tegrations to compute the total cross section. Our preliminary runs using Gaussian quadrature gave answers which were too low by approximately 30%. Marinov *et al.* (Ref. 4) seem to have run into the same trouble because their answers are consistently 10% lower than those of Czyż *et al.* (Ref. 3). In these problems mappings should be made to reduce the peaking in the integration variables.

[7]K. Koike, M. Konuma, K. Karata, and K. Sugano, Progr. Theoret. Phys. (Kyoto) 46, 1150 (1971); 46, 1799 (1971).

[8]A. V. Berkov, L. M. Voronina, and E. P. Shabalin, Yad. Fiz. 14, 416 (1971) [Sov. J. Nucl. Phys. 14, 233 (1972)].

[9]S. Weinberg, Phys. Rev. Letters 19, 1264 (1967).

[10]G. 't Hooft, Nucl. Phys. B35, 167 (1971).

[11]B. W. Lee, Phys. Rev. D 5, 823 (1972); see also B. W. Lee and J. Zinn-Justin, *ibid.* 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972).

[12]S. Weinberg, Phys. Rev. Letters 27, 1688 (1971).

[13]G. 't Hooft, Phys. Letters 37B, 195 (1971).

[14]H. H. Chen and B. W. Lee, Phys. Rev. D 5, 1874 (1972). The term $-2(p\cdot p')(k\cdot k')$ in Eq. (11) of this reference should be deleted. Even with this correction the total cross section calculated from Eq. (11) is much too large. The approximation $m_e{}^2 \ll q^2$ is not very good because the important region in the q^2 integration is near $q^2=(q^2)_{\min}=4m_e{}^4/\omega^2$.

[15]R. Reines and H. S. Gurr, Phys. Rev. Letters 24, 1448 (1970). H. S. Gurr, F. Reines, and H. W. Sobel, Phys. Rev. Letters 28, 1406 (1972).

[16]Our basic notation and conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964); and *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). In particular, $\hbar=c=1$, $\alpha=e^2/4\pi$, and $\not{p}\equiv\gamma^\mu p_\mu$.

[17]M. Veltman, CERN report, 1967 (unpublished).

[18]D. J. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).

[19]D. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Letters 38B, 519 (1972).

[20]J. Dooher and M. Tausner, Phys. Rev. 142, 1018 (1966).

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Parametric Sum Rules for Electroproduction and Tests of Complex Scaling and Regge Behavior*

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Parametric dispersion relations which separately test complex scaling and Regge fits to the inelastic structure function $W_2(\nu, q^2)$ are found and evaluated. The numerical results indicate that precocious complex scaling is consistent with the present electroproduction data. The sum rule for the Regge fits is very restrictive and eliminates many of the fits proposed in the literature. The two fits which satisfy the sum rules are ones made by Pagels and by Preparata.

I. INTRODUCTION

Recently Khuri[1] has derived a new class of sum rules, or parametric dispersion relations, for off-shell Compton scattering. These sum rules follow from analyticity in two complex variables and complex scaling within the analyticity domain. Khuri and the present author[2] have evaluated several of these parametric dispersion relations; the results suggest that the sum rules are consistent with the concept of precocious complex scaling and that they provide tests of the Regge fits to the SLAC-MIT electroproduction data which go beyond the constraints imposed by the FESR (finite-energy sum rules). However, while some of these sum rules are more sensitive to the complex scaling hypothesis and others to the Regge fit used, they do not allow totally independent tests of the two.

The purpose of this paper is to find and evaluate additional parametric dispersion relations which will furnish more stringent constraints on the behavior of the inelastic structure function νW_2 and which do not use both Regge behavior and complex scaling in the same sum rule. As in I, the sum rules we will consider also follow from analyticity in two complex variables. We will divide them into two classes. Those in class A will depend on the hypothesis of complex scaling, but not on the Regge fit used; those in class B will use only Regge input and analyticity and will provide new and rigorous restrictions on the Regge fits.

We have evaluated our sum rules using the SLAC-MIT electroproduction data[3,4] on νW_2 and our results confirm that precocious complex scaling is indeed consistent with the data. The sum rules of type A are satisfied to within 15%. Our test of the Regge fits is very powerful and excludes all but two of the Regge fits studied. The two which do survive were obtained using partial information on the analyticity of the Regge residues in the virtual photon mass variable. Both contain only Pomeranchukon and f-A_2 Regge contributions and are dominated by the Pomeranchukon; both have $J=0$ fixed poles with nonpolynomial residues. We have also examined Regge fits which have additional Regge contributions and polynomial fixed-pole residues; none of these fits satisfy our sum rule. Since all of the fits tested satisfy the FESR in the scaling limit, it is clear that our parametric dispersion relation provides new restrictions on the Regge behavior of the inelastic structure function νW_2.

II. NEW PARAMETRIC SUM RULES

It was shown in I that there exist analytic closed contours, C, in the four-dimensional ν-q^2 complex space such that

$$\oint \nu(z) T_2(\nu(z), q^2(z))dz = 0 . \tag{2.1}$$

The invariant off-shell Compton amplitude T_2 is related to the inelastic structure function W_2 by $\mathrm{Im}T_2 = \pi W_2$. q is the virtual photon four-momentum and $\nu = q\cdot p/M_p$, where p is the photon four-momentum and M_p is the proton mass; $q^2(z)$ and $\nu(z)$ are analytic functions of z. The contours remain inside the analyticity domain of T_2, namely, in this case the forward tube. They consist of two parts: Along the first (which we call L), ν and q^2 are real, q^2 is spacelike, and T_2 is determined by the data on W_2 through the fixed-q^2 forward dispersion relation. Along the second part of the contour (which we call S), ν and q^2 are complex. We seek sum rules such that on S, ν, and q^2 lie either (a) in the complex scaling region or (b) in the (complex) Regge region.

