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# Relativistic Treatment of Low-Energy Nucleon-Nucleon Scattering*

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An improved version of the one-boson-exchange model is presented. In addition to $\pi$, $\rho$, $\omega$, and $X^0$, the scalar mesons $\epsilon$ and $\delta$ are used with their physical masses. The broad width of the $\rho$ and $\epsilon$ mesons is taken into account. The relativistic Blankenbecler-Sugar equation is utilized over the laboratory scattering energy range 0 to 425 MeV. A better fit to the experimental $N$-$N$ data is obtained than in previous one-boson-exchange models. Reasonable values of the coupling constants are obtained, which agree qualitatively with other experiments. The fitting procedure was done including and excluding $S$ waves in order to test the influence of the core region. An approximate version of the above model is developed using the Schrödinger equation in place of the Blankenbecler-Sugar equation. A good fit to the experimental scattering data and the deuteron parameters is obtained with coupling constants similar to those of the relativistic model.

## I. INTRODUCTION

This work is a continuation of previous attempts[1,2] to describe the nucleon-nucleon interaction with the aid of the one-boson-exchange model in conjunction with the relativistic Blankenbecler-Sugar equation or the Schrödinger equation. It was found in Refs. 1 and 2 that the nucleon-nucleon interaction can be well approximated by one-meson-exchange contributions of the $\pi$, $\eta$, $\rho$, and $\omega$ mesons for laboratory energies up to about 400 MeV. However, it was also necessary to include the contributions from two scalar mesons with masses of about 500 to 600 MeV. Recently scalar mesons with these predicted quantum numbers have been found, but with masses somewhat higher than those employed in the models of Refs. 1 and 2. A $J=0^+$, $T=0$, $\epsilon$ meson with mass about 700 MeV and width possibly as great as 400 MeV is now listed in the Particle Data Group tables.[3] A rather narrow $J=0^+$, $T=1$, $\delta$ meson with mass 962 MeV is also listed.[3] In the present work we have fit the nucleon-nucleon data with these one-meson-exchange contributions, taking the experimental values for the masses of the scalar mesons into account. (The width of the $\epsilon$ meson is treated in a rather crude way, described below.) We find that including widths greatly improves the fit; this is because the width gives the potential of the exchanged meson an effectively greater range. Thus we can report a fit to the $N$-$N$ data which is superior to the fits achieved previously[1,2] while using only physical masses for the exchanged mesons.

It is known that the partial waves with high value of the angular momentum $l$ are well described by the long-range exchanges. This is due to the large centrifugal barrier which screens the effects of the short-range interaction. This centrifugal barrier is lacking for $S$ waves. Accordingly we would expect the very heavy mesons (both known and not yet discovered) to contribute to $S$ waves. As it is presently impossible to calculate the contributions of these mesons, we have to somehow parametrize the effect of the very short-range interaction. This we have done by readjusting the fitting parameters. It appears that only the coupling constants of the heaviest mesons and the cutoff parameter $\Lambda$ are greatly affected by this procedure. This is perhaps not surprising since these parameters mainly affect the short-range interaction. On the other hand, we could get a truer indication of the values of the coupling constants of the mesons exchanged by fitting the experimental data excluding $S$ waves.

The fit which includes the $S$ waves, although more entangled with parametrizing the unknown short-range interaction, is of course useful in describing the low-energy interaction and the properties of the deuteron. These results may also serve as a starting point for nuclear matter calculations. Therefore, in addition to the calculations carried out with the relativistic Blankenbecler-

Sugar (BBS) equation, we have carried out calculations with the Schrödinger equation using the potential of Ref. 2 (BS-III). This can be considered a nonrelativistic approximation to the BBS equation. It appears that this approximation mainly affects the $S$ waves, where differences can be quite large. However, by readjusting the coupling constants, especially those affecting the $S$ waves (i.e., of the heaviest mesons), we obtain an equally good fit to the experimental data.

## II. CALCULATIONS WITH THE BBS AND SCHRÖDINGER EQUATIONS

We shall describe the $N$-$N$ interaction utilizing a particular approximation to the Bethe-Salpeter equation. This approximation was suggested by Blankenbecler and Sugar for spinless particles[4] and its $N$-$N$ version (used in our calculations) was worked out by Thompson.[5] The details of handling the exchanges of pseudoscalar, scalar, and vector mesons as well as the fitting procedure to the $N$-$N$ data is described in Ref. 1. The numerical methods for solving the BBS equation are described in the paper of Thompson, Gersten, and Green.[6] As these references completely cover the above subjects, we will not repeat the work here. The new features of the present work are the inclusion of the widths of the wide mesons $\epsilon$ and $\rho$, calculations with and without the inclusion of $S$ waves, and fitting the $S$-wave effective-range parameters.

