High-Frequency Sound Waves to Eliminate a Horizon in the Mixmaster Universe*

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From the linear wave equation for small-amplitude sound waves in a curved space-time, there is derived a geodesiclike differential equation for sound rays to describe the motion of wave packets. These equations are applied in the generic, nonrotating, homogeneous closed-model universe (the "mixmaster universe," Bianchi type IX). As for light rays described by Doroshkevich and Novikov (DN), these sound rays can circumnavigate the universe near the singularity to remove particle horizons only for a small class of these models and in special directions. Although these results parallel those of DN, different Hamiltonian methods are used for treating the Einstein equations.

I. INTRODUCTION

The present-day universe can be described very well by the Robertson-Walker cosmological models. The extrapolation of these models for the early times of the universe gives rise to the problem of particle horizons.[1] A particle horizon at a particular epoch bounds each finite part of the universe which could have been spanned by a causal signal during the time available since the initial singularity. Since the Robertson-Walker models possess particle horizons, only a finite part of such universe could have been causally connected. Thus, we are faced with the observation of the microwave background radiation having precisely ($\leqslant 0.2\%$) the same temperature[2] in widely different directions even though the regions of plasma which scattered the radiation last had no prior causal relationship. The Robertson-Walker models therefore are too simplified to describe the early phase of the universe. Here we would consider a more general model of the universe – the nonrotating Bianchi type IX model. It has a very different singularity behavior,[3,4] but it could evolve into the closed Robertson-Walker model at the present epoch. Misner[3] first pointed out the possibility of mixing by light in these models.

Doroshkevich and Novikov[5] (DN) quote the results of their investigation of the propagation of light in the mixmaster universe. Doroshkevich, Lukash, and Novikov[6] in a recent report apply these results for finding the likelihood of horizon vanishing and find it to be very low. Our results are in substantial agreement with theirs. In a future paper, we will show how our formulation and treatment of the problem gives us a natural probabilistic estimate for horizon vanishing. Here we will derive the equations for rays of high-frequency sound waves in these generic models and study their behavior in a certain class of solutions to Einstein's equations. The Hamiltonian methods which we use to obtain information about the relevant solutions to Einstein's equations are quite different from the ones employed by Belinski *et al.*[7] or Doroshkevich and Novikov.[5] Also we do not reject the application of our calculations to epochs where quantum effects could enter. We look forward to calculations in which quantum effects might be included and would meaningfully modify the interpretation of these small perturbations.

The metric of the Bianchi type IX for an anisotropic nonrotating universe can be written as

$$ds^2 = -dt^2 + (6\pi)^{-1} e^{-2\Omega} (e^{2\beta})_{ij}\, \sigma_i \sigma_j \,, \qquad (1.1)$$

where

$$\sigma_1 = \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi \,,$$

$$\sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi \,,$$

$$\sigma_3 = -(d\psi + \cos\theta\, d\phi)$$

satisfy $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$ and are differential forms on the three-sphere parametrized by Euler angles ψ, θ, ϕ with $0 \leqslant \psi \leqslant 4\pi$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \phi \leqslant 2\pi$. The quantities Ω and β_{ij} depend only on time, with Ω determining the volume and β_{ij} a diagonal traceless 3×3 matrix

$$\beta = \mathrm{diag}(\beta_1, \beta_2, \beta_3)$$

governing the anisotropy (shape). Note that for $\beta_{ij} = 0$ this metric is one form for the positive-curvature Robertson-Walker metric. As two independent shape parameters choose