In the sum rules we discuss, L will always be the interval of the real axis between $z=-1$ and $z=1$, and S the unit semicircle $0 \leq \arg z \leq \pi$. To simplify our notation we define:

$$L = 2\int_{-1}^{1} dx\, \nu(x) \int_{\nu_t(x)}^{\infty} d\nu' \frac{\nu' W_2(\nu', q^2(x))}{\nu'^2 - [\nu(x)]^2} , \tag{2.2a}$$

$$S = -i \int_0^{\pi} d\phi\, e^{i\phi} F(\nu(\phi), q^2(\phi)) . \tag{2.2b}$$

For sum rules of type A,

$$F(\nu(\phi), q^2(\phi)) = 2\omega(e^{i\phi}) \int_1^{\infty} d\omega' \frac{F_2(\omega')}{\omega'^2 - [\omega(e^{i\phi})]^2} , \tag{2.3a}$$

where $\omega = 2M_p\nu/(-q^2)$ and $F_2(\omega)$ is the usual Bjorken scaling function defined by

$$\lim_{\nu\to\infty;\,\omega \text{ fixed}} \nu W_2(\nu, q^2) = F_2(\omega) .$$

For sum rules of type B,

$$F(\nu, q^2) = R(\nu, q^2) , \tag{2.3b}$$

where $R(\nu, q^2)$ is the Regge fit to νT_2. We can now write our sum rules in the following form:

$$L \cong S . \tag{2.4}$$

To find the desired parametric dispersion relations it is convenient to define $\zeta = q_0$ and $\eta = |\vec{q}|$, in terms of which $q^2 = \zeta^2 - \eta^2$ and, in the lab system, $\nu = \zeta$. With these variables the forward tube is defined by $\mathrm{Im}\zeta > |\mathrm{Im}\eta|$. If we also define $u = \frac{1}{2}(\zeta + \eta)$ and $v = \frac{1}{2}(\zeta - \eta)$, we see that $u(z)$ and $v(z)$ must be analytic and satisfy $\mathrm{Im}u(z) > 0$ and $\mathrm{Im}v(z) > 0$ within the contour C in order that $\nu(z)$ and $q^2(z)$ lie in the forward tube. These requirements will be satisfied automatically if $u(z)$ and $v(z)$ are chosen to be Herglotz functions.[5]

We first consider sum rules of type A. Bloom and Gilman[6] have shown that the scaling limit of the structure function νW_2 interpolates the behavior of the measured structure function $\nu W_2(\nu, q^2)$ as a function of the variable $\omega' = (2M_p\nu + M_p^2)/(-q^2)$ for fixed $|q^2| > 1$ BeV^2 in the resonance region. However, for $|q^2| \lesssim 1$ BeV^2, $\nu W_2(\nu, q^2)$ drops below the scaling limit curve. To obtain the strongest test of complex scaling we should therefore look for a sum rule such that ν and q^2 have the following behavior along the contour C: (i) Along L, ν and q^2 start in the real scaling region at $z=-1$, pass through the low-q^2 ($|q^2| < 1$ BeV^2) part of the resonance region, and end in the real scaling region at $z=1$; (ii) along S, ν and q^2 lie in the complex scaling region ($|q^2|$ large). In Appendix A we prove that it is impossible to find $\nu(z)$ and $q^2(z)$ satisfying these conditions and such that within the contour C, ν and q^2 are in the forward tube.

Nonetheless, we can find nontrivial tests of complex scaling. It is easy enough to find sum rules such that along S, ν and q^2 lie entirely in the complex scaling region and on L, ν and q^2 pass through the resonance region with $|q^2| > 1$ BeV^2. The mass q^2 is variable along the contours in our sum rules, and the scaling limit curve does not interpolate the behavior of $\nu W_2(\nu, q^2)$ along all contours through the resonance region. Clearly, contours along which q^2 varies rapidly compared with ν in the resonance region are most likely to give nontrivial tests of complex scaling.

As an example of a sum rule of this type we give the following:

$$\text{(A1)}\quad \begin{cases} u(z) = 1.1z + 1.2\,, \\ v(z) = 1.1z - 1.2\,, \end{cases} \tag{2.5a}$$

with

$$\begin{aligned} q^2(z) &= 4u(z)v(z) \quad (\text{in BeV}^2)\,, \\ \nu(z) &= u(z) + v(z) \quad (\text{in BeV})\,. \end{aligned} \tag{2.6}$$

Here the integral L is effectively an integral of νT_2 along the contour in the ν-q^2 plane for these Fig. 1. Only the imaginary part of the sum rule is nontrivial; the real part of L vanishes because $\mathrm{Re}(\nu T_2)$ is antisymmetric in ν and the contour L is symmetric about the q^2 axis. To get nontrivial sum rules for the real part of νT_2 we allow one endpoint of the contour L to lie near the Bjorken-Johnson-Low region ($|q^2| \to \infty$ with $|\omega| \to 0$) where complex scaling is also expected to hold. Two examples of this type are

$$\text{(A2)}\quad \begin{cases} u(z) = 0.55z + 1.75\,, \\ v(z) = 0.55z - 0.65 \end{cases} \tag{2.5b}$$

and

$$\text{(A3)}\quad \begin{cases} u(z) = 0.6z + 1.95\,, \\ v(z) = 0.6z - 0.75\,. \end{cases} \tag{2.5c}$$