Let us deal first with the wide mesons. We replace the propagator of a stable meson, $1/(q^2+m^2)$, by[7]

$$P(q^2)=1/[m^2+q^2+\gamma(q^2+4m_\pi{}^2)^{1/2}]\,, \tag{1}$$

where

$$\gamma=\Gamma m/(m^2-4m_\pi{}^2)^{1/2}\,. \tag{2}$$

$m$ and $m_\pi$ are the meson and pion masses (the $\epsilon$ and $\rho$ are two pion resonances), and $\Gamma$ is, to a very good approximation, the experimental width. Equation (1) can be represented by the following dispersion integral:

$$P(q^2)=\int_{4m_\pi{}^2}^{\infty}\frac{\rho(m'^2)dm'^2}{q^2+m'^2}\,, \tag{3}$$

where

$$\rho(m'^2)=\frac{1}{\pi}\frac{\gamma(m'^2-4m_\pi{}^2)^{1/2}\theta(m'^2-4m_\pi{}^2)}{(m'^2-m^2)^2+\gamma^2(m'^2-4m_\pi{}^2)} \tag{4}$$

and $\theta$ is the step function. In addition, $\rho(m'^2)$ satisfies the following relation[7]:

$$\int_{4m_\pi{}^2}^{\infty}\rho(m'^2)dm'^2=1. \tag{5}$$

Thus $\rho(m'^2)$ can be interpreted as the mass-squared density distribution of the $\epsilon$ or the $\rho$ meson, having the proper threshold (two-pion mass squared), normalization, and width.

Equation (1) is a crude approximation to the propagator, but the existing experimental data do not allow for a more accurate one. At least it satisfies the gross features of a resonance. As it is difficult to handle Eq. (1) in practical calculations, we have derived a two-pole approximation to it which is described in the Appendix.

As in Refs. 1 and 2 mentioned above, we have modified the propagator with the aid of the regulating parameter $\Lambda$ by replacing the usual meson propagator $1/(q^2+m^2)$ by

$$\frac{1}{q^2+m^2}\,\frac{\Lambda^2}{q^2+\Lambda^2}=\frac{1}{1-(m^2/\Lambda^2)}\left(\frac{1}{q^2+m^2}-\frac{1}{q^2+\Lambda^2}\right). \tag{6}$$

For the wide mesons we have adopted the following regularization:

$$\frac{1}{1-(m^2/\Lambda^2)}\left(\frac{1}{q^2+m^2+\gamma(q^2+4m_\pi{}^2)^{1/2}}-\frac{1}{q^2+\Lambda^2}\right). \tag{7}$$

The parameter $\Lambda$ is used here as a phenomenological parameter which modifies the short-range interaction. For reasons of economy we have a common $\Lambda$ for all mesons.

From our list of mesons we have excluded the $\eta$ meson as it has a small effect on the nucleon-nucleon interaction and there are indications that its coupling constant is negligible.[8] Instead we have used the $X^0$ meson.

In fitting the one-boson-exchange model to the nucleon-nucleon data, we adjusted the coupling constants to yield a best fit to the Arndt-MacGregor reduced-error matrices at 25, 50, 95, 142, 210, 330, and 425 MeV (Ref. 9). At each energy, these $A_{ij}$ error matrices give $\chi^2$ in approximation

$$\chi^2=\chi_{\min}{}^2+\sum_{ij}A_{ij}(\delta_i-\delta_i^{\exp})(\delta_j-\delta_j^{\exp})\,.$$

Thus it is a parabolic approximation. The $\delta_j^{\exp}$ are the values of the phase shifts which yielded the lowest value of $\chi^2$ in the Arndt-MacGregor phase-shift analysis. In the analysis, the higher partial-wave phase shifts were set to the one-pion-exchange contribution. The matrices at each energy, $A_{ij}$, also automatically include renormalization of the experimental data to give the model phase parameters the lowest possible $\chi^2$ consistent with the uncertainty in the over-all normalization of each experiment. This is explained in Ref. 10.

Three fits, searching the coupling constants and $\Lambda$, were performed with the aid of the relativistic

TABLE I. List of meson-nucleon coupling constants for the various fits described in the text. Coupling constants defined by interaction Hamiltonians of Ref. 1.[a] Also listed are the meson masses employed. The widths of the $\rho$ and the $\epsilon$ were taken to be 130 and 400 MeV, respectively. The nucleon mass was set equal to 938.5 MeV.

| Meson mass | | Fit A[b] | Fit B[c] | Fit C[d] | Fit D[e] |
|---|---|---|---|---|---|
| 138.7 MeV | $g_\pi^2/4\pi$ | 13.84 | 14.64 | 14.43 | 14.13 |
| 958 MeV | $g_{X_0}^2/4\pi$ | 13.81 | 8.69 | 10.01 | 7.01 |
| 782.8 MeV | $g_\omega^2/4\pi$ | 10.91 | 8.30 | 8.30 | 9.39 |
| | $(f/g)_\omega$ | 0.293 | 0.465 | 0.438 | 0.483 |
| 763 MeV | $g_\rho^2/4\pi$ | 0.767 | 0.712 | 0.691 | 0.658 |
| | $(f/g)_\rho$ | 4.98 | 5.23 | 5.21 | 5.12 |
| 715 MeV | $g_\epsilon^2/4\pi$ | 14.35 | 12.49 | 12.46 | 12.69 |
| 962 MeV | $g_\delta^2/4\pi$ | 2.41 | 0.828 | 0.900 | 1.18 |
| | $\Lambda$ | 1188 MeV | 1087.5 MeV | 1083 MeV | 1051 MeV |
| | $\chi^2$/data | 2.0 | 2.6 | 3.3 | 3.6 |

[a]There is an error in the definition of $f_V$ in Ref. 1, Eq. (30a). $f_V$ should be divided by $4m$, not $2m$.