$$\beta_+ = \tfrac{1}{2}(\beta_1 + \beta_2)$$

and

$$\beta_- = (\beta_1 - \beta_2)/2\sqrt{3}\,.$$

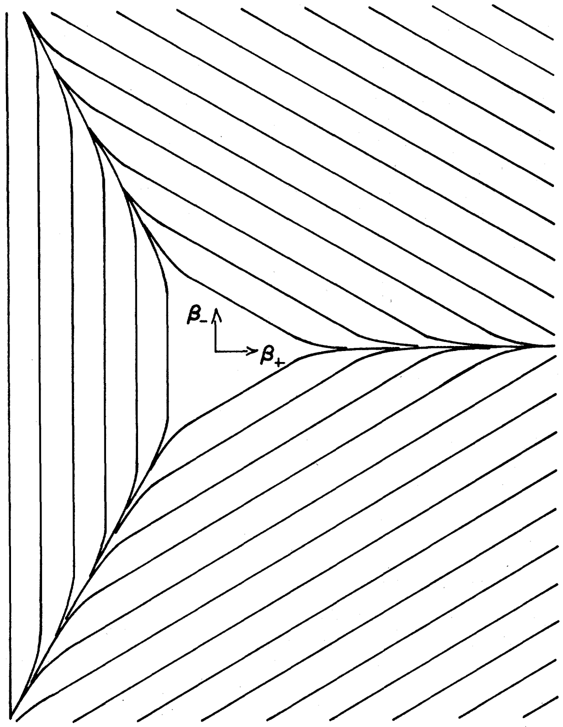

FIG. 1. The equipotentials of $V(\beta_+,\beta_-)$ for large "β". (Figure courtesy of C. Misner).

The variational principle for Einstein's equations $\delta I = 0$ with

$$I=(16\pi)^{-1}\int R(-g)^{1/2}d^4x$$

can be cast into a canonical form to obtain[8] the Hamiltonian

$$H=[p_+{}^2+p_-{}^2+e^{-4\Omega}(V-1)]^{1/2}. \qquad (1.2)$$

p_+ and p_- are the momenta conjugate to the field amplitudes β_+ and β_-, respectively, with Ω as the choice for the independent (coordinate time) variable. An equation giving Ω as a function of the cosmic time t is

$$dt=-\tfrac{2}{3}\frac{1}{H}e^{-3\Omega}d\Omega. \qquad (1.3)$$

The "anisotropy potential" $V(\beta_+,\beta_-)$ arises due to the anisotropy of the curvature of the three-dimensional space sections of the universe. The potential walls rise steeply away from $\beta=0$, with the equipotentials asymptotically forming equilateral triangles in the $\beta_+\beta_-$ plane as shown in Fig. 1. One of the three equivalent sides of the triangle is described by the asymptotic form

$$V\sim\tfrac{1}{3}e^{-8\beta_+},\quad \beta_+\to-\infty, \qquad (1.4)$$

which is valid in the sector $|\beta_-|<-\sqrt{3}\,\beta_+$. The corners of this triangular potential are flared open; for instance if $\beta_+\to\infty$ with $|\beta_-|\ll 1$, one finds

$$V(\beta)\sim 16\beta_-{}^2e^{4\beta_+}+1. \qquad (1.5)$$

The evolution of the universe is described by the motion of the system point $\beta\equiv(\beta_+,\beta_-)$ as a function of the time coordinate Ω. When β is well away from the potential walls, the universe point moves with velocity

$$\beta'\equiv\frac{d\beta}{d\Omega}=\left[\left(\frac{d\beta_+}{d\Omega}\right)^2+\left(\frac{d\beta_-}{d\Omega}\right)^2\right]^{1/2}$$

of unit magnitude in straight lines and it can be parametrized as

$$\frac{d\beta_+}{d\Omega}=\frac{u^2+u-\frac{1}{2}}{u^2+u+1}, \qquad (1.6)$$

$$\frac{d\beta_-}{d\Omega}=\frac{\sqrt{3}\,(u+\frac{1}{2})}{u^2+u+1}, \qquad (1.7)$$

where the parameter u goes from $-\infty$ to ∞. The potential walls move outward with velocity (in the sense of $d\beta_{\rm wall}/d\Omega$) $\frac{1}{2}$. The system point β would thus move in one direction with unit velocity till it comes close to one of the walls and feels the potential and would then bounce off the wall changing its direction. Furthermore, Belinski and Khalatnikov[4] have shown that all solutions would come arbitrarily close to the values $u=-2, -1, -\frac{1}{2}, 0, 1, \infty$ after rattling back and forth between the walls. These values of u correspond to the system point moving parallel to the three corner axes.