Again, ν and q^2 are given by (2.6). The paths of integration along L in the ν-q^2 plane for these sum rules are shown in Fig. 2. For each of these contours, q^2 varies rapidly as q^2 and ν pass through the resonance region. The path L in the ν-q^2 plane for sum rule (A1) is equivalent to the path for sum rule (A2) combined with its image in the q^2 axis. The evaluation of these sum rules is presented in the next section.

We would now like to mention some of the theoretical motivations for exploring the Regge behavior of νT_2. The high-energy behavior of both on- and off-shell Compton scattering seems to be well approximated by a fairly simple Regge parametrization. T_2 is usually considered to be dominated by the Pomeranchukon and f-A_2 trajectories. In addition, there is evidence for the existence of a right-signatured fixed pole at $J = 0$ in the on-mass-shell Compton amplitude,[7] and this fixed pole is also expected to be present in the off-shell amplitude. In the scaling region, the residue of this fixed pole should be nearly linear in q^2, and it should vanish with q^2 in the limit $q^2 \to 0$. If in the scaling limit $\nu W_2(\nu, q^2)$ is dominated by the Pomeranchukon and f-A_2 trajectories for $\omega > 12$, then the sign of the fixed-pole residue in this limit is opposite that found for the on-shell amplitude.[8,9] This behavior seems rather undesirable; in fact, if the fixed-pole residue is a polynomial in q^2, as has been suggested theoretically,[10] then it must be linear in q^2 for scaling to hold and it could not change sign between the low- and high-q^2 regions. Therefore, either the fixed-pole residue is not a polynomial in q^2, or the asymptotic expansion including only the Pomeranchukon and f-A_2 trajectories is too simple. If there is a substantial nonleading Regge contribu-

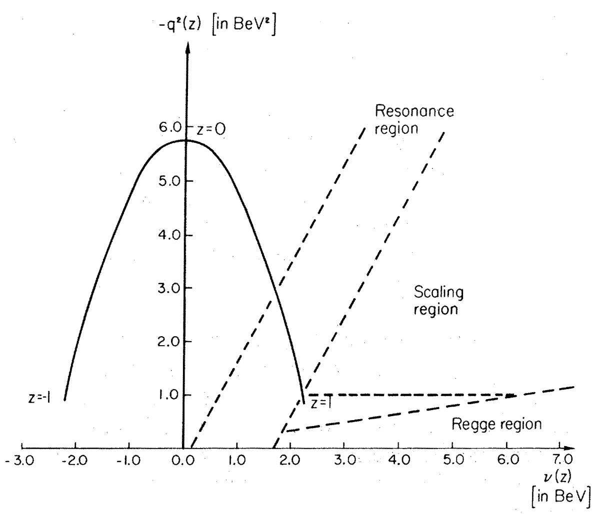

FIG. 1. Path of integration in the ν-q^2 plane for sum rule (A1).

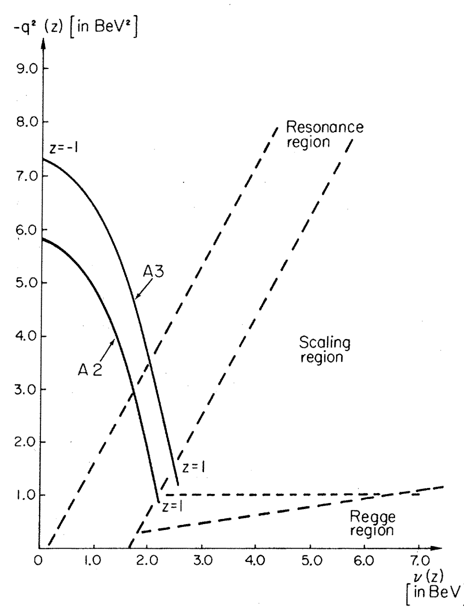

FIG. 2. Paths of integration in the ν-q^2 plane for sum rules (A2) and (A3).

tion to the asymptotic behavior in addition to the conventional leading Regge form, then it is possible to have a fixed pole with residue linear in q^2 and with the same sign in both the $q^2 \to 0$ and scaling limits.[9]

The possibility of a fixed-pole contribution to νT_2 and the question of the behavior of its residue make it especially important to find ways of discriminating between the various Regge fits to νT_2. All of these fits have been constrained to satisfy the FESR, so additional tests must be found. To that end, we look for a parametric dispersion relation of type B, namely, such that along the contour S, ν and q^2 lie in the complex Regge region (which we take to be defined by $|\omega| = |2M_p \nu/q^2| > 12$, with $|q^2| \leq 1$ BeV^2 and $|\nu| > 2$ BeV). An example of this type is

$$\text{(B)} \quad \begin{cases} u(z) = 4.5z - 2.5 \, , \\ v(z) = (z-1)/8(4.5z-2.5) \, . \end{cases} \tag{2.7}$$

