz group is interesting and physically relevant precisely because there are other invariants specifying the character of the separation between points. In particular, the sign of $z_i^0 - z_j^0$ for null or timelike separations is invariant under the Lorentz group; other such invariants can also be constructed.[51] Unfortunately, for any invariants which are not polynomials in the particle variables the expansion method used here is not applicable.

Possible future uses of the formalism developed here, apart from those mentioned before, include an approximation of variational principles describing particles with intrinsic angular momentum, as well as dipole and higher multipole interactions, and a generalization of the results of Refs. 21 and 23, using the Lagrangian (75); some of these problems are currently being investigated.

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[17]F. Coester and P. Havas (unpublished).

[18]Such an expansion is popular despite the fact that nothing is known of its convergence properties. However, since the limit $c \to \infty$ is sufficient to effect the transition from the Lorentz group to the Galilei group, c^{-1} does seem to be a natural parameter in which to expand.

[19]C. G. Darwin, Phil. Mag. 39, 537 (1920).

[20]A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. 39, 66 (1938).

[21]R. B. Hoffman and P. Havas, Phys. Rev. 140, B1162 (1965).

[22]S. Coleman and J. H. Van Vleck, Phys. Rev. 171, 1370 (1968).

[23]J. E. Krizan and P. Havas, Phys. Rev. 128, 2916 (1962); J. E. Krizan, *ibid.* 140, A1155 (1965); 152, 136 (1966); 177, 376 (1969); T. E. Dengler and J. E. Krizan, Phys. Rev. A 2, 2388 (1970).

[24]P. Havas and J. Stachel, Phys. Rev. 185, 1636 (1969); hereafter referred to as HS.

[25]F. J. Kennedy, Am. J. Phys. 40, 63 (1972) has given an alternative construction of approximate Lagrangians

which are symmetric in the particle variables, but did not relate them to Lorentz-invariant variational principles involving direct-particle interaction. His approximate Lagrangians, which are of the form proposed by HS, constitute only a subclass of the approximate Lagrangians considered here.

[26]The approach can be made unique by requiring the existence of a canonical formalism to order c^{-2}; the resulting Hamiltonian agrees with that obtained here. This approach is taken by J. Stachel and P. Havas (to be published).

[27]See, e.g., C. Møller, *The Theory of Relativity* (Oxford Univ. Press, London, 1952).

[28]For a review see P. Havas, Rev. Mod. Phys. 36, 938 (1964).

[29]See Sec. IV for a class of such interactions.

[30]L. Schwartz, *Théorie des distributions* (Hermann et Cie., Paris, 1950–51). For a detailed treatment more attuned to a physicist's needs see I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964).

[31]See, e.g., C. Lanczos, *The Variational Principles of Mechanics* (Univ. of Toronto Press, Toronto, 1966).

[32]For a concrete application of a Lagrange expansion in a form similar to that used here, see L. Page and N. I. Adams, Jr., *Electrodynamics* (Van Nostrand, New York, 1940), p. 171.

[33]An excellent discussion of Lagrange expansions, including many references, is given by K. G. Dedrick and E. L. Chu [Arch. Rat. Mech. and Anal. 16, 385 (1964)]. However, both of the forms they consider are different from that used here.

[34]The preceding discussion of the field-related interactions generalizes and supersedes the discussion given in H; Eq. (76) replaces Eq. (52L) of that paper. Equations (77) and (78) are identical with Eqs. (53L) and (54L) of H (apart from notation) except that in Eq. (54L) the coupling constant g_k was omitted erroneously.

[35]E. Noether, Nachr. Akad. Wiss. Göttingen, Math.-Physik. Kl. 235 (1918); for a concise formulation and discussion see Ref. 24, Sec. II.

[36]J. W. Dettman and A. Schild, Phys. Rev. 95, 1057 (1954); see also Ref. 11 and the second paper of Ref. 2.