When the system point is well inside the walls, the potential V can be neglected. But $V=0$ just gives the Einstein equations $R_{\mu\nu}=0$ for Bianchi type I. One finds then[3] that these epochs parallel Kasner solutions using $\Omega=-\frac{1}{3}\log t+\text{constant}$ as the independent variable; the Kasner metric being given by

$$ds^2=-dt^2+R_0{}^2(t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2),$$

where the exponents p_1, p_2, and p_3 are connected by the following two relations:

$$p_1+p_2+p_3=p_1{}^2+p_2{}^2+p_3{}^2=1.$$

Thus the β point shifts from one Kasner-like model to another at each collision with a potential wall. For the Kasner solution with $p_1=p_2=0$, $p_3=1$, there exist no horizons for causal propagation in the z direction.[9] Similarly, there is absence of horizons in the other two directions for Kasner metrics with $p_1=p_3=0$, $p_2=1$ and $p_2=p_3=0$, $p_1=1$, respectively. This motivates us to study the epochs of Bianchi type IX model which approximate these Kasner solutions for a long period of time. These epochs can be seen to be the ones when the system point is moving parallel to one of the axes of the equipotential triangle and is either running towards a corner or following an inclined wall. When the system point is running towards a corner on the β_+ axis, the parameter "u" designating the direction of the velocity is asymptotically ∞ and β_+ is very large; $|\beta_-| \ll 1$ giving the universe a pancake-shaped anisotropy corresponding to a relative compression of the 3 axis (ψ axis) with the other two axes approximately equal. While near the inclined walls, say for $\beta_1 \to \infty$, the anisotropy is cigar-shaped with the stretching of the 1 axis relative to the others. So we expect the null geodesics in the ψ direction to go around the universe during the $u=\infty$ epochs.

In Sec. II we will derive the equations for the propagation of high-frequency sound waves and in the following sections we will study their behavior during the epochs when u is very large. It will be seen that there exists a set of initial conditions for which the special Kasner-like behavior persists long enough for these sound waves to go around the universe in the ψ direction. This possibility of communication either by sound waves or light rays along a certain direction during the evolution of a universe will be called the removal of horizon in that direction for that universe.

II. THE PROPAGATION OF HIGH-FREQUENCY SOUND WAVES

Let $\bar{\epsilon}$, $\bar{p}$, and $\bar{u}^\mu$ be the energy density, pressure, and the four-velocity of the fluid, and let ϵ', p', and u'^μ be the small amplitude, high-frequency perturbations on the above solution. The propagation of the disturbance is governed by the energy equation

$$\epsilon_{,\mu} u^\mu + (p+\epsilon) u^\mu{}_{;\mu} = 0 \tag{2.1}$$

and the Euler equation

$$(p+\epsilon) u^\mu{}_{;\nu} u^\nu = -(g^{\mu\nu} + u^\mu u^\nu) p_{,\nu} . \tag{2.2}$$

Substituting $p = \bar{p} + p'$, $\epsilon = \bar{\epsilon} + \epsilon'$, and $u = \bar{u} + u'$ in Eqs. (2.1) and (2.2) and linearizing we obtain

$$\epsilon'_{,\mu} \bar{u}^\mu + \epsilon_{,\mu} u'^\mu + (\epsilon' + p') \bar{u}^\mu{}_{;\mu} + (\bar{\epsilon} + \bar{p}) u'^\mu{}_{;\mu} = 0 \tag{2.3}$$

and

$$(\bar{\epsilon}+\bar{p})(\bar{u}^\mu{}_{;\nu} u'^\nu + u'^\mu{}_{;\nu} \bar{u}^\nu) + (\epsilon' + p')(\bar{u}^\mu{}_{;\nu} \bar{u}^\nu) = -(g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) p'_{,\nu} - u'^\mu \bar{u}^\nu \bar{p}_{,\nu} - \bar{u}^\mu u'^\nu \bar{p}_{,\nu} . \tag{2.4}$$

Differentiating Eqs. (2.4) with respect to μ and substituting for $u'^\mu{}_{;\mu}$ from Eq. (2.3) we get

$$(g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) p'_{,\nu\mu} - \epsilon'_{,\mu\nu} \bar{u}^\mu \bar{u}^\nu = F , \tag{2.5}$$

where F is a scalar function which contains the high-frequency perturbations ϵ', p', and u'^μ only up to their first derivatives.