Here ν and q^2 are defined in terms of u and v by (2.6). The path of integration along the contour L in the ν-q^2 plane is shown in Fig. 3; note that it passes through the interesting low-q^2 part of the resonance region and that on the entire contour C, q^2 varies in the range $|q^2| \leq 1$ BeV^2. Since $\nu(z)$ has a pole at $z = 2.5/4.5$, we must make an infinitesimal distortion of the contour L around the pole. However, νT_2 is assumed to be finite as $\nu \to \infty$, so this infinitesimal distortion will make a negligible contribution to the integral. To the accuracy of our evaluation (and of the data), the integral L is completely determined by the data on the structure function νW_2. Sum rule (B) therefore allows us to compare the integral S, which depends only on the Regge fit, with a number computed directly from the data with no assumption about real or complex scaling. This parametric dispersion relation provides a very severe restriction on the Regge fits. Its evaluation is also presented in the next section.

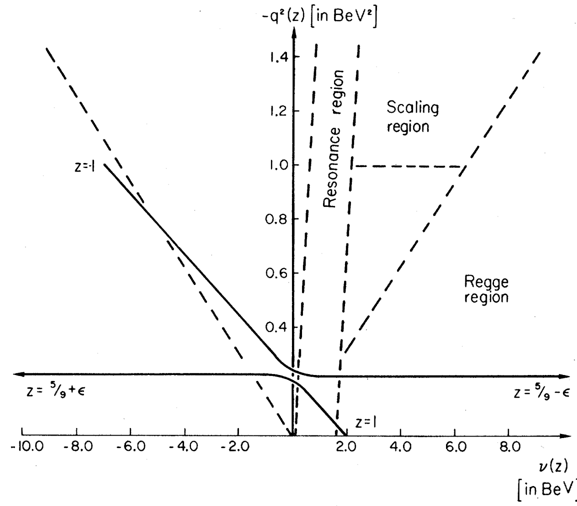

FIG. 3. Path of integration in the ν-q^2 plane for sum rule (B).

III. EVALUATION OF SUM RULES

Our evaluation of the sum rules described in the last section has been somewhat different from our evaluation of Khuri's original sum rules in II. First we discuss sum rules A. In the imaginary part of L, the ν' integration is trivial and we have only to evaluate a single integral over νW_2 along the paths shown in Figs. 1 and 2. The structure function was obtained from the electroproduction data using the interpolation program (SEARCH) provided by the SLAC-MIT collaboration.[4] The real part of L involves a principal-value integration of νW_2 over ν' which we cut off at $\nu' - \nu_t(x) = 20$ BeV; the real part of L did not vary more than 1% when this cutoff was varied between 20 and 40 GeV. We also evaluated L using the fits to νW_2 used in II and found that to within the accuracy of the data the results agreed with those found using the interpolation program. Both the real and imaginary parts of S were determined using the dispersion integral (2.3a) with $F_2(\omega')$ given by[2,3]:

$$\begin{aligned} F_2(\omega) &= 2.33(1-1/\omega)^3 - 2.67(1-1/\omega)^4 \\ &\quad + 0.91(1-1/\omega)^5 \, , \qquad 1<\omega<4 \\ F_2(\omega) &= 0.35 \, , \quad 4<\omega<6 \\ F_2(\omega) &= 0.369 - 0.0033\omega \, , \quad 6<\omega<12 \\ F_2(\omega) &= \frac{1}{\pi} \lim_{-q^2\to\infty;\, \omega \text{ fixed}} \text{Im}R(q^2,\nu) \, , \quad \omega > 12 \, . \end{aligned} \tag{3.1}$$

The integral over ω' was cut off at $\omega_{max} = 20$; the results were insensitive to both the cutoff and the Regge fit used. Table I contains the results of sum rules (A).

To evaluate the integral L in sum rules (B) we used the same procedure as in sum rules (A) for $||\nu| - \nu_t(x)| \leq 12$ BeV. However, for $||\nu| - \nu_t(x)| > 12$ BeV, we replaced the dispersion integral for νT_2 with the Regge fit to νT_2; this region makes a negligible contribution to L, and to the accuracy

TABLE I. Results of evaluation of sum rules (A).

Sum rule	ReL	ReS	ImL	ImS
(A1)	0.00	0.00	0.24	0.22
(A2)	0.43	0.47	0.24	0.22
(A3)	0.37	0.42	0.14	0.13

of our evaluation, the integral L is independent of which Regge fit is used. The integral S is now a single integral of the Regge fit $R(q^2, \nu)$ to the off-shell Compton amplitude νT_2. The Regge fits we tested were all of the form

$$R(q^2, \nu)=i\beta_1(q^2)+(i-1)\beta_2(q^2)\nu^{-1/2} + (i+1)\beta_3(q^2)\nu^{-3/2}+\beta_{FP}(q^2)\nu^{-1}. \quad (3.2)$$

Here β_1, β_2, and β_3 are the residues of the Pomeranchukon, f-A_2, and effective nonleading trajectories, respectively; β_3 is zero in some of the fits. β_{FP} is the residue of the $J=0$ fixed pole. With one exception the Regge residues for these fits have the form

$$\begin{aligned}\beta_1(q^2)&=\frac{-\pi q^2}{-q^2+\mu^2}\alpha\,,\\ \beta_2(q^2)&=\frac{-\pi q^2}{-q^2+\mu^2}\left(\frac{-q^2+m^2}{2M_p}\right)^{1/2}\beta\,,\\ \beta_3(q^2)&=\frac{-\pi q^2}{-q^2+\mu^2}\left(\frac{-q^2+m^2}{2M_p}\right)^{3/2}\gamma\,.\end{aligned} \quad (3.3)$$

TABLE II. Parameters for Regge fits 1–7.

