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Generalization of the Einstein Theory

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The assumption that $T^{\nu}{}_{\mu;\nu}=0$ in curved space-time is questioned. Field equations are given which are consistent with the assumption $T^{\nu}{}_{\mu;\nu}=\lambda R_{,\mu}$, and which reduce to the Einstein equations when $\lambda=0$. The equations are equivalent to the Einstein equations in empty space-time, but differ from them in the presence of matter. Applications to cosmology, stellar structure, and collapsing objects are suggested.

I. INTRODUCTION

A fundamental assumption of the Einstein theory of gravitation is that the covariant divergence of the energy-momentum vanishes, or in symbols, $T^{\nu}{}_{\mu;\nu}=0$. The usual, textbook argument for this equation is that it is valid in special relativity, and therefore valid in general because of the principle of equivalence. It is, however, well known that this principle is an ambiguous guide: Consider the motion of a spinning particle in a gravitational field, or even Maxwell's equations if one uses the "wrong" variables.[1] Another way of deriving $T^{\nu}{}_{\mu;\nu}=0$ is from a variational principle.[2] Here one must assume that the Lagrangian density can be written as a sum of two terms, the first independent of the derivatives of the metric, and the second independent of the nongravitational field variables. Third, one can argue that $T^{\nu}{}_{\mu;\nu}=0$ on the basis of a classical, statistical model of matter. One assumes that matter consists of particles that collide with one another, but move geodesically and with constant rest mass between collisions.[3]

The assumptions from which one derives $T^{\nu}{}_{\mu;\nu}=0$ are all questionable, so one should not accept this equation without further investigation. All one can assert with fair confidence is that $T^{\nu}{}_{\mu;\nu}=a_{\mu}$, where the functions a_{μ} vanish in flat space-time. We must ask whether there are a_{μ} which do not vanish in curved space-time, and which are in agreement with present observations.

It is plausible that $T_{\mu\nu}$, and hence a_{μ}, should depend on the curvature. As a simple, classical model of an elementary particle, we may take an elastic sphere. When the curvature does not vanish, gravitational "tidal" forces are present which distort the sphere and so change its energy and rest mass (in contradiction to the assumptions of the statistical model in the first paragraph). It would therefore be surprising if $T_{\mu\nu}$ did not depend on the curvature. We will show that one can take $a_{\mu}=\lambda R_{,\mu}$, where λ is a constant, $R=g^{\mu\nu}R_{\mu\nu}=g^{\mu\nu}g^{\lambda\pi}R_{\lambda\mu\nu\pi}$ is the curvature invariant, and the comma denotes the partial derivative.

II. FIELD EQUATIONS

Since $(R_{\mu\nu}-\frac{1}{2}\delta_{\mu\nu}R)_{;\nu}=0$, the assumption that $T^{\nu}{}_{\mu;\nu}=\lambda R_{,\mu}$ is consistent with the field equations $R^{\nu}{}_{\mu}-\frac{1}{2}\delta_{\mu\nu}R=\kappa(T^{\nu}{}_{\mu}-\lambda\delta_{\mu\nu}R)$, or with

$$R_{\mu\nu}+(\kappa\lambda-\tfrac{1}{2})g_{\mu\nu}R=\kappa T_{\mu\nu}\,, \tag{1}$$

where κ is a constant. Contracting (1) gives $(4\kappa\lambda - 1)R = \kappa T^{\mu}{}_{\mu}$. We exclude the case $\kappa\lambda = \frac{1}{4}$, since it implies that $T^{\mu}{}_{\mu} = 0$, which is not always true. In empty space-time one has $T_{\mu\nu} = 0$, and hence $R = 0$. The field equations then reduce to $R_{\mu\nu} = 0$, just as in the Einstein theory.

If the source of the gravitational field is an ideal fluid with energy density ϵ, pressure p, and 4-velocity n, one has $g^{\mu\nu} n_{\mu} n_{\nu} = -1$ and $T_{\mu\nu} = -(\epsilon + p) n_{\mu} n_{\nu} - p g_{\mu\nu}$. It follows that $n^{\mu} n_{\mu;\pi} = 0$,

$$n^{\mu} T^{\nu}{}_{\mu;\nu} = [(\epsilon + p) n^{\nu}]_{;\nu} - n^{\nu} p_{,\nu} = \lambda n^{\nu} R_{,\nu}, \tag{2}$$

and

$$T^{\nu}{}_{\mu;\nu} = -(\epsilon + p) n^{\nu} n_{\mu;\nu} - [(\epsilon + p) n^{\nu}]_{;\nu} n_{\mu} - p_{,\mu} = \lambda R_{,\mu},$$

$$(\epsilon + p) n^{\nu} n_{\mu;\nu} = -(\lambda R_{,\nu} + p_{,\nu})(\delta_{\mu\nu} + n_{\mu} n^{\nu}). \tag{3}$$