[b]These coupling constants least affected by core-region uncertainties, as $S$ waves are not fit.

[c]Improved set of coupling constants for model of Ref. 1.

[d]Best relativistic model for $S$ waves plus all higher partial waves.

[e]Improved set of coupling constants for model of Ref. 2 (BS-III).

BBS equation:

*Fit A.* All phase shifts, with the exception of $\delta(^1S_0)$, $\delta(^3S_1)$, and $\epsilon_1$, were searched against the phase-shift error matrices.[9] [In calculating $\chi^2$, the experimental values of $\delta(^1S_0)$, $\delta(^3S_1)$, and $\epsilon_1$ were taken. As discussed earlier, excluding these three parameters from the fit gives truer information about the meson-nucleon coupling constants.]

*Fit B.* All phase shifts were searched against the phase-shift error matrices, but $\delta(^1S_0)$ and $\delta(^3S_1)$ were not fit to the low-energy effective-range parameters. (Same search as in Ref. 1.)

*Fit C.* All phase shifts were searched against the phase-shift error matrices, and $\delta(^1S_0)$ and $\delta(^3S_1)$ were adjusted to fit the low-energy $n$-$p$ effective-range parameters as well.

A fourth fit was obtained using the nonrelativistic Schrödinger equation:

*Fit D.* Same as Fit C, except that the Schrödinger equation was employed in place of the BBS equation. (Updated BS-III.)

The parameters for Fits A through D are given in Table I.

In Fig. 1 we display the results of Fit C. In the figures we indicate the Livermore group data by heavy error lines.[9] For comparison we also include some representative results of the energy-dependent analyses of the Yale group,[11] indicated by circles and crosses. Phase shifts correspond-

