Monopole Theory with Potentials*

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The Cabibbo-Ferrari two-potentials formalism is used to construct a representation of the gauge-independent electrically and magnetically charged Mandelstam field. This representation is applied to convert the Cabibbo-Ferrari-Coleman formulation of monopole theory into a more conventional one in terms of gauge-dependent fields and potentials.

Mandelstam,[1] years ago, proposed an alternative to the conventional ways of quantizing electrodynamics which use potentials and suffer from disadvantages that result from constructing the theory in particular gauges, as the indefinite metric of the Lorentz gauge or the lack of manifest Lorentz invariance of the Coulomb gauge. His formulation of the theory entirely avoids potentials and gauge-dependent quantities at the expense of attaching a path dependence to the matter fields. Complications that arise from this path dependence are argued to be physical rather than a feature of the formal apparatus and would directly reflect the essence of the experiment of Aharonov and Bohm.

Soon after, Cabibbo and Ferrari[2] and Coleman[3] noted that Mandelstam's approach to ordinary electrodynamics could be extended to allow for magnetically as well as electrically charged particles and gave "en passant" an elegant derivation of the quantization condition of Dirac.[4,5]

Of course, although a gauge-independent formulation of electrodynamics is nice and desirable to have, our long acquaintance with potentials renders them irreplaceable tools of any theoretical laboratory and makes it of much practical importance to be able, at one stage, to eliminate the path dependence and recover the conventional formulation of the theory corresponding to a particular gauge. While Mandelstam, to achieve this purpose, provided a representation of the path-dependent field in terms of the auxiliary gauge-dependent field and potential, the paper of Cabibbo and Ferrari was lacking in this respect. They suggested the use of potentials which should be appropriate to the case in which both charges and poles are present, but did not express in terms of them the gauge-independent Mandelstam field. Our simple considerations are aimed at supplementing the work of these authors by exhibiting such a representation.

Let us recall the usual definition of potentials, A and $\tilde{A}$, for the electromagnetic tensor F and its dual $\tilde{F}$ (Ref. 6):

$$A_{\nu,\mu}-A_{\mu,\nu}=F_{\mu\nu},\quad \tilde{A}_{\nu,\mu}-\tilde{A}_{\mu,\nu}=\tilde{F}_{\mu\nu}\ . \tag{1}$$

The group of gauge transformations,

$$A_\mu\to A_\mu+A^0_\mu,\quad \tilde{A}_\mu\to\tilde{A}_\mu+\tilde{A}^0_\mu, \tag{2}$$

is determined by the solutions

$$A^0_\mu=\Lambda_{,\mu},\quad \tilde{A}^0_\mu=\tilde{\Lambda}_{,\mu} \tag{3}$$

to the homogeneous equations

$$A^0_{\nu,\mu}-A^0_{\mu,\nu}=0,\quad \tilde{A}^0_{\nu,\mu}-\tilde{A}^0_{\mu,\nu}=0\ . \tag{4}$$

In a world with both electrons and monopoles,

$$F_{\mu\nu,\nu}=j_\mu\neq 0,\quad \tilde{F}_{\mu\nu,\nu}=\tilde{j}_\mu\neq 0\ , \tag{5}$$

the two systems of partial differential equations (1) will in general not be soluble everywhere. The appearance of unphysical singularities ("Dirac's strings") will be taken to suggest that a different choice of potentials might be convenient.[7] Following Cabibbo and Ferrari, we define the regular potentials B and $\tilde{B}$:

$$B_{\nu,\mu}-B_{\mu,\nu}+\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{B}_{\sigma,\rho}=F_{\mu\nu}\ . \tag{6}$$

The group of gauge transformations

$$B_\mu\to B_\mu+B^0_\mu,\quad \tilde{B}_\mu\to\tilde{B}_\mu+\tilde{B}^0_\mu \tag{7}$$

is now determined by the solutions of the homogeneous equations:

$$B^0_{\nu,\mu}-B^0_{\mu,\nu}+\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{B}^0_{\sigma,\rho}=0\ . \tag{8}$$

These imply, by differentiation,

$$B^0_{\nu,\mu\nu}-B^0_{\mu,\nu\nu}=0\ , \tag{9'}$$

while integration of the dual of Eq. (8), along the direction n from infinity to the point x, in the axial gauge $n_\nu\tilde{B}_\nu=0$, yields

$$\tilde{B}^0_\mu(x)=-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_{-\infty}^{s(x)}ds\ n_\nu B^0_{\sigma,\rho}\ .$$

Equation (9′) and the generalization of the latter constraint to an arbitrary gauge,[8]

$$\tilde{B}^0_\mu = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_{P(x)}^x d\xi_\nu B^0_{\sigma,\rho}(\xi) + \Lambda_{,\mu} \;, \tag{9''}$$

are equivalent to the system (8); yet, they provide a more immediate specification of the gauge freedom associated with B and $\tilde{B}$.