Fit	α	β	γ	δ	μ (BeV)	m (BeV)
1	0.12	0.462	4.02	1.0	0.37	0.22
2	0.06	0.618	4.64	1.0	0.37	0.22
3	0.05	0.645	4.75	1.0	0.37	0.22
4	0.07	0.663	3.67	0.0	0.37	0.22
5	0.17	0.113	3.42	1.0	0.44	0.44
6a	0.28	0.18	0.0	−0.6	0.5	0.5
6b	0.28	0.18	0.0	1.0	0.5	0.5
7a	0.11	0.68	0.0	−1.25	0.316	0.316
7b	0.11	0.68	0.0	1.0	0.316	0.316

The fixed-pole residue was assumed to be linear in q^2 over the range $|q^2| \leq 1$ BeV^2:

$$\beta_{FP}(q^2)=q^2\delta/M_p\,. \quad (3.4)$$

The constants α, β, γ, δ, μ, and m for these fits are listed in Table II. Fits 1–5 were obtained by Close and Gunion.[9] Fits 6 and 7 were made by Pagels[11] for large q^2 and extended to low q^2 by us. The fixed-pole residues for fits 6a and 7a were calculated by Elitzur[8] using a FESR for large q^2; fits 6b and 7b have fixed-pole residue determined by the FESR in the $q^2\to 0$ limit.[7,9] Fit 6 corresponds to $\sigma_s/\sigma_T=0.18$ and fit 7 to $\sigma_s/\sigma_T=0$. These fits are discussed in more detail in II and in the original references.

We have also tested another Regge fit which has residues of a different form; namely,

$$\begin{aligned}\beta_1(q^2)&=-\pi q^2\left(\frac{\alpha_1}{-q^2+m_\rho^2}+\frac{\alpha_2}{-q^2+M_1^2}\right)\,,\\ \beta_2(q^2)&=\begin{cases}-\pi q^2\left(\dfrac{\gamma_1}{-q^2+m_\rho^2}+\dfrac{\gamma_2}{-q^2+M_1^2}+\dfrac{1}{M_2^2}\right)\dfrac{1}{(M_p)^{1/2}}, & |q^2|\leq 2\text{ BeV}^2\\ \pi\gamma_3(-q^2/2M_p)^{1/2}, & |q^2|>2\text{ BeV}^2\end{cases}\\ \beta_3(q^2)&=0\,,\\ \beta_{FP}(q^2)&=q^2\delta_1/M_p\,.\end{aligned} \quad (3.5)$$

This fit was made by Preparata[12] using the formalism of mass dispersion relations and light-cone dominance for large masses. The constants for fit 8 are tabulated in Table III. The fixed-pole residue for fit 8b is that determined by the FESR in the $q^2\to 0$ limit[7,9]; that for fit 8a was calculated by us, following Elitzur,[8] using the FESR in the scaling limit.

These Regge fits were all determined for real spacelike q^2 and large values of $\nu/(-q^2)$. We use the same functional form for complex q^2 and $|\nu/q^2|$ large with $|q^2|\lesssim 1$ BeV^2.

The results of sum rule (B) evaluated with each of these Regge fits are tabulated in Table IV.

IV. DISCUSSION AND CONCLUSIONS

From Table I we see that sum rule (A1) is satisfied to within 10%. As pointed out earlier, this sum rule depends only on complex Bjorken scaling; it shows that complex scaling is valid to the same accuracy and down to the same low values of $|\nu|$ and $|q^2|$ as is real Bjorken scaling. Sum rules (A2) and (A3) require complex scaling in both the Bjorken scaling region and in the Bjorken-Johnson-Low region. Both real and imaginary parts of these sum rules are satisfied to within 15% of the magnitude of L. In addition, sum rule

TABLE III. Parameters for Regge fits 8.

Fit	α_1	α_2	γ_1	γ_2	M_2^2 (BeV^2)	γ_3	δ_1	m_ρ (BeV)	M_1^2 (BeV^2)
8a	0.424	−0.124	0.0412	0.0088	50.0	0.07	−0.38	0.765	2.0
8b	0.424	−0.124	0.0412	0.0088	50.0	0.07	1.0	0.765	2.0

(A2) shows that on each half of the contour L the real part of sum rule (A1) dominates the imaginary part in magnitude. However, the antisymmetry of the real part of νT_2 causes the real part of L to vanish. Only the imaginary part of sum rule (A1) is nontrivial, and its accuracy is even better than we had a right to expect. We reiterate that the parametric dispersion relations (A1), (A2), and (A3) are by no means fixed-q^2 sum rules. In each case, q^2 varies rapidly compared with ν as L passes through the resonance region, and it would be surprising indeed if the scaling limit curve provided a reasonable approximation to the inelastic structure function $\nu W_2(\nu, q^2)$ along these contours through the resonance region. We therefore feel that these parametric dispersion relations provide clear and nontrivial confirmation for the concept of precocious complex scaling.