Writing $p' = Ae^{i\phi}$, where ϕ is a rapidly varying function, and setting the dominant terms in Eq. (2.5) equal to zero, we obtain

$$(g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) \phi_{,\mu} \phi_{,\nu} - \left(\frac{\partial \epsilon}{\partial p}\right)_s \phi_{,\mu} \phi_{,\nu} \bar{u}^\mu \bar{u}^\nu = 0 , \tag{2.6}$$

where $v_s = [(\partial p/\partial \epsilon)_s]^{1/2}$ is the sound velocity. The Eq. (2.6) is a Hamilton-Jacobi equation corresponding to

$$H = \tfrac{1}{2}(g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) p_\mu p_\nu - \frac{1}{v_s^2} \tfrac{1}{2} \bar{u}^\mu \bar{u}^\nu p_\mu p_\nu \tag{2.7}$$

as a particle Hamiltonian. To obtain the corresponding Lagrangian, we solve for p_μ from one set of Hamilton's equations:

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu} = (g^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) p_\nu - \frac{1}{v_s^2} \bar{u}^\mu \bar{u}^\nu p_\nu , \tag{2.8}$$

where $x^\mu \equiv (t, \theta, \phi, \psi)$.

Noting that $\bar{u} = \partial/\partial t$ for comoving coordinates, we can invert (2.8) to obtain

$$p_\nu = \frac{dx^\mu}{d\lambda} [g_{\mu\nu} + (1 - v_s^2) \bar{u}_\mu \bar{u}_\nu] . \tag{2.9}$$

Thus, we get the Lagrangian L as follows:

$$\begin{aligned} L &= p_\mu \frac{dx^\mu}{d\lambda} - H \\ &= \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} [g_{\mu\nu} + (1 - v_s^2) \bar{u}_\mu \bar{u}_\nu] \\ &\quad - \tfrac{1}{2} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} [g_{\mu\nu} + (1 - v_s^2) \bar{u}_\mu \bar{u}_\nu] \end{aligned} \tag{2.10}$$

$$= \tfrac{1}{2} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} [g_{\mu\nu} + (1 - v_s^2) \bar{u}_\mu \bar{u}_\nu] . \tag{2.11}$$

The propagation of rays is then given by the Lagrange equations

$$\frac{d}{d\lambda}\left[2 g_{\mu\nu} \frac{dx^\nu}{d\lambda} + 2(1 - v_s^2) \bar{u}_\mu \bar{u}_\nu \frac{dx^\nu}{d\lambda}\right] = \frac{dx^\rho}{d\lambda} \frac{dx^\nu}{d\lambda} \frac{\partial g_{\rho\nu}}{\partial x^\mu} . \tag{2.12}$$

Consider a possible set of solutions with θ = constant and ϕ = constant. Then the Lagrange equations reduce to

$$\frac{d}{d\lambda}\left[-2 \frac{dt}{d\lambda} + 2(1 - v_s^2)\left(\frac{dt}{d\lambda}\right)\right] = \left(\frac{d\psi}{d\lambda}\right)^2 \frac{\partial g_{\psi\psi}}{\partial t} , \tag{2.13}$$

$$\frac{\partial g_{\psi\psi}}{\partial\theta}\left(\frac{d\psi}{d\lambda}\right)^2=0, \qquad (2.14)$$

$$\frac{d}{d\lambda}\left(g_{\psi\phi}\frac{d\psi}{d\lambda}\right)=0, \qquad (2.15)$$

$$\frac{d}{d\lambda}\left(g_{\psi\psi}\frac{d\psi}{d\lambda}\right)=\left(\frac{d\psi}{d\lambda}\right)^2\frac{\partial g_{\psi\psi}}{\partial\psi}. \qquad (2.16)$$

Since $g_{\psi\psi}$ is a function of t only, Eq. (2.14) is identically satisfied, while (2.15) and (2.16) reduce to

$$\frac{d}{d\lambda}\left(\cos\theta\, e^{-2\Omega}e^{2\beta_3}\frac{d\psi}{d\lambda}\right)=0 \qquad (2.17)$$

and

$$\frac{d}{d\lambda}\left(e^{-2\Omega}e^{2\beta_3}\frac{d\psi}{d\lambda}\right)=0. \qquad (2.18)$$

For $\theta=$constant, (2.17) reduces to (2.18). So the Lagrange equations now reduce to Eq. (2.13) and Eq. (2.18) which can be solved for $dt/d\lambda$ and $d\psi/d\lambda$. Putting $H=0$ in Eq. (2.10) we obtain

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}[g_{\mu\nu}+(1-v_s^2)\bar{u}_\mu\bar{u}_\nu]=0$$

or

$$-\left(\frac{dt}{d\lambda}\right)^2+g_{33}\left(\frac{d\psi}{d\lambda}\right)^2+\left(\frac{dt}{d\lambda}\right)^2(1-v_s^2)=0 \qquad (2.19)$$

for $\theta=$constant, $\phi=$constant class of solutions. From Eq. (2.19) we obtain

$$\frac{d\psi}{dt}=\frac{v_s}{\sqrt{g_{33}}}=\sqrt{6\pi}\, v_s e^{\Omega}e^{-\beta_3}. \qquad (2.20)$$

By putting $v_s=1$, we get the law of propagation for light going in the ψ direction.