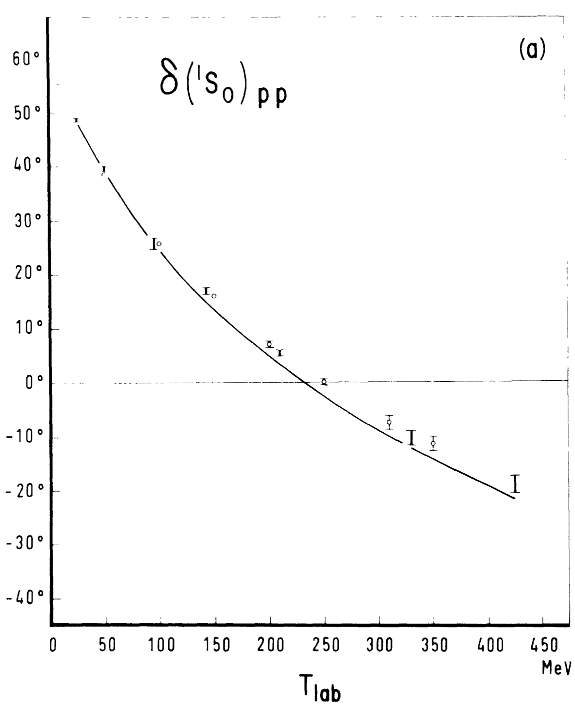


FIG. 1. *(continued on following page)*

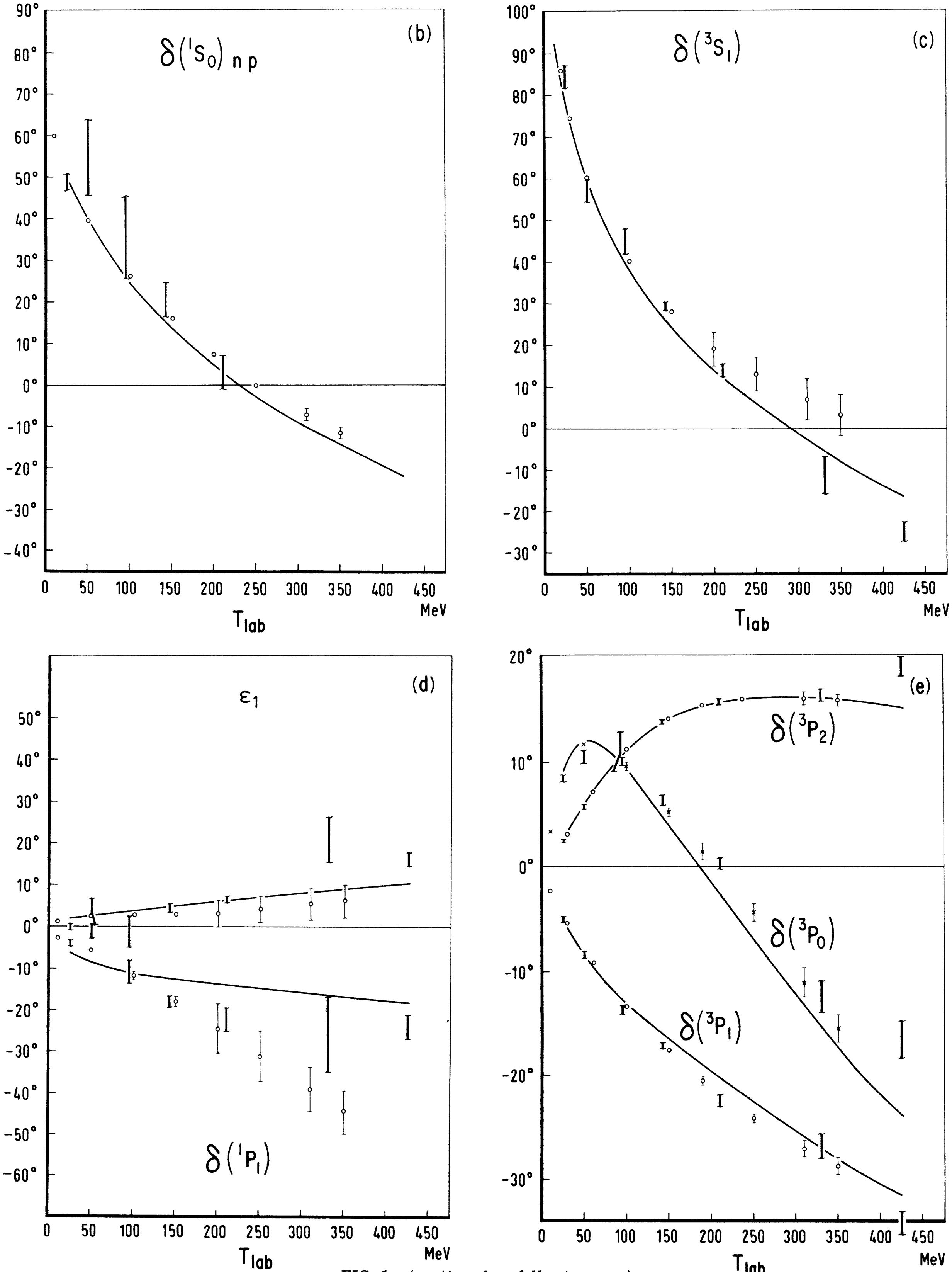


FIG. 1. *(continued on following page)*

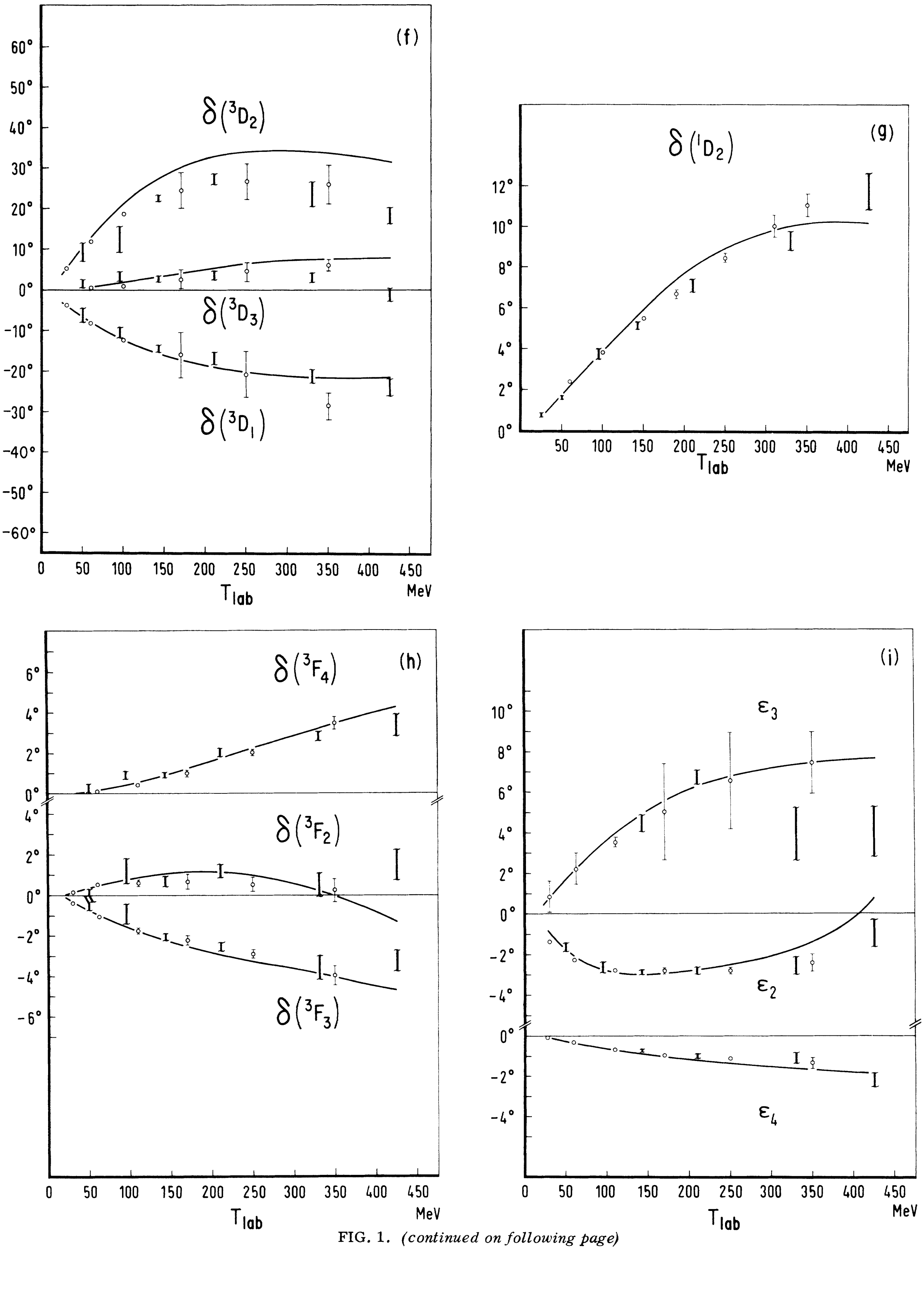


FIG. 1. *(continued on following page)*

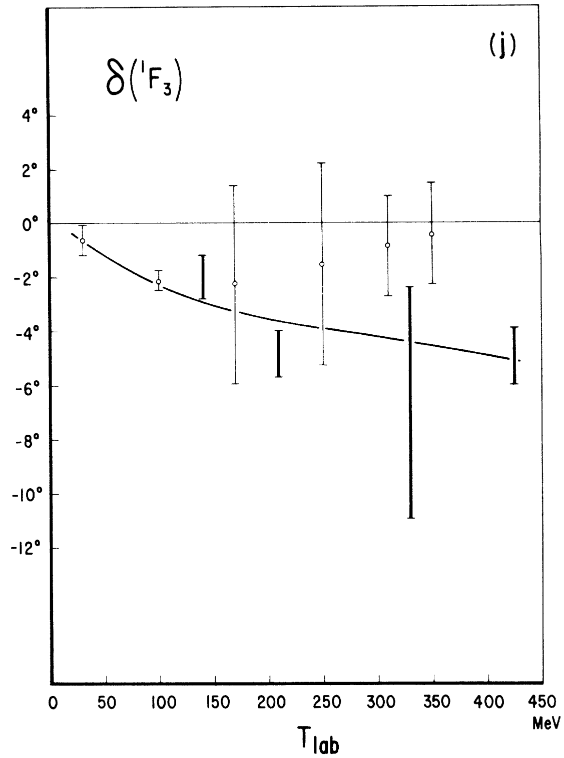


FIG. 1. Nucleon-nucleon nuclear-bar phase shifts predicted by one of the relativistic one-boson-exchange models (Fit C) described in the text. The heavy error lines depict experimental phase shifts found by the Livermore group (Ref. 9), while the circles and crosses correspond to an energy-dependent solution of the Yale group (Ref. 11).

ing to Fits C and D are listed in Tables II and III, respectively.

All calculations were carried out ignoring the Coulomb interaction, hence are $n$-$p$ calculations. However the reader will note that in Fig. 1 the $p$-$p$ $^1S_0$ phase shift is graphed as well as the $n$-$p$ $^1S_0$ phase shift. We determined the $p$-$p$ $^1S_0$ phase shift at each energy by taking the difference between the $p$-$p$ and $n$-$p$ $^1S_0$ phase shifts that had been calculated by the Yale group using the Yale potential, and adding it to our calculated $n$-$p$ $^1S_0$ phase shift. The Yale phase shift differences are listed in Table IV. Also, this derived $p$-$p$ phase shift is the one that we used in fitting the Arndt-MacGregor phase-shift error matrices.

Low-energy parameters were computed for Fits C and D according to the method of Ref. 12. These parameters are displayed in Table V.

It might be appropriate at this point to indicate explicitly how the calculations involving both the narrow and wide mesons with cutoff were carried out. Consider the nonrelativistic Schrödinger-equation model (Fit D). Before introducing width and cutoff, the one-boson-exchange potential to be inserted in the Schrödinger equation,

$$-\frac{\hbar^2}{M}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t},$$

is

$$V(r) = \sum_\nu V^{(\nu)}(r, m_\nu), \quad \nu = \pi, X^0, \rho, \omega, \delta, \epsilon,$$

where $V^{(\nu)}(r)$ signifies the one-boson-exchange potential for the meson $\nu$, listed (through order $p^2/M^2$) in Ref. 2 in Eqs. (1), (2), or (3). For narrow mesons, the momentum-space cutoff factor $\Lambda^2(\Lambda^2+q^2)^{-1}$ of Eq. (6) results in the replacement of $V^{(\nu)}(r, m_\nu)$ by