Next, we relate A and $\tilde{A}$ to B and $\tilde{B}$:

$$A_\mu = B_\mu - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_{P(x)}^x d\xi_\nu \tilde{B}_{\sigma,\rho}(\xi) + \Lambda_{,\mu} \;, \tag{10'}$$

$$\tilde{A}_\mu = \tilde{B}_\mu + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_{P(x)}^x d\xi_\nu B_{\sigma,\rho}(\xi) + \tilde{\Lambda}_{,\mu} \;. \tag{10''}$$

Needless to say, since B and $\tilde{B}$ do not in general satisfy the condition of zero source (9′) and the condition of zero source "dual" to (9′), Eqs. (10) should not be regarded as gauge transformations. However, the connection between the two sets of potentials *is* in general a gauge transformation *almost* everywhere. Equations (10) show directly how nonintegrable physical singularities of the potentials $\tilde{B}$ and B, at the position of pointlike monopoles and electrons, generate unphysical singularity lines in A and $\tilde{A}$.

We are now prepared to generalize Mandelstam's treatment of electrodynamics to a world in which dually charged particles exist, and the first step is constructing the gauge-independent field of a particle of electric charge Q_k and magnetic charge $\tilde{Q}_k$. This is readily done in terms of the gauge-dependent matter field $\psi_k(x)$ and the potentials (1):

$$\Psi_k(x,P) = \psi(x)\exp\left(-i\int_P^x d\xi_\mu(Q_k A_\mu + \tilde{Q}_k\tilde{A}_\mu)\right) \;. \tag{11}$$

Then, in terms of the Cabibbo-Ferrari potentials (6), the Mandelstam field is seen, from Eqs. (10), to become a surface-dependent quantity:

$$\Psi_k(x,\Sigma) = \psi(x)\exp\left[-i\int_P^x d\xi_\mu\left((Q_k B_\mu + \tilde{Q}_k\tilde{B}_\mu) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_{P(\xi)}^\xi d\eta_\nu(Q_k\tilde{B}_{\sigma,\rho} - \tilde{Q}_k B_{\sigma,\rho})\right)\right] \;. \tag{12}$$

The surface Σ is determined by the integration path P, which is part of the boundary of Σ, and by the subintegration lines $P(\xi)$. Not surprisingly, however, such *surface dependence is unphysical and reduces to a path dependence if the quantization condition is satisfied.*

This conclusion is easily reached after computing the response $\delta\Psi_k$ to a change in the surface Σ by an infinitesimal area at the point z off the boundary line. Denoting by V_μ the infinitesimal volume delimited by the two surfaces, Gauss's theorem and Maxwell's equations entail

$$\begin{aligned}\delta\Psi_k &= \psi(x)\exp\{-iV_\mu[Q_k(\tilde{B}_{\nu,\mu\nu} - \tilde{B}_{\mu,\nu\nu}) \\ &\qquad - \tilde{Q}_k(B_{\nu,\mu\nu} - B_{\mu,\nu\nu})]\} \\ &= \psi(x)\exp\{-iV_\mu[Q_k\tilde{j}_\mu(z) - \tilde{Q}_k j_\mu(z)]\} \;. \end{aligned} \tag{13}$$

Hence, the vanishing of $\delta\Psi_k$ is indeed equivalent to the quantization condition, in the chiral-invariant form of Zwanziger[5,9]:

$$Q_k\tilde{Q}_h - \tilde{Q}_k Q_h = 2\pi n \;. \tag{14}$$

The representation (11), or (12), defines the path dependence of the gauge-independent field Ψ_k and, as such, fully specifies the interaction of electric and magnetic charges with the radiation. The Lagrangian density of the system that one postulates is, in fact, just the sum of free-field Lagrangians for Ψ_k and F,

$$L = \sum_k \bar{\Psi}_k(i\gamma\cdot\partial - m_k)\Psi_k - \tfrac{1}{4}(F_{\mu\nu})^2 \;, \tag{15}$$

with the interaction being completely buried in the functional dependence of Ψ_k on F (Ref. 10), specified (11) or (12). In perfect analogy with the case of ordinary electrodynamics, the equations of motion for Ψ_k and F,

$$\begin{aligned} F_{\mu\nu,\nu} &= \sum_k Q_k\bar{\Psi}_k\gamma_\mu\Psi_k \;, \\ \tilde{F}_{\mu\nu,\nu} &= \sum_k \tilde{Q}_k\bar{\Psi}_k\gamma_\mu\Psi_k \;, \\ (i\gamma\cdot\partial - m_k)\Psi_k &= 0 \;, \end{aligned} \tag{16}$$

are then derived directly from an action principle or from making use of more familiar techniques and adopting the representation (11) as a mathematical aid in the intermediary steps. Also in perfect analogy with Mandelstam's case, all the commutation relations between the gauge-independent variables are obtained. These equations of motion and commutation relations, together with the dependence of the operators on the paths and the quantization condition, give us the theory of Cabibbo and Ferrari[2] and Coleman.[3]