The results of sum rule (B), listed in Table IV, show that this parametric dispersion relation is very sensitive to the Regge fit used and that analyticity in two variables provides further restrictions on the Regge residues. For Regge fits 6b and 8b the agreement is by far the best; both real and imaginary parts of the sum rule are satisfied to within about 13% of the magnitude of L. These fits were made by Pagels and by Preparata, respectively. With each of the other fits the sum rule fails by more than 25%. Fits 6b and 8b contain only the Pomeranchukon and f-A_2 Regge trajectories and have fixed-pole residues determined by the FESR in the $q^2 \to 0$ limit. The same Regge fits, but with fixed-pole residues determined by the FESR in the scaling limit (fits 6a and 8a) fail badly. This is not surprising given that our parametric dispersion relation requires the fixed-pole residue only for low $|q^2|$.

TABLE IV. Results of evaluation of sum rule (B).

Regge fit	ReL	ReS	ImL	ImS
1	0.05	−0.37	1.30	1.65
2		−0.57		1.57
3		−0.61		1.55
4		−0.90		1.28
5		−0.33		1.73
6a		−0.39		1.02
6b		0.08		1.29
7a		−1.23		0.63
7b		−0.55		1.01
8a		−0.21		0.93
8b		0.21		1.16

While we have assumed the fixed-pole residues for fits 6–8 to be linear in q^2 for $|q^2| \leq 1$ BeV^2, they cannot be polynomials in q^2 for all q^2 if scaling is to hold; the fixed-pole residues must change sign on passing from $q^2=0$ to the scaling limit. The fits made by Close and Gunion (fits 1–5), which do have polynomial fixed-pole residues, do not satisfy our sum rules; the errors for these fits are more than 35% of the magnitude of L. Our analysis assumes of course that the real part of νT_2 is determined from the Regge fit to νW_2 by the standard Regge signature factors, and it is possible that this assumption is not applicable to the effective $\nu^{-3/2}$ term in the fits of Close and Gunion.

We also point out that Regge fits 6 and 8 have much larger Pomeranchukon contributions than any of the other fits tested. The q^2 behavior of the Regge residues for fits 6 and 8 is, however, quite different. We have also evaluated sum rule (B) using fit 6b, but varying the masses μ and m. We found that for $0.4 \leq \mu \leq 0.6$ and $0.4 \leq m \leq 0.8$ the sum rule is still satisfied to within 15%; this is about as wide a variation of these masses as the data in the low-q^2 part of the Regge region allow. These results suggest that, while our parametric dispersion relation is very sensitive to the relative magnitude of the Pomeranchukon contribution and to the fixed-pole residue, it does not depend on the precise form chosen for the q^2 behavior of the Regge residues. Our tentative conclusion is, therefore, that the correct Regge fit is dominated by the Pomeranchukon, with relatively small contributions from the f-A_2 and any other Regge trajectories. This would of course require that the fixed-pole residue have nonpolynomial behavior in q^2.

It is interesting to note that analyticity in q^2 was used to obtain both fit 6 and fit 8. Pagels[11] obtained fit 6 by using the analyticity properties of the partial-wave amplitudes in the virtual photon mass, which follow from the Froissart-Gribov definition of these amplitudes and from the DGS[13] (Deser-Gilbert-Sudarshan) representation, to extrapolate the high-energy behavior of the total photon-proton cross section to the electroproduction region. Preparata[12] employed relations between photoproduction and deep-inelastic electroproduction derived from light-cone dominance for large masses and mass dispersion relations for Regge residues to derive fit 8.

While our parametric dispersion relation does not solve the intriguing problem of the existence and q^2 behavior of a $J=0$ fixed pole in the off-mass-shell Compton amplitude, it does severely limit the allowable Regge fits to the inelastic structure functions νW_2. Of the fits we have tested, the only two which satisfy our sum rule must have fixed poles with nonpolynomial residues. It would be interesting to see if Regge fits can be found which do have polynomial fixed-pole residues and which satisfy the parametric dispersion relation.

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APPENDIX A

We prove here the assertion made in Sec. II that there exist no parametric dispersion relations such that: (i) within the contour C, consisting of the interval (L) of the real axis from $z=-1$ to $z=1$ and the unit semicircle (S) $0 \leqslant \arg z \leqslant \pi$, q^2 and ν lie in the forward tube; (ii) on the contour L, q^2 and ν are real with q^2 spacelike and pass through the low-q^2 ($|q^2| < 1$ BeV^2) part of the resonance region; (iii) on the contour S, q^2 and ν lie in the complex scaling region ($|q^2|$ large).

The simplest way to proceed is to conformally transform the interior of the unit semicircle onto the upper-half plane. This is accomplished by the transformation $z' = 2z/(1+z^2)$, which maps the interval $-1 \leqslant x \leqslant 1$ onto itself and the unit semicircle onto the remainder of the real axis. The conditions (i)–(iii) translate to the following: (i′) In the upper-half z plane, q^2 and ν lie in the forward tube; (ii′) on the interval $-1 \leqslant x \leqslant 1$, q^2 and ν are real with q^2 spacelike and pass through the low-q^2 part of the resonance region; (iii′) on the rest of the real z axis, $|q^2|$ is large. Conditions (ii′) and (iii′) imply that for z on the real axis, $|q^2|$ takes its minimum on the interval $-1 < x < 1$. We will show that (i′) and (ii′) are not compatible with this requirement.

