PHYSICAL REVIEW D VOLUME 6, NUMBER 11 1 DECEMBER 1972

Extension of Boundary Dispersion Relations*

G. E. Hite
University of Kaiserslautern, Kaiserslautern, Germany

and

Richard J. Jacob†
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

and

Frank Steiner
CERN, Geneva, Switzerland

(Received 2 August 1972)

By introducing a variable corresponding to the Leader-Pennington variable, we are able to write dispersion relations which are an extension of boundary dispersion relations to the interior of the physical regions.

In a recent article[1] we showed how dispersion relations could be written, for certain inelastic processes, which would include both t-channel and s-channel contributions, the latter evaluated on the contour representing the boundary of the s-channel physical region. For the elastic scattering case, these boundary dispersion relations (BDR) become the ordinary backward dispersion relations.[2] BDR are limited however in that they incorporate information only from forward ($Z_s = 1$) and backward ($Z_s = -1$) scattering or production data. Dispersion relations which could be written for contours interior to the boundary curves, i.e., using data away from the forward or backward directions, would be quite useful, especially as the data in the region somewhat off the scattering polar axis are often more numerous and reliable. In this note, we wish to demonstrate how such an extension of BDR away from the boundary curve can be effected.

We are concerned, in particular, with reactions of the type

$$a + b \to c + d, \tag{1}$$

where $m_b = m_d$. Reactions in which $m_a = m_c$ we term elastic (irrespective of quantum-number exchange); others ($m_a \neq m_c$) are inelastic. BDR are obtained by applying the Cauchy theorem in the complex t plane for a suitable amplitude at fixed $\sin\theta_t = 0$. This is equivalent in the regions of interest to fixing the Kibble boundary function

$$\phi = 4t(p_t p_t' \sin\theta_t)^2 \tag{2}$$

at zero. (See Ref. 1 for kinematical details.) The resultant contour consists of the entire boundary curve for the s-channel physical region, a contribution extending from the lowest two-body t-channel threshold, t_2, to the threshold of the t-channel reaction

$$a + \bar{c} \to \bar{b} + d, \tag{3}$$

and the branch of the t-channel boundary corresponding to $\cos\theta_t = -1$. This choice of fixed and dispersion variables is convenient in that it allows us to remove easily contributions from kinematical cuts. There are no kinematical singularity contributions for amplitudes $\tilde{A}$ with even s-u crossing symmetry, i.e., $\tilde{A} = A$ for even amplitudes and $A/(s-u)$ for odd amplitudes.

As was mentioned above, the BDR for elastic reactions are backward dispersion relations, i.e., dispersion relations at fixed angles $\theta_s = \pi$. For inelastic reactions, θ_s is piecewise-fixed around the boundary. Fixed-angle dispersion relations are convenient because data are usually taken at fixed angles. There is as yet no way of writing fixed-angle dispersion relations for angles away from the polar axis, from which kinematical singularity contributions can easily be eliminated and which do not involve unphysical contributions from a double-spectral function.

As an alternative, we introduce the variable

$$C = 4\phi/t^2, \tag{4}$$

which we will fix in order to write dispersion relations. (This is essentially the Leader-Pennington variable[3] for t-channel reactions.) An advantage in working with fixed C is that in the Mandelstam plane the physical scattering regions are characterized by positive values of ϕ and thus of C, and the unphysical regions by negative values of ϕ and C. This ensures that a fixed-C curve passing through the direct-channel physical region will

also pass through the crossed-channel physical region. From the relationship

$$s-u=4p_t p_t' z_t \equiv \nu_t z_t \tag{5}$$

we can write

$$C=[\nu_t^2-(s-u)^2]/t. \tag{6}$$

Thus,

$$2s=\Sigma-t+(\nu_t^2-Ct)^{1/2}, \tag{7a}$$

$$2u=\Sigma-t-(\nu_t^2-Ct)^{1/2}, \tag{7b}$$

where $s+t+u=\Sigma$. Here we have adopted the sign convention which reduces to that for the BDR when $C=0$.