III. THE REMOVAL OF HORIZONS

Let us now study the behavior of the above high-frequency sound waves during the $u=\infty$ epochs. First consider the axial case when the system point is very close to the β_+ axis and is running towards the corner. This is the case which Belinski *et al.*[7] call the case of small oscillations. The appropriate solution to Einstein's equations as derived in Sec. IV is

$$\beta_-=Z_0\left(\frac{2e^{2\beta_0}}{K}\right), \qquad [\text{see } (4.8)]$$

where Z_0 is the Bessel function of order zero. K is a constant and β_0 is defined as

$$\beta_0\equiv\beta_+-\Omega.$$

The variation of β_0 is given by

$$\frac{d\beta_0}{d\Omega}=-\frac{K}{H}. \qquad [\text{see } (4.3)]$$

Writing $\beta_3=-2\beta_+=-2(\beta_0+\Omega)$ in Eq. (2.20), we reexpress the equation of the sound-wave propagation in the ψ direction as

$$\frac{d\psi}{dt}=\sqrt{6\pi}\, v_s e^{3\Omega}e^{2\beta_0}.$$

Using Eq. (1.3), the change in ψ can be given in terms of the variable Ω as

$$\frac{d\psi}{d\Omega}=\frac{d\psi}{dt}\frac{dt}{d\Omega}$$

$$=-v_s\frac{2}{H}e^{2\beta_0}. \qquad (3.1)$$

Hence, the change in ψ along the sound wave between the epochs Ω_1 and Ω_2 is given by

$$\Delta\psi=\int_{\Omega_1}^{\Omega_2}d\psi$$

$$=\int_{\Omega_1}^{\Omega_2}v_s\frac{2}{H}e^{2\beta_0}d\Omega$$

$$=v_s\int_{\Omega_1}^{\Omega_2}\frac{2}{H}e^{2\beta_0}\frac{d\Omega}{d\beta_0}d\beta_0. \qquad (3.2)$$

Substituting Eq. (4.3) in Eq. (3.2), we obtain

$$\Delta\psi=-\frac{2}{K}v_s\int_{\Omega_1}^{\Omega_2}e^{2\beta_0}d\beta_0 \qquad (3.3)$$

$$=v_s\Delta\left(\frac{e^{2\beta_0}}{K}\right). \qquad (3.4)$$

Therefore, a change of $4\pi/v_s$ in $e^{2\beta_0}/K$ would give a change of 4π in ψ. Since $\beta_-=Z_0(2e^{\beta_0}/K)$, which for small K goes roughly as

$$\left(\frac{K}{\pi e^{2\beta_0}}\right)^{1/2}\cos\left(\frac{2e^{2\beta_0}}{K}-\tfrac{1}{4}\pi\right)$$

or

$$\left(\frac{K}{\pi e^{2\beta_0}}\right)^{1/2}\sin\left(\frac{2e^{2\beta_0}}{K}-\tfrac{1}{4}\pi\right),$$

β_- would go through four cycles as its argument changes by 8π. Thus setting $v_s=1$, we see that the light ray would circumnavigate the universe in the ψ direction during four cycles of β_-. This corresponds to the DN result of $N_e=\frac{1}{4}N_m$. For radiation-filled universe, the velocity v_s of the sound-wave propagation will be $1/\sqrt{3}$; as a result these waves would go around the universe in the ψ direction during seven cycles of β_-. Similarly, when the system point is running towards the other two corners, the causal and the high-frequency sound wave influence would circumnavigate in the other two principal directions.