[37]Several misprints (in H) in the conservation law for $L^{\mu\nu}$ [Eq. (60)] should be noted: The last square bracket should be multiplied by $\frac{1}{2} g_i g_j$; s^{ν}_{ij} and s^{μ}_{ij} should be replaced by $z^{\nu}_i + z^{\nu}_j$ and $z^{\mu}_i + z^{\mu}_j$, respectively; and the last term within the square brackets should have the opposite sign.

[38]The very lengthy calculations are given in H. W. Woodcock, Temple University thesis, 1972 (unpublished), Appendix D. This thesis contains the main results of this paper, as well as an alternative derivation of Eq. (75).

[39]For the particular case of Fokker's action principle of electrodynamics (Ref. 11), it has been established recently that, at least in special cases, there are more solutions (and thus more degrees of freedom) for given initial positions and velocities than in Newtonian mechanics; see C. M. Andersen and H. C. von Baeyer, Ref. 15, and D.-C. Chern and P. Havas (unpublished). However, it is still not known whether there exist several relativistic solutions which reduce to the same Newtonian solution in the limit $c \to \infty$, but differ to order c^{-2}.

[40]F. Coester (private communication).

[41]For a detailed survey see K. Nordtvedt, Science (to be published).

[42]P. Havas (unpublished).

[43]Stated originally by W. Shockley and R. P. James, Phys. Rev. Letters 18, 876 (1967).

[44]These calculations, in addition to the Darwin Lagrangian describing the electromagnetic interaction, assume additional interaction terms for other interactions. These are constructed in the Appendix of Ref. 22 from the requirement of ten conservation laws. However, these terms apply only to fields described by a four-vector [i.e., $l = 1$ in our Eq. (81c)], as noted by E. Bagge, who suggested the same terms some time ago [Z. Naturforsch. 1, 361 (1946)].

[45]P. Havas, in *Proceedings of the 1964 International Congress of Logic, Methodology, and Philosophy of Science* (North-Holland, Amsterdam, 1965), p. 347; Synthese 18, 75 (1968).

[46]In Eq. (67) of HS, the factor $\frac{1}{2}$ was omitted erroneously. Similarly, a factor $\frac{1}{2}$ is missing from the last term of their Eq. (66), and in front of $(v_k/c)^2$ in their Eq. (49). However, their discussion is appropriate for the correct equations.

[47]Eq. (132) of the second paper of Ref. 16; it should be noted that Breit's J is the negative of V.

[48]In Ref. 25, F. J. Kennedy comments on the application of his approximate Lagrangian [which corresponds to Eq. (81c)] to the case of the EIH Lagrangian, and concludes that it implies $1 - l = -3$ or a special relativistic interaction mediated by a tensor field of fourth rank. However, applying Eqs. (116), with $m = 0$, we note that a linear combination of a scalar field ($l = 0$) and a second-rank tensor field ($l = 2$), with $a^{(00)}_{ij} = -1$ and $a^{(20)}_{ij} = 2$, will also give -3. Furthermore, this latter interpretation follows directly from the variational principle for the equations of motion obtained in a Lorentz-invariant approximation method in general relativity by P. Havas and J. N. Goldberg [Phys. Rev. 128, 398 (1962)].

[49]In Ref. 14, Katz considered the interaction (76) with $l = m = 0$ and showed that (in our notation) the potentials $V_{ij}(r_{ij})$ and $\phi_{ij}(s_{ij})$ determine one another uniquely. Clearly, as shown by Eq. (66), V_{ij} is uniquely determined from any $U_{ij}(s_{ij}{}^2, \omega_{ij}, \kappa_i, \kappa_j)$; on the other hand, U_{ij} could be determined from V_{ij} in Ref. 14 only because its functional dependence on (or rather independence of) κ_i, κ_j and ω_{ij} was assumed.

[50]L. Königsberger, *Die Principien der Mechanik* (B. G. Teubner, Leipzig, 1901).

[51]P. Havas (unpublished).