We shall use Eq. (3) to discuss the motion of a fluid body in an otherwise empty space-time. Let γ be the path of a material particle of the body, and q be a point on γ. On some neighborhood of q there is a chart (or coordinate system) $x_{(q)}$ in which the Christoffel symbols vanish at q, and in which

$$g_{mn}(q) = \delta_{mn}, \quad g_{\mu 0}(q) = -\delta_{\mu 0},$$

$$n^{\mu}(q) = \delta_{\mu 0}, \quad n_{\mu}(q) = -\delta_{\mu 0}.$$

(Our convention is that lower-case Latin indices have the range $\{1, 2, 3\}$, and lower-case Greek indices have the range $\{0, 1, 2, 3\}$.) In $x_{(q)}$, Eq. (3) reduces at q to

$$(\epsilon + p) n_{m,0}(q) = -(\lambda R_{,m} + p_{,m})(q),$$

$$(\epsilon + p) n_{0,0}(q) = 0.$$

If one can choose the material particle so that $(\lambda R_{,m} + p_{,m})(q) = 0$ in $x_{(q)}$ for all q on γ, then $n^{\nu} n_{\mu;\nu} = 0$ on γ, and since $n(q)$ is the tangent to γ at q, it follows that γ is a geodesic.

To apply this result to planetary motions, we make the reasonable idealization that a planet is a roughly ellipsoidal ball of fluid (i.e., it is roughly ellipsoidal at each instant in a suitable instantaneous rest frame). In general, at each instant there will be a point q in the planet where the function $\lambda R + p$ is stationary, and hence $(\lambda R_{,m} + p_{,m})(q) = 0$. Unless there is an exceedingly strong, inhomogeneous, external gravitational field acting on the planet, the point q will be very close to its center. Further, a material particle which is initially at the center will stay there, or move away only very slowly. It follows that, to a good approximation, the center of a planet follows a geodesic. This derivation is plausible, but not rigorous. It seems to be difficult to give a more rigorous account without complicated series expansions.[4]

If we let the speed of a particle tend to the speed of light, while its proper mass tends to zero in such a manner that its energy stays bounded, then we get the simplest model of a photon in the Einstein theory. If the path of the particle is a geodesic, then in the limiting case the path is a null geodesic. The same argument holds in our theory, and we conclude that Eqs. (1) agree with present observations to the same extent as the Einstein equations.

Equation (2) tells something about conservation of energy for an ideal fluid. We apply Gauss's theorem to a regular, 4-dimensional domain ω whose surface is $\partial\omega$. If d^4v and d^3v are the invariant volume elements on ω and $\partial\omega$, respectively, we find (Ref. 3, p. 46)

$$\int_{\partial\omega} (\epsilon + p) n^{\mu} \operatorname{sgn}(g_{\pi\rho} \alpha^{\pi} \alpha^{\rho}) \alpha_{\mu} d^3v = \int_{\omega} n^{\mu} (\lambda R_{,\mu} + p_{,\mu}) d^4v, \tag{4}$$

where α is the outward-pointing unit normal on $\partial\omega$, and where $\operatorname{sgn} z = 1, -1, 0$ for $z > 0$, $z < 0$, $z = 0$, respectively. If $\partial\omega$ consists of two spacelike hypersurfaces A_1 and A_2 and a hypersurface A_3 on which $n^{\mu} \alpha_{\mu} = 0$, Eq. (4) reduces to

$$\int_{A_2} (\epsilon + p) n^{\mu} \beta_{\mu} d^3v - \int_{A_1} (\epsilon + p) n^{\mu} \beta_{\mu} d^3v = -\int_{\omega} n^{\mu} (\lambda R_{,\mu} + p_{,\mu}) d^4v, \tag{5}$$

where $\beta = \alpha$ on A_2 and $\beta = -\alpha$ on A_1. When $p = 0$ (the case of "dust"), the right-hand side of (5) vanishes if $n^{\mu} R_{,\mu} = 0$, that is if R is constant along the timelike vector field n. An interesting question, although beyond the scope of our classical theory, is whether elementary particles are created in a region where $n^{\mu} R_{\mu} \neq 0$. Another possibility, mentioned earlier, is that their proper masses are not constant in such a region. In the case of photons, we may ask whether there is an additional frequency change when $n^{\mu} R_{,\mu} \neq 0$ (anomalous red shifts?).