Next consider the off-axial case with u very large and $\beta_->1$. The appropriate solution to Einstein's equations as derived in Sec. IV again gives

$$\frac{d\beta_0}{d\Omega} = -\frac{K}{H}, \qquad \text{[see (4.11)]}$$

while the total change in β_0 during one bounce with the inclined potential wall for large u is given by

$$\Delta\beta_0 = \frac{1}{2u_i}, \qquad \text{[see (4.17)]}$$

where u_i is the value of u before the bounce. The change in ψ along the high-frequency sound wave ray going in the ψ direction is again given by Eq. (3.3)

$$\Delta\psi = \frac{2v_s}{K}\int e^{2\beta_0}d\beta_0 .$$

So during one collision with the wall, the change in ψ would be

$$\Delta\psi = \frac{v_s}{K}(e^{2\beta_0})_i\,[1-e^{2\Delta(\beta_0)}]$$
$$= \frac{v_s}{K}(e^{2\beta_0})_i\,\frac{1}{u_i}, \qquad (3.5)$$

where the subscript i denotes the values of the variables before the collision. The value of the constant K can be obtained in terms of u_i and H_i from Eq. (1.6):

$$\frac{d\beta_+}{d\Omega} = \frac{u^2+u-\frac{1}{2}}{u^2+u+1}$$

and Eq. (4.11)

$$\frac{d\beta_0}{d\Omega} = -\frac{K}{H}.$$

Then $\Delta\psi$ is given in terms of the initial values as

$$\Delta\psi = v_s\frac{2}{3}\frac{u_i^2+u_i+1}{H_i}(e^{2\beta_0})_i\frac{1}{u_i}$$
$$= \left(\frac{2}{3}v_s\frac{e^{2\beta_0}}{H}u\right)_i \quad \text{for large } u_i . \qquad (3.6)$$

As the system point evolves, consider the epoch when the system point had its first collision with the inclined wall for large u. So the system point has just bounced back off the vertical wall and is going towards the inclined wall at say $\Omega=\Omega_b$. The position of the potential wall is then given by

$$H^2 = e^{-4\Omega_b}V(\beta)$$
$$= e^{-4\Omega_b}\tfrac{1}{3}e^{-8(\beta_+)_{\text{wall}}}$$
$$= \tfrac{1}{3}e^{-12\Omega_b}\,e^{-8(\beta_0)_{\text{wall}}}.$$

Substituting the expression for H in Eq. (3.6) and dropping the subscripts, we get

$$\Delta\psi = (v_s\tfrac{2}{3}e^{2\beta_0}u)\Big/\left(\frac{1}{\sqrt{3}}e^{-6\Omega_b}e^{-4\beta_0}\right)$$
$$= \frac{2}{\sqrt{3}}uv_s\,e^{6(\beta_+)_{\text{wall}}}. \qquad (3.7)$$

Therefore, for all solutions for which at the beginning of the series of collisions with the inclined wall, the value of u is such that

$$u > \frac{\sqrt{3}}{2v_s}4\pi e^{-6(\beta_+)_{\text{wall}}},$$

then the high-frequency sound wave communication has an open channel in the ψ direction. Since $(\beta_+)_{\text{wall}}$ is negative (it goes as: $\beta_+ \simeq -\frac{1}{2}\Omega+\text{constant}$), we find that there exist small sectors around the lines parallel to the β_+ axis such that when the system point is running along these sectors at Ω_b, a horizon is removed in the ψ direction during the next bounce with the inclined potential wall. The angular extent of these sectors depends upon Ω and it goes to zero as Ω goes to ∞.

One concludes, therefore, that at each epoch there exist certain subsets of initial conditions $[\beta_+,\beta_-;u(\Omega)]$, such that some rays of high-frequency sound waves and null geodesics will proceed to circumnavigate the corresponding universe. It will be shown in a future publication that the universe point wanders about in a truly ergodic fashion and that by finding a measure on initial conditions, one can compute the probability for a typical solution to have no horizon along one axis.

IV. $u=\infty$ SOLUTIONS OF EINSTEIN EQUATIONS

In this section we will derive the relevant information about $u=\infty$ solutions which we used in Sec. III. First consider the axial case when the system point is very close to one of the corner axes and is running towards the corner. For the corner on the β_+ axis, the asymptotic form of the potential is

$$V(\beta)\sim 16\beta_-^2e^{4\beta_+}+1, \quad \beta_+\to\infty \text{ and } |\beta_-|\ll 1.$$