We may now, as before,[1] write the invariant amplitudes for the process (1) as functions of t and

$$\nu=s-u$$
$$=(\nu_t^2-Ct)^{1/2}. \tag{8}$$

Therefore, one may write fixed-C dispersion relations for amplitudes, $\tilde{A}$, which are even in ν, i.e., are even under s-u crossing, without contributions arising from kinematical singularities.

It now remains to discuss the contours of integration for fixed C. From (7a) and the expression

$$\nu_t^2=(\Sigma-t)^2-4(m_a^2-m_b^2)(m_c^2-m_b^2)$$
$$-4m_b^2(m_a^2-m_c^2)^2/t \tag{9}$$

we find that the equation for $t(s, C)$ is a quadratic; it reduces for elastic reactions to a linear equation with solution

$$t=-\frac{s^2-\Sigma s+(m_a^2-m_b^2)^2}{s+\frac{1}{4}C}$$
$$=-\frac{4sp_s^2}{s+\frac{1}{4}C}\ \text{(elastic)}. \tag{10}$$

In either event, for positive values of C the s-channel unitary cut maps onto the negative t axis, $-\infty<t<0$. Figures 1(a) and 1(b) show these contours for various values of C in the Mandelstam diagrams for elastic πN scattering and $\pi N\to\eta N$, respectively. The fixed-C dispersion relation is thus written

$$\tilde{A}(t, s(t, C))=\tilde{A}^{\text{Born}}(t, s(t, C))+\frac{1}{\pi}\int_{-\infty}^{0} dt'\,\frac{\operatorname{Im}\tilde{A}(t', s(t', C))}{t'-t}+\frac{1}{\pi}\int_{t_2}^{\infty} dt'\,\frac{\operatorname{Im}\tilde{A}(t', s(t', C))}{t'-t}, \tag{11}$$