$$[1-(m_\nu^2/\Lambda^2)]^{-1}[V^{(\nu)}(r, m_\nu) - V^{(\nu)}(r, \Lambda)],$$

where $m_\nu$ is the mass of meson $\nu$ and $\Lambda$ is the cutoff mass. For wide mesons, the momentum-space cutoff function, given by Eq. (7), plus the two-pole approximation to the wide-meson propagator, given in the Appendix, results in the replacement of $V^{(\nu)}$ by

$$[1-(m_\nu^2/\Lambda^2)]^{-1} \times [AV^{(\nu)}(r, m_1) + (1-A)V^{(\nu)}(r, m_2) - V^{(\nu)}(r, \Lambda)],$$

where $A$, $m_1$, and $m_2$ are defined in the Appendix. [Observe that all coupling constants $g_\nu^2$ of Ref. 2 are to be replaced by $g_\nu^2/4\pi$ in this work and in Ref. 1, because of the different definition of the interaction Lagrangian. Of course, the ratios $(f/g)_\nu$ remain unaltered.]

Finally, before discussing our results, we should point out how one might interpret the coupling constants of the $\rho$ and the $\epsilon$ mesons, which appear with different masses in various papers. According to our experience the phase shifts are reproduced quite well if the propagators (6) or (7) are changed in such a way that the approximation (valid for low $q^2$)

$$g^2P(q^2) \simeq A + Bq^2, \tag{8}$$

where $A$ and $B$ are some constants, remains the same even though some parameters are changed. For the propagator (6) the approximation of Eq. (8) is

$$g^2P(q^2) \simeq g^2/m^2 - [g^2(\Lambda^2+m^2)/(m^4\Lambda^2)]q^2. \tag{9}$$

From Eqs. (8) and (9) we see that similar results are obtained if

$$g^2/m^2 = \text{const.} \tag{10}$$

TABLE II. Neutron-proton nuclear-bar phase shifts (in degrees) predicted by one of the relativistic one-boson-exchange models (Fit C) described in the text.

| $T_{lab}$ | 25 MeV | 50 MeV | 95 MeV | 142 MeV | 210 MeV | 330 MeV | 425 MeV |
|---|---|---|---|---|---|---|---|
| $\delta(^1S_0)$ | 50.52 | 39.61 | 25.83 | 15.19 | 3.29 | −12.26 | −21.75 |
| $\delta(^3P_0)$ | 9.24 | 11.97 | 9.72 | 4.74 | −3.00 | −15.32 | −23.63 |
| $\delta(^1P_1)$ | −6.25 | −8.76 | −10.81 | −12.14 | −13.76 | −16.38 | −18.18 |
| $\delta(^3P_1)$ | −5.22 | −8.51 | −12.57 | −15.96 | −20.26 | −26.91 | −31.49 |
| $\delta(^3S_1)$ | 79.17 | 60.06 | 40.29 | 26.60 | 12.34 | −5.38 | −15.99 |
| $\epsilon_1$ | 2.11 | 2.78 | 3.72 | 4.73 | 6.22 | 8.66 | 10.31 |
| $\delta(^3D_1)$ | −3.01 | −6.83 | −12.16 | −15.93 | −19.24 | −21.58 | −21.38 |
| $\delta(^1D_2)$ | 0.72 | 1.73 | 3.65 | 5.61 | 7.99 | 10.28 | 10.58 |
| $\delta(^3D_2)$ | 4.13 | 10.37 | 20.37 | 27.60 | 32.96 | 34.11 | 31.96 |
| $\delta(^3P_2)$ | 2.61 | 6.00 | 10.77 | 13.71 | 15.63 | 15.97 | 15.11 |
| $\epsilon_2$ | −0.86 | −1.81 | −2.74 | −2.99 | −2.72 | −1.69 | −0.84 |
| $\delta(^3F_2)$ | 0.11 | 0.34 | 0.75 | 1.04 | 1.11 | 0.20 | −1.31 |
| $\delta(^1F_3)$ | −0.45 | −1.20 | −2.25 | −2.96 | −3.60 | −4.42 | −5.08 |
| $\delta(^3F_3)$ | −0.24 | −0.73 | −1.55 | −2.21 | −2.93 | −3.92 | −4.69 |
| $\delta(^3D_3)$ | 0.07 | 0.41 | 1.70 | 3.38 | 5.56 | 7.69 | 7.89 |
| $\epsilon_3$ | 0.59 | 1.72 | 3.55 | 4.96 | 6.28 | 7.36 | 7.66 |
| $\delta(^3G_3)$ | −0.06 | −0.27 | −0.91 | −1.69 | −2.78 | −4.15 | −4.65 |
| $\delta(^1G_4)$ | 0.04 | 0.16 | 0.40 | 0.65 | 1.02 | 1.71 | 2.26 |
| $\delta(^3G_4)$ | 0.18 | 0.77 | 2.16 | 3.67 | 5.73 | 8.79 | 10.70 |
| $\delta(^3F_4)$ | 0.02 | 0.11 | 0.42 | 0.89 | 1.70 | 3.19 | 4.21 |
| $\epsilon_4$ | −0.05 | −0.20 | −0.53 | −0.84 | −1.20 | −1.61 | −1.75 |
| $\delta(^3H_4)$ | 0.00 | 0.03 | 0.10 | 0.20 | 0.34 | 0.56 | 0.62 |

Thus Eq. (10) can be used as a crude approximation to compare coupling constants corresponding to different meson masses.