Then the Hamiltonian of the system is

$$H = (p_+^2+p_-^2+16\beta_-^2e^{-4\Omega}e^{4\beta_+})^{1/2}. \qquad (4.1)$$

To get a time-independent Hamiltonian, substitute

$$\beta_+ = \beta_0+\Omega$$

in the action integrand

$$\omega = p_+d\beta_+ + p_-d\beta_- - Hd\Omega$$

to give

$$\omega = p_+d\beta_0 + p_-d\beta_- - (H-p_+)d\Omega .$$

So the new Hamiltonian is

$$K = (p_+^2+p_-^2+16\beta_-^2e^{4\beta_0})^{1/2} - p_+ . \qquad (4.2)$$

The corresponding Hamilton's equations give

$$\frac{d\beta_0}{d\Omega}=\frac{\partial K}{\partial p_+}=\frac{p_+}{K+p_+}-1=-\frac{K}{H}, \qquad (4.3)$$

$$\frac{d\beta_-}{d\Omega}=\frac{\partial K}{\partial p_-}=\frac{p_-}{H}, \qquad (4.4)$$

$$\frac{dp_+}{d\Omega}=\frac{-\partial K}{\partial \beta_0}=\frac{-32\beta_-^2}{H}e^{4\beta_0}, \qquad (4.5)$$

$$\frac{dp_-}{d\Omega}=\frac{-\partial K}{\partial \beta_-}=\frac{-16\beta_- e^{4\beta_0}}{H}, \qquad (4.6)$$

and

$$\frac{dK}{d\Omega}=\frac{\partial K}{\partial \Omega}=0. \qquad (4.7)$$

Equation (4.7) tells us that K is a constant while Eqs. (4.3), (4.4), and (4.6) can be manipulated to give

$$\frac{d\beta_-}{d\beta_0}=-\frac{p_-}{K},$$

$$\frac{dp_-}{d\beta_0}=\frac{dp_-}{d\Omega}\Big/\frac{d\beta_0}{d\Omega}=\frac{16\beta_- e^{4\beta_0}}{K}.$$

Hence

$$\frac{d^2\beta_-}{d\beta_0^2}=-\frac{1}{K}\frac{dp_-}{d\beta_0}=\frac{-16\beta_- e^{4\beta_0}}{K^2}$$

or

$$\left(\frac{d^2\beta_-}{d\beta_0^2}\right)+\left(\frac{16e^{4\beta_0}}{K^2}\right)\beta_-=0,$$

which has the solution

$$\beta_-=Z_0\left(\frac{2e^{2\beta_0}}{K}\right), \qquad (4.8)$$

where Z_0 is a Bessel function of order zero. Note from Eqs. (4.1) and (4.2) that K and H are strictly positive. Then from Eq. (4.3) β_0 is always decreasing, so Eq. (4.8) is valid starting from some initial value of β_0 until β_0 decreases to the point where the argument of the Bessel function gets small and β_- gets large contradicting the $|\beta_-|\ll 1$ assumption.

Next consider the off-axial case ($\beta_->1$). When the system point is almost parallel to the β_+ axis (large u) and is following one of the inclined potential walls, the asymptotic form of the potential is

$$V(\beta)\sim\tfrac{1}{3}e^{4(\beta_++\sqrt{3}\,\beta_-)}.$$

Then the Hamiltonian of the system is

$$H=[p_+^2+p_-^2+\tfrac{1}{3}e^{-4\Omega}e^{4(\beta_++\sqrt{3}\,\beta_-)}]^{1/2}. \qquad (4.9)$$

Substituting $\beta_+=\beta_0+\Omega$ in the action, we get the time-independent Hamiltonian

$$K=[p_+^2+p_-^2+\tfrac{1}{3}e^{4\beta_0}e^{4\sqrt{3}\,\beta_-}]^{1/2}-p_+. \qquad (4.10)$$

The Hamilton's equations give

$$\frac{d\beta_0}{d\Omega}=\frac{\partial K}{\partial p_+}=\frac{p_+}{K_++p_+}-1=-\frac{K}{H}, \qquad (4.11)$$

$$\frac{d\beta_-}{d\Omega}=\frac{\partial K}{\partial p_-}=\frac{p_-}{H}, \qquad (4.12)$$

$$\frac{dp_+}{d\Omega}=-\frac{\partial K}{\partial \beta_0}=-\frac{2}{3}\frac{e^{4(\beta_0+\sqrt{3}\,\beta_-)}}{H}, \qquad (4.13)$$

$$\frac{dp_-}{d\Omega}=\frac{-\partial K}{\partial \beta_-}=\frac{-2}{\sqrt{3}}\frac{e^{4(\beta_0+\sqrt{3}\,\beta_-)}}{H}, \qquad (4.14)$$