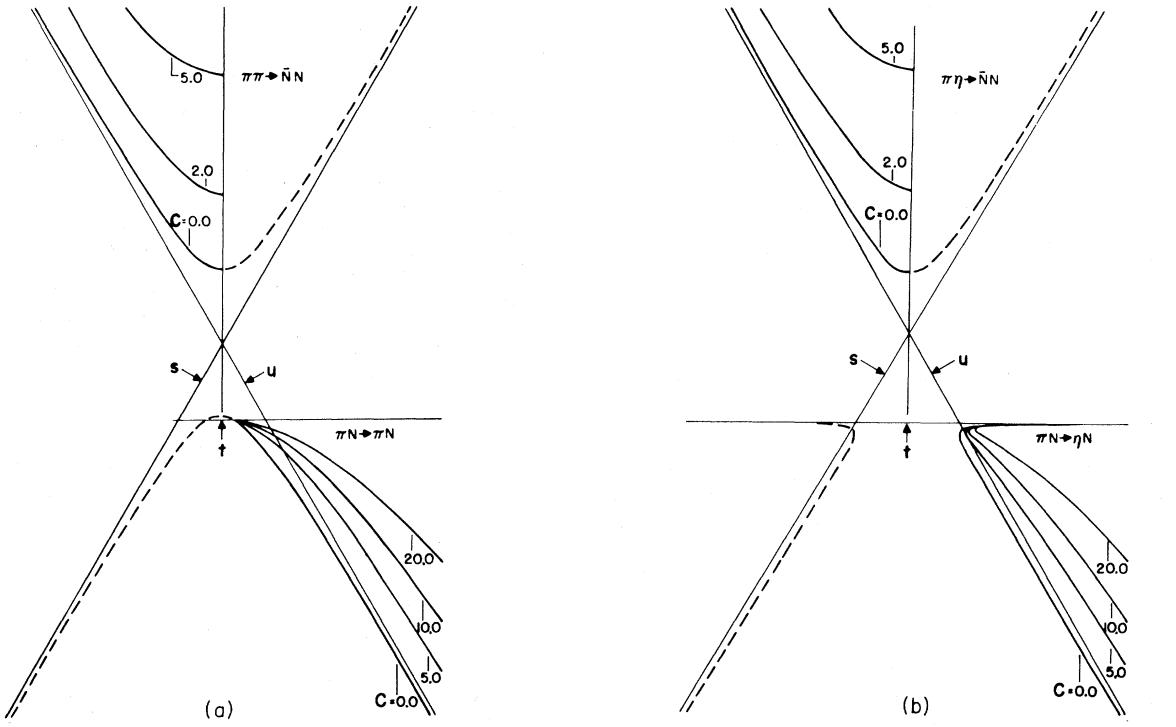

FIG. 1. Mandelstam diagram for (a) $\pi N\to\pi N$, (b) $\pi N\to\eta N$, showing curves of fixed C; dashed lines are boundaries of physical regions which do not contribute to $C=0$ dispersion relations (BDR).

where $s(t, C)$ is given by (7a) with the assistance of (9).

In the region $t_2 \leq t < t_{th}(C)$, $s(t, C)$ is complex; here the vertical path in the figures indicates only the range of the t variable. [$t_{th}(C)$ is that value of t at which, for given C, $\nu = 0$.] Other than this, however, all curves with $C > 0$ lie within physical regions. Double-spectral functions are not encountered.[4] In this region t is, of course, real and one can use a t-channel partial-wave expansion of the appropriate t-channel reaction for the amplitude. In such cases, one must establish whether or not z_t is contained in the large Lehmann ellipse of the reaction.

Curves within the s-channel boundary all pass through the threshold point $s = (m_b + m_a)^2$ for the elastic reaction and approach $s = \infty$ as $t \to 0_-$ in the inelastic case. In both cases, the curves are asymptotic to $u = -\frac{1}{4}C$ as $t \to \pm\infty$. [Following the phase convention in Ref. 1, we have $\nu < 0$ as $t \to \infty$.]

We show in Fig. 2 the variation of Z_s with t within the s-channel physical region for the reaction $\pi N \to \eta N$. We see that for any value of C, other than zero, knowledge of the amplitude at all angles is required. The results of analyses using fixed-C dispersion relations are therefore likely to be very sensitive to differences in phase-shift solutions.

It should be mentioned here that it is not necessary to use the variable C to avoid the introduction of double-spectral-function contributions. An obvious alternative extension of BDR is provided by constant-ϕ contours. Unfortunately, at least for elastic reactions, fixed-ϕ dispersion relations lead to much more complicated kinematics than

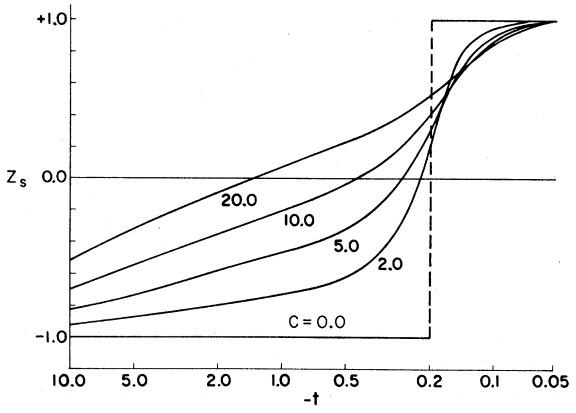

FIG. 2. Z_s as a function of t for various values of C for the reaction $\pi N \to \eta N$. The units of t are (nucleon mass)^2.

afforded by Eq. (10). In particular, whereas all fixed-C curves pass through the physical threshold point, fixed-ϕ curves do not.

Finally, we mention the possibility of writing dispersion relations with the variable $C' = Ct$ fixed. This variable offers some advantages in simplifying kinematical calculations but there are encounters with the double-spectral region for sufficiently large $-C'$. C' also does not have the advantage of being always positive in physical regions and negative in unphysical regions [see the discussion following Eq. (4)].

One of us (GEH) would like to thank Elliot Leader for discussions and valuable comments.

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*Work supported in part by Associated Western Universities and the U. S. Atomic Energy Commission.

†Permanent address: Department of Physics, Arizona State University, Tempe, Arizona 85281.

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[4]A general investigation of dispersion relations which do not receive contributions from double-spectral functions has been made by G. E. Hite and Frank Steiner, CERN report, 1972 (unpublished).