## III. DISCUSSION

Our results show that the inclusion of the widths of the $\epsilon$ and $\rho$ mesons helps to get a better fit to the experimental nucleon-nucleon scattering data. In terms of $\chi^2$/data point, our fit is better than the similar fit of Gersten *et al.*,[1] who used the same data in conjunction with the BBS equation, but did not take the widths of the $\epsilon$ and $\rho$ mesons into account. Our fit has $\chi^2$/data point = 2.6 (Fit B) compared to $\chi^2$/data point = 4.6 of Gersten *et al.*[1] or $\chi^2$/data point = 4.8 of Ueda and Green.[13] We use the experimental masses of the $\epsilon$ and $\rho$ mesons. However, taking a lower-mass meson can be also justified, as a very broad mass $\epsilon$ can act effectively as if it has a lower mass. We do not have enough information on the properties of a propagator of a very unstable particle in the spacelike region of the momentum. We only know its behavior for timelike momenta near the pole of the propagator, which is determined from the mass and width of the $\epsilon$ meson. It is difficult to make the extrapolation to the spacelike region of the momenta. However, the propagator given by Eq. (1) is an attempt in that direction.[7,14] For low momentum transfers it acts as if it has a lower effective mass. Other attempts, for example those of Gounaris[15] or Schwinger,[16] also indicate that for low momentum transfers an effectively lower meson mass should

TABLE III. Neutron-proton nuclear-bar phase shifts (in degrees) predicted by the nonrelativistic Schrödinger-equation model (Fit D) described in the text. (Updated BS-III.)

| $T_{lab}$ | 25 MeV | 50 MeV | 95 MeV | 142 MeV | 210 MeV | 330 MeV | 425 MeV |
|---|---|---|---|---|---|---|---|
| $\delta(^1S_0)$ | 50.17 | 39.33 | 25.85 | 15.47 | 3.80 | −11.66 | −21.32 |
| $\delta(^3P_0)$ | 9.00 | 11.63 | 9.37 | 4.33 | −3.64 | −16.59 | −25.51 |
| $\delta(^1P_1)$ | −6.12 | −8.62 | −10.71 | −12.22 | −14.38 | −17.76 | −19.91 |
| $\delta(^3P_1)$ | −5.12 | −8.43 | −12.67 | −16.43 | −21.52 | −29.59 | −35.20 |
| $\delta(^3S_1)$ | 78.65 | 59.71 | 40.59 | 27.47 | 13.41 | −4.39 | −14.53 |
| $\epsilon_1$ | 1.81 | 2.10 | 2.29 | 2.56 | 2.99 | 3.76 | 4.44 |
| $\delta(^3D_1)$ | −2.91 | −6.59 | −11.91 | −16.34 | −21.24 | −26.54 | −29.69 |
| $\delta(^1D_2)$ | 0.71 | 1.71 | 3.56 | 5.33 | 7.26 | 8.60 | 8.14 |
| $\delta(^3D_2)$ | 4.07 | 10.34 | 20.78 | 28.37 | 34.41 | 37.92 | 36.71 |
| $\delta(^3P_2)$ | 2.64 | 6.00 | 10.57 | 13.31 | 15.00 | 15.15 | 14.37 |
| $\epsilon_2$ | −0.85 | −1.78 | −2.66 | −2.86 | −2.53 | −1.27 | −0.23 |
| $\delta(^3F_2)$ | 0.11 | 0.34 | 0.76 | 1.07 | 1.21 | 0.39 | −1.18 |
| $\delta(^1F_3)$ | −0.44 | −1.19 | −2.25 | −2.98 | −3.72 | −4.70 | −5.49 |
| $\delta(^3F_3)$ | −0.24 | −0.72 | −1.55 | −2.27 | −3.16 | −4.63 | −5.93 |
| $\delta(^3D_3)$ | 0.09 | 0.51 | 1.99 | 4.01 | 6.60 | 8.41 | 8.48 |
| $\epsilon_3$ | 0.58 | 1.70 | 3.55 | 5.01 | 6.30 | 7.17 | 7.45 |
| $\delta(^3G_3)$ | −0.05 | −0.27 | −0.95 | −1.80 | −3.00 | −5.00 | −6.61 |
| $\delta(^1G_4)$ | 0.04 | 0.16 | 0.40 | 0.65 | 1.03 | 1.68 | 2.14 |
| $\delta(^3G_4)$ | 0.18 | 0.76 | 2.18 | 3.73 | 5.87 | 9.44 | 11.83 |
| $\delta(^3F_4)$ | 0.02 | 0.11 | 0.43 | 0.91 | 1.75 | 3.20 | 4.15 |
| $\epsilon_4$ | −0.05 | −0.20 | −0.53 | −0.85 | −1.24 | −1.68 | −1.85 |
| $\delta(^3H_4)$ | 0.00 | 0.03 | 0.10 | 0.20 | 0.36 | 0.57 | 0.62 |

be taken. For example, we have calculated that for low momentum the Gounaris $\rho$-meson propagator is approximately equal to a propagator of a stable particle with a mass of about 712 MeV. By taking into account the broad width of the $\epsilon$ and $\rho$ mesons we have incorporated in our model some contributions coming from the correlated two-pion exchanges. Of course, approximate ladder diagrams are generated by the BBS equation.