III. SOLVING THE FIELD EQUATIONS

A reasonable way to find solutions of Eqs. (1) is to transform them into the Einstein equations, of which many solutions are known. Again we assume that the source of the gravitational field is an ideal fluid with $T_{\mu\nu} = -(\epsilon + p) n_{\mu} n_{\nu} - p g_{\mu\nu}$. We first rewrite (1) in the form

$$R_{\mu\nu} = \kappa T_{\mu\nu} + \kappa(\kappa\lambda - \tfrac{1}{2})(1 - 4\kappa\lambda)^{-1} T^{\lambda}{}_{\lambda} g_{\mu\nu},$$

where $T^{\lambda}{}_{\lambda} = g^{\mu\nu} T_{\mu\nu} = \epsilon - 3p$. At any point q, we define a chart $x_{(q)}$, as in Sec. II, and we find that at q in $x_{(q)}$

$$R_{00}=\kappa(1-4\kappa\lambda)^{-1}[(3\kappa\lambda-\tfrac{1}{2})\epsilon+3(\kappa\lambda-\tfrac{1}{2})p],$$
$$R_{m0}=0, \qquad (6)$$
$$R_{mn}=\kappa(1-4\kappa\lambda)^{-1}\delta_{mn}[(\kappa\lambda-\tfrac{1}{2})\epsilon+(\kappa\lambda+\tfrac{1}{2})p].$$

If we now define ϵ' and p' by

$$\epsilon'=(1-4\kappa\lambda)^{-1}[(1-3\kappa\lambda)\epsilon-3\kappa\lambda p],$$
$$p'=(1-4\kappa\lambda)^{-1}[-\kappa\lambda\epsilon+(1-\kappa\lambda)p], \qquad (7)$$

Eqs. (6) reduce to

$$R_{00}=\kappa(-\tfrac{1}{2}\epsilon'-\tfrac{3}{2}p'),$$
$$R_{mn}=\tfrac{1}{2}\kappa\delta_{mn}(-\epsilon'+p').$$

Hence Eqs. (1) are equivalent to the equations of Einstein type $R_{\mu\nu}=\kappa(T'_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T'^{\lambda}{}_{\lambda})$, where $T'_{\mu\nu}=-(\epsilon'+p')n_\mu n_\nu-p'g_{\mu\nu}$. The energy-momentum $T'_{\mu\nu}$ is that of an ideal fluid with energy density ϵ' and pressure p'. The equation of state of this fluid can be found from Eqs. (7) and the equation of state of the original fluid.

If the first of Eqs. (6) is to reduce to Poisson's equation in the static, weak-field limit, κ and λ must satisfy

$$\kappa(1-4\kappa\lambda)^{-1}(3\kappa\lambda-\tfrac{1}{2})=-4\pi Gc^{-4}, \qquad (8)$$

where G is the Newtonian gravitational constant, and c is the speed of light. When $\lambda=0$, κ becomes the Einstein gravitational constant $\kappa_e=8\pi Gc^{-4}$. If we wish we can define $\overline{\epsilon}=\kappa\kappa_e{}^{-1}\epsilon'$, $\overline{p}=\kappa\kappa_e{}^{-1}p'$, $\overline{T}_{\mu\nu}=\kappa\kappa_e{}^{-1}T'_{\mu\nu}$, so that $\overline{T}_{\mu\nu}=-(\overline{\epsilon}+\overline{p})n_\mu n_\nu-\overline{p}g_{\mu\nu}$, and the expression for $R_{\mu\nu}$ in the last paragraph becomes $R_{\mu\nu}=\kappa_e(\overline{T}_{\mu\nu}-g_{\mu\nu}\overline{T}^{\lambda}{}_{\lambda})$.

IV. CONCLUSION

We have shown that Eqs. (1) and the Einstein equations are in equally good (i.e., fair) agreement with present observations. There is no reason to suppose that they will give equivalent results in other problems of current interest: gravitational collapse, cosmology, or the generation of gravitational waves. Even in a down-to-earth subject like stellar evolution, there may be significant differences, which could set quite stringent bounds for λ. We hope that experts in the various fields will explore these possibilities.

[1]A. Trautman, in *Lectures on General Relativity*, edited by Stanley Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), Sec. 6.2.
[2]W. Pauli, *Theory of Relativity* (Pergamon, London, 1958), Sec. 57.
[3]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), Chap. IV.
[4]K. S. Thorne and C. M. Will, Astrophys. J. 163, 595 (1971).

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Gravitational Field of a Charged Mass Point in the Scalar-Tensor Theory*

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The gravitational field of a static, electrically charged mass point is obtained exactly in the framework of the scalar-tensor theory. It is the counterpart of the Nordström-Reissner solution in the Einstein theory.

In this note we put on record an exact calculation which yields the gravitational field of a charged particle at rest in the scalar-tensor (Brans-Dicke) theory. The result may be of some interest in astrophysics, but it also has an intrinsic interest because it is an exact solution in which all three of the simplest classical fields (scalar, vector, and tensor) are present. It corresponds to the Nordström-Reissner solution in the Einstein theory. An approximate solution of the same problem has been obtained recently by Mahanta and Reddy.[1]

The set of field equations in the scalar-tensor theory[2] are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\frac{8\pi T_{\mu\nu}}{c^4\phi}+\frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\eta}\phi^{,\eta})+\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi), \qquad (1)$$