As in Sec. II we define

$$\zeta = q_0 ,$$

$$\eta = |\vec{q}| ,$$

and

$$u = \tfrac{1}{2}(\zeta + \eta) ,$$

$$v = \tfrac{1}{2}(\zeta - \eta) .$$

The forward tube is defined by $\mathrm{Im}\zeta > |\mathrm{Im}\eta|$, or equivalently, $\mathrm{Im}u > 0$ and $\mathrm{Im}v > 0$. We must now show that it is not possible to find two functions $u(z)$ and $v(z)$ such that: (a) $u(z)$ and $v(z)$ are analytic for $\mathrm{Im}z > 0$ and continuous for $\mathrm{Im}z \geqslant 0$, except possibly at isolated points on the real axis outside the interval $-1 \leqslant x \leqslant 1$; (b) $\mathrm{Im}u(z) > 0$ and $\mathrm{Im}v(z) > 0$ for $\mathrm{Im}z > 0$; (c) on the interval $-1 \leqslant x \leqslant 1$, $q^2(z) = 4u(z)v(z)$ is real and spacelike; (d) $|q^2(z)|$ takes its minimum, for real z, in the interior of the interval $-1 \leqslant x \leqslant 1$.

Conditions (a) and (b) imply that $u(z)$ and $v(z)$ must be Herglotz functions. This immediately allows us to show that q^2 cannot have a zero on the interval $-1 < x < 1$. To see this, note that Herglotz functions can have only simple zeros on the real axis. Now (c) implies that if q^2 has a zero on the unit interval, it must be of even order; it must in fact be of order 2 and both u and v must have simple zeros at some point x_0 with $-1 < x_0 < 1$. Since u and v are Herglotz functions, for z near x_0 they must have the form

$$u(z) \approx a(z - x_0) ,$$

$$v(z) \approx b(z - x_0) ,$$

with a and b positive. This requires that $q^2 = 4uv$ be positive on both sides of x_0, which violates condition (c).

Now we will prove that $|q^2|$ cannot reach a nonzero minimum on the real axis in the interval $-1 < x < 1$. Consider the functions $u'(z) = -1/u(z)$, $v'(z) = -1/v(z)$, and $q'^2 = 4u'v'$. $u'(z)$ and $v'(z)$ are also Herglotz functions. We must show that $|q'^2|$ cannot reach a maximum on the interval $-1 < x < 1$. If it did, then $|q'^2|$ would be bounded by a constant on the entire real axis. Since a Herglotz function cannot have any singularities in the upper-half plane and cannot blow up faster than $|z|$ as $|z| \to \infty$ in complex directions, we can apply the Phragmén-Lindelöf theorem to show that $|q'^2|$ would be bounded by the same constant throughout the entire upper-half plane. Using the Schwarz reflection principle, we can analytically continue the function $q'^2(z)$ into the lower-half plane where

TABLE V. Results of sum rule (A) of Ref. 2 evaluated with $m_0^2 = 10$ BeV^2, $b = 2.5$, $q_c^2 = 1.0$ BeV^2.

	Regge fit 1	2	3	5
ImL	0.20	0.20	0.20	0.20
ImC	0.17	0.17	0.17	0.13
ImC_R	−0.02	−0.02	−0.02	−0.06
ReL	0.17	0.17	0.17	0.17
ReC	0.08	0.06	0.06	0.08
ReC_R	0.03	0.01	0.01	0.03

it would be bounded by the same constant. This would require that $q'^2(z)$ take its maximum at a point interior to its region of analyticity, which violates the maximum-modulus theorem. Therefore, sum rules of the type desired are impossible.

APPENDIX B

In this appendix we would like to reevaluate some of the sum rules of Ref. 2. The Regge fits obtained from Ref. 4 in II had an error in the fixed-pole residue. In that paper the normalization of R_p, the coefficient of q^2 in the fixed-pole residue, defined in the text is not the same as the normalization of R_p used in tabulating the fits. The relevant factor is $1/\pi M_p$. As a result, in Table I of II the constant δ should have units of BeV^{-1} (notice that the definition of δ in the present paper differs from that used in II) and, for Regge fits 1–3 and 5, $\delta = 1$ should be replaced by $\delta = 1/\pi M_p$.

We have reevaluated the sum rule (A) of II with this correction for the Close and Gunion fits. The results are found in Table V of this paper. The results for the other Regge fits tested remain unchanged and are found in Table II of Ref. 2. Clearly some of the conclusions drawn in II are no longer quantitatively correct for the fits 1–3 and 5. The imaginary part of the sum rule is still roughly independent of the Regge fit used and is satisfied to within 25% of the magnitude of L for each Regge fit. This still gives qualitative support for the concept of complex scaling. However, the real part of the sum rule is no longer satisfied for fits 1–3 and 5. There are two possible reasons for this failure. One is that these particular fits are bad. The other is that this sum rule uses the Regge fits for relatively small $|\nu|$ ($|\nu| < 2$ BeV) and $|q^2| < 1$ BeV^2 and only part of this domain is in the Regge region.

To distinguish these possibilities we evaluated the same sum rule, but with $q_c^2 = 1.0$ BeV^2 replaced by $q_c^2 = 0.5$ BeV^2. That is, on the unit semicircle we used complex scaling for $|q^2| > 0.5$ BeV^2 and the Regge fits for $|q^2| \leq 0.5$ BeV^2. With this choice of q_c^2 the Regge fits are still used at times for smaller values of $|\omega|$ than we would like, but not at the very small values of $|\omega|$ where they were used in II. This new choice of q_c^2 also requires that we assume complex scaling extends to even lower values of $|q^2|$ at least in an average sense, and this sum rule should provide a stronger test of the precocious nature of complex scaling.