One may raise many objections to the use of our model. First of all our equations, based on quantum field theory, do not include many important diagrams. One may argue that the inclusion of meson resonances may in some way simulate multi-pion exchanges; others may claim that the main contribution to the interaction comes somehow from the meson resonances. In any case, there is one striking feature of our model; namely, our coupling constants are in a good qualitative agreement with other experiments and model theories. For example, the pion-nucleon coupling constants, $g_\pi^2/4\pi$ of Table I, should be compared with the recommended value of $14.64^{+0.54}_{-0.72}$ obtained from interpretation of the $\pi N$ scattering experiments.[17] On the basis of the vector-meson-dominance model the value of about 0.6 is obtained for

TABLE IV. Coulomb corrections for the $^1S_0$ phase shift calculated by the Yale group.[a]

| $T_{lab}$ | 25 MeV | 50 MeV | 95 MeV | 142 MeV | 210 MeV | 330 MeV |
|---|---|---|---|---|---|---|
| $\delta(^1S_0)_{np} - \delta(^1S_0)_{pp}$ | 1.82° | 0.83° | 0.43° | 0.25° | 0.15° | 0.03° |

[a]See Ref. 11. Obtained by subtracting values for $\delta(^1S_0)_{pp}$, given in Table II of that paper, from values for $\delta(^1S_0)_{np}$, given in Table III of the same paper.

the $g_p^2/4\pi$ coupling constant.[17] (Note a difference of a factor of 4 between our definition of the coupling constant and that of Ref. 17.) This coupling constant has values close to 0.6 for many different experiments and is close to our results. For other coupling constants the interpretation of various experiments does not lead to as well-defined values, but our results qualitatively agree with them. For further discussion see Ref. 18.

We have worked with both the relativistic BBS equation and the Schrödinger equation; differences appear mainly in the $S$ waves and $\epsilon_1$. Incidentally, one interesting difference is that at 330 MeV $\epsilon_1 = 8.7°$ in the BBS model (Fit C) and only 3.8° in the Schrödinger model (Fit D).[19] (See Tables II and III.) However, as far as the fit to the data is concerned, there is little significant difference between these models and a similar $\chi^2$/data point is obtained.

## ACKNOWLEDGMENTS

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## APPENDIX

The propagator

$$P(q^2) = 1/[m^2 + q^2 + \gamma(q^2 + 4m_\pi^2)^{1/2}]$$

is approximated by

$$P(q^2) \simeq \frac{A}{m_1^2 + q^2} + \frac{B}{m_2^2 + q^2}\,. \tag{A1}$$

The four constants $A$, $B$, $m_1^2$, $m_2^2$ are determined by imposing the following four conditions on the approximation (A1). We require that the approximation be exact for

(1) $q^2 \to \infty$,

(2) $q^2 = 0$,

(3) small $q^2$ $\left(\text{i.e., for } \left.\frac{dP}{dq^2}\right|_{q^2=0}\right)$, (A2)

and

(4) $q^2 = m^2$.

From condition (1) we have

$$A + B = 1.$$

Let us write

$$A = (m_1^2 - m_3^2)/(m_1^2 - m_2^2)$$

TABLE V. Neutron-proton low-energy parameters obtained in Fits C and D.

| | Fit C | Fit D | Experiment |
|---|---|---|---|
| $a_s$ | −23.71 F | −23.77 F | −23.715 ± 0.015 F [a] |
| $\rho_s(0,0)$ | 2.70 F | 2.73 F | 2.73 ± 0.03 F [a] |
| $a_t$ | 5.39 F | 5.41 F | 5.414 ± 0.005 F [b] |
| $\rho_t(0,0)$ | 1.81 F | 1.84 F | |
| Binding energy B | 2.3 MeV | 2.24 MeV | 2.2245 ± 0.0002 MeV [c] |
| $\rho_t(-B,-B)$ | | 1.84 F | 1.82 ± 0.05 F [c] |
| Quadrupole moment Q | | 0.277 $F^{-2}$ | 0.282 $F^{-2}$ [c] |

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and

$$B=(m_3^2-m_2^2)/(m_1^2-m_2^2).$$

Then conditions (2), (3), and (4) of (A2) lead to

$$m_3^2=\frac{m^2[(m^2+4m_\pi^2)^{1/2}-2m_\pi]}{m^2/(4m_\pi)+2m_\pi-(m^2+4m_\pi^2)^{1/2}},$$

$$m_1^2m_2^2=(m^2+2m_\pi\gamma)m_3^2,$$

and

$$m_1^2+m_2^2=m^2+m_3^2(1+\gamma/4m_\pi)+2m_\pi\gamma,$$

from which $m_1^2$, $m_2^2$, and $m_3^2$ can be easily determined.

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