With an explicit representation for the path-dependent quantities, Eq. (12), we may proceed paralleling Mandelstam's discussion to obtain a formulation of monopole theory in terms of potentials. The equations of motion are quickly secured by putting (6) and (12) into (16) (Ref. 11):

$$B_{\nu,\mu\nu} - B_{\mu,\nu\nu} = \sum_k Q_k \bar{\Psi}_k \gamma_\mu \Psi_k \ ,$$
$$\tilde{B}_{\nu,\mu\nu} - \tilde{B}_{\mu,\nu\nu} = \sum_k \tilde{Q}_k \bar{\Psi}_k \gamma_\mu \Psi_k \ , \tag{17}$$
$$\left\{\gamma_\mu\left[i\partial_\mu - Q_k\left(B_\mu - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_P^x d\xi_\nu \tilde{B}_{\sigma,\rho}\right) - \tilde{Q}_k\left(\tilde{B}_\mu + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int_P^x d\xi_\nu B_{\sigma,\rho}\right)\right] - m\right\}\psi = 0 \ .$$

Then, after picking a gauge in which to write expressions for the potentials in terms of the electromagnetic tensor F, one could use these, the representation (12), and the commutation relations between the gauge-independent quantities to deduce the commutation relations for B, $\tilde{B}$, and ψ_k in that gauge. There are gauges in which B and $\tilde{B}$ are local and relatively local fields:

$$[B_\mu(t,\vec{x}), B_\nu(t,\vec{0})] = [\tilde{B}_\mu(t,\vec{x}), \tilde{B}_\nu(t,\vec{0})] = [B_\mu(t,\vec{x}), \tilde{B}_\nu(t,\vec{0})] = 0 \ , \tag{18}$$
$$[\dot{B}_\mu(t,\vec{x}), B_\nu(t,\vec{0})] = [\dot{\tilde{B}}_\mu(t,\vec{x}), \tilde{B}_\nu(t,\vec{0})] = -i\delta_{\mu\nu}\delta^3(\vec{x}) \ .$$

Notice, however, the nonlocal character of the equation of motion for ψ, peculiar of monopole theory. It is true that by a change of variables, as in (10), we could cast the equation of motion for the matter into a local form and retain the locality of the equations of motion for the radiation, but our variables would then not be local fields. What their commutation relations would look like may be straightforwardly derived from (10) and (18):

$$[A_\mu(t,\vec{x}), A_\nu(t,\vec{0})] = [\tilde{A}_\mu(t,\vec{x}), \tilde{A}_\nu(t,\vec{0})] = -i(g_{4\mu}n_\nu + g_{4\nu}n_\mu) \times \int_{-\infty}^0 ds\,\delta^3(\vec{x} - \vec{n}s) \ , \tag{19}$$
$$[A_\mu(t,\vec{x}), \tilde{A}_\nu(t,\vec{0})] = i\epsilon^{\mu\nu\rho 4}n_\rho \int_{-\infty}^0 ds\,\delta^3(\vec{x} - \vec{n}s) \ .$$

These are the commutation relations which appear in Zwanziger's formulation of the theory.

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[5]Monopole theory has been discussed by J. Schwinger, Phys. Rev. 144, 1087 (1966); 151, 1048 (1966); 151, 1055 (1966); Science 165, 757 (1969); 166, 690 (1969); T.-M. Yan, Phys. Rev. 150, 1349 (1966); 155, 1423 (1967); B. Zumino, in *Theory and Phenomenology in Particle Physics*, edited by A. Zichichi (Academic, New York, 1968); D. Zwanziger, Phys. Rev. 176, 1489 (1968); Phys. Rev. D 3, 880 (1971).

[6]We use rationalized units with $\hbar = c = 1$, and a metric $g_{\mu\nu} = \mathrm{diag}(1,1,1,-1)$. We define $\tilde{F}_{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric Ricci pseudotensor, $\epsilon^{1234} = i$.

[7]We shall later find out that A and $\tilde{A}$ are not local and relatively local fields.

[8]We also generalize from straight lines to arbitrary (spacelike) paths.

[9]Schwinger and Zwanziger are actually in favor of the stronger "integer quantization condition."

[10]Notice that the path for the representation (12) is an infinite line going through the point x rather then a semi-infinite line ending at the point x.

[11]Notice the invariance of the last equation under the gauge transformations defined by (7) and (9). With such requirement in mind, one could have guessed its form directly from the corresponding equation in conventional electrodynamics.