and

$$\frac{dK}{d\Omega}=\frac{\partial K}{\partial \Omega}=0. \qquad (4.15)$$

From Eqs. (4.13) and (4.14) we get

$$\frac{\sqrt{3}\,dp_+}{d\Omega}-\frac{dp_-}{d\Omega}=0$$

or

$$\sqrt{3}\,p_+-p_-=\text{constant}=\alpha, \qquad (4.16)$$

let us say. Substituting for p_+ and p_- in Eq. (4.16) from Eqs. (4.11) and (4.12) we obtain

$$H\left(\sqrt{3}\,\frac{d\beta_+}{d\Omega}-\frac{d\beta_-}{d\Omega}\right)=\alpha\,.$$

Also from Eq. (4.15) $K=H-p_+=H(1-d\beta_+/d\Omega)$ is a constant. These two constants of motion enable us to find β_+', β_-' after the bounce in terms of their values before. Let u_i and u_f be the values of the parameter u, characterizing the velocities of the system point well before and well after the bounce. Then the constancy of $K=H(1-\beta_+')$ and $\sqrt{3}\,p_+-p_-=H(\sqrt{3}\,\beta_+'-\beta_-')$ give, respectively,

$$H_i\left(1-\frac{u_i^2+u_i-\frac{1}{2}}{u_i^2+u_i+1}\right)=H_f\left(1-\frac{u_f^2+u_f-\frac{1}{2}}{u_f^2+u_f+1}\right)$$

and

$$H_i\left(\frac{u_i^2-1}{u_i^2+u_i+1}\right)=H_f\left(\frac{u_f^2-1}{u_f^2+u_f+1}\right).$$

Hence,

$$\frac{H_f}{H_i}=\frac{u_f^2+u_f+1}{u_i^2+u_i+1}$$

and $u_f=-u_i$, where H_i and H_f are the values of H before and after the bounce, respectively.

During the collision with the wall,

$$H^2=p_+^2+p_-^2+e^{-4\Omega}\tfrac{1}{3}e^{4(\beta_++\sqrt{3}\,\beta_-)}$$
$$=H^2\left(\frac{d\beta_+}{d\Omega}\right)^2+H^2\left(\frac{d\beta_-}{d\Omega}\right)^2+e^{-4\Omega}\tfrac{1}{3}e^{4(\beta_++\sqrt{3}\,\beta_-)}\,.$$

So the equation $dH/d\Omega=\partial H/\partial\Omega$ gives

$$H\frac{dH}{d\Omega}=-\tfrac{2}{3}e^{-4\Omega}e^{4(\beta_++\sqrt{3}\,\beta_-)}\,.$$

Using this result and solving for $d\beta_+/d\Omega$, $d\beta_-/d\Omega$ in terms of H, K, and α, one obtains

$$H^2=H^2\left(1-\frac{K}{H}\right)^2+H^2\sqrt{3}\left(1-\frac{K}{H}\right)-\frac{\alpha^2}{H}-\frac{1}{2}H\frac{dH}{d\Omega}$$

or

$$H\frac{dH}{d\Omega}=-6(H_i-H)(H-H_f)\,.$$

Hence

$$\frac{d\beta_0}{dH}=\frac{d\beta_0}{d\Omega}\Big/\frac{dH}{d\Omega}$$
$$=-\frac{K}{H}\Big/\frac{dH}{d\Omega}$$
$$=\frac{K}{6(H_i-H)(H-H_f)}\,.$$

A lower limit on the change in β_0 during the collision can be computed as

$$\Delta(\beta_0)=(H_i-H_f)(\text{minimum value of } d\beta_0/dH)\,.$$

The minimum value of $d\beta_0/dH$ is at that value of H where $d^2\beta_0/dH^2$ vanishes, i.e., at

$$H=\tfrac{1}{2}(H_i+H_f)\,.$$

Therefore,

$$\Delta(\beta_0)=\frac{2K}{3(H_i-H_f)}\,.$$

But

$$(H_i-H_f)=\tfrac{2}{3}K[(u_i^2+u_i+1)-(u_f^2+u_f+1)]$$
$$=\tfrac{4}{3}Ku_i\,,$$

since $u_f=-u_i$. Hence,

$$\Delta(\beta_0)=\frac{1}{2u_i}\,. \qquad (4.17)$$

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