We have evaluated this sum rule with all the Regge fits examined in the present paper. The results are presented in Table VI. The imaginary part of the sum rule is satisfied to within about 15% of the magnitude of L, independent of which Regge fit is used. This provides further confirmation for the concept of precocious complex scaling. The real part of the sum rule is satisfied to within about 20% for all of the Regge fits, which is about as good as can be expected given that we are extending our use of complex scaling and Regge behavior beyond their expected ranges of validity. Therefore, while the integral on the unit semicircle does depend on which Regge fit is used, we cannot use this sum rule to exclude any of the fits. However, the qualitative conclusions drawn in II are correct. These sum rules suggest that complex scaling is good and that parametric dis-

TABLE VI. Results of sum rule (A) of Ref. 2 evaluated with $m_0^2 = 10$ BeV^2, $b = 2.5$, $q_c^2 = 0.5$ BeV^2. Regge fits are those described in the present paper.

	Regge fit 1	2	3	4	5	6a	6b	7a	7b	8a	8b
ImL	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
ImC	0.17	0.18	0.18	0.17	0.16	0.16	0.16	0.17	0.18	0.16	0.16
ImC_R	−0.01	−0.01	−0.01	−0.02	−0.03	−0.03	−0.03	−0.01	−0.01	−0.03	−0.03
ReL	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17
ReC	0.12	0.12	0.12	0.12	0.12	0.12	0.14	0.15	0.17	0.11	0.12
ReC_R	0.03	0.03	0.03	0.03	0.03	0.03	0.05	0.06	0.08	0.02	0.03

persion relations can be used to discriminate between Regge fits.

Tables III–V of Ref. 2 also contain the same error for the fits of Ref. 4 and should be ignored.

We have not reevaluated these sum rules here because the sum rules presented in the present paper are more powerful, and the results presented in this paper supersede those found in II.

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[1]N. N. Khuri, Phys. Rev. D 5, 462 (1972); this paper will be referred to as I.

[2]J. B. Healy and N. N. Khuri, Phys. Rev. D 5, 2763 (1972). This paper will be referred to as II. Some of the Regge fits in it have a factor $1/\pi M_p$ missing in the residue of the fixed pole. While the conclusions reached there are qualitatively correct, we refer the reader to Appendix B of the present paper for detailed numerical results.

[3]M. Breidenbach, MIT Report No. MIT-2098-635, 1970 (unpublished); E. D. Bloom *et al.*, MIT-SLAC Report No. SLAC-PUB-796, 1970 (unpublished), presented at the Fifteenth International Conference on High Energy Physics, Kiev, U. S. S. R., 1970.

[4]SEARCH is an interpolation program which uses data from experiments reported in the following papers: E. D. Bloom *et al.*, Phys. Rev. Letters 23, 930 (1969); M. Breidenbach, MIT, LNS Report No. 89, 1970 (unpublished); M. Breidenbach *et al.*, Phys. Rev. Letters 23, 935 (1969); G. Miller, SLAC Report No. 129, 1971 (unpublished); G. Miller *et al.*, Phys. Rev. D 5, 528 (1972). We thank E. D Bloom and R. Early for sending us the SEARCH package.

[5]J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, Providence, R.I., 1943), p. 23. See also Dyson's proof in Appendix A of I.

[6]E. D. Bloom and F. J. Gilman, Phys. Rev. D 4, 2901 (1971).

[7]M. J. Creutz, S. D. Drell, and E. A. Paschos, Phys. Rev. 178, 2300 (1969); M. Damashek and F. J. Gilman, Phys. Rev. D 1, 1319 (1970); C. A. Dominguez, C. Ferro Fontan, and R. Suaya, Phys. Letters 31B, 365 (1970).

[8]C. H. Llewellyn Smith, SLAC Report No. SLAC-PUB-843, 1970 (unpublished). See also S. Rai Choudhury and S. Rajaraman, Phys. Rev. D 2, 2728 (1970); M. Elitzur, *ibid.* 3, 2166 (1971).

[9]F. Close and J. Gunion, Phys. Rev. D 4, 742 (1971).

[10]T. P. Cheng and Wu-Ki Tung, Phys. Rev. Letters 24, 851 (1970). See also J. M. Cornwall, D. Corrigan, and R. Norton, *ibid.* 24, 1141 (1970).

[11]H. Pagels, Phys. Letters 34B, 299 (1971).

[12]G. Preparata, Phys. Letters 36B, 53 (1971). There is a typographical error in the residue of the f-A_2 Regge pole in that paper. The correct form for the Regge residue in Eq. (12) is

$$\beta_{2T}(q^2) = -q^2 \lambda_T \frac{2}{\pi}\left(\frac{\Lambda}{2}\right)^{1/2}\left[\frac{M^2(1-\Lambda/3M^2)}{(M^2-q^2)(m_\rho{}^2-q^2)} + \frac{1}{M^2-q^2} + \frac{1}{\Lambda}\right]$$

We thank Dr. Preparata for bringing it to our attention.

[13]S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. 115, 731 (1959); N. Nakanishi, Progr. Theoret. Phys. (Kyoto) 26, 337 (1967).