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Connection of "Parabolic" Mass and Width Trajectories with High-Energy Scattering

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An infinite sequence of direct-channel resonances is explicitly summed by the Sommerfeld-Watson method. This is quite different from the problem heretofore considered of summing t-channel resonances and requires different physical assumptions. In fact, the "parabolic" mass trajectory $\mu_J \propto J$ and a slow growth of width (on resonance) $\Gamma_J \propto J$ as $J \to \infty$ are necessary to reproduce in the Regge limit the form const$\times J_{\Delta\lambda}(R\sqrt{-t})$, $R \equiv$ interaction range, $\Delta\lambda \equiv$ total helicity change, for the imaginary part of this (nondiffractive) part of the c.m. two-body reaction amplitude, in agreement with experiment, and Harari's qualitative theory, for πN and KN entrance channels. The above width ansatz is of central importance in the derivation and is also experimentally supported. Linear and parabolic mass trajectories in the direct channel are compared. Small s-dependent deviations from fixed crossover $-t \approx 0.2$ $(\text{GeV}/c)^2$ and first dip $-t \approx 0.6$ $(\text{GeV}/c)^2$ points are predicted.

I. INTRODUCTION

Harari, synthesizing much previous work, has recently formulated a qualitative theory[1,2] of hadronic two-body reactions at high energy which explains the detailed t structure of both elastic and inelastic cross sections remarkably well. The imaginary part of the c.m. scattering amplitude f is supposed to consist of two parts:

$$\text{Im}f = \text{Im}f_{\text{I}} + \text{Im}f_{\text{II}}, \tag{1.1}$$

where f_{II} is the diffractive part ("Pomeranchukon exchange"), which has a structureless t dependence, and f_{I} is the sum of nonexotic resonances (or of nonexotic exchanges, in the sense of duality) with t structure contained in Bessel-like functions,

$$\text{Im}f_{\text{I}}(s,t) = \beta(s)\,\text{“}J_{|\Delta\lambda|}(R\sqrt{-t})\text{”}. \tag{1.2}$$

"$J_{|\Delta\lambda|}(z)$" has zeros, maxima, and minima in the same places as $J_{|\Delta\lambda|}(z)$; here $\Delta\lambda$ is the total helicity change $\Delta\lambda \equiv \lambda_a - \lambda_b - (\lambda_c - \lambda_d)$ for the reaction $ab \to cd$, R is the "radius of interaction" ≈ 1 F $\equiv 10^{-13}$ cm, and the s dependence of $\beta(s)$ is not specified in detail except that asymptotically it goes like const (elastic case)[3] while $\text{Im}f_{\text{II}} = O(\sqrt{s})$.

Our purpose is to derive the form (1.2) by explicitly performing the sum over resonances by means of the SW (Sommerfeld-Watson) transform. A parabolic (mass)^2 trajectory[4] $\mu_J \propto J$ rather than the usual linear one $\mu_J{}^2 \propto J$ for the resonance families of interest is assumed, as already suggested in Refs. 1 and 2. But far from neglecting the widths, as previous work has done, we find that the particular width-spin relation assumed is of central importance; and in fact the "parabolic" width trajectory[4] $\Gamma_J \propto J$ asymptotically is strongly suggested. A more precise form of (1.2) is derived in which $\beta(s)$ is specified and there are small s-dependent corrections to the argument of $J_{|\Delta\lambda|}$ which should allow an experimental check of the crucial width-spin ansatz.

In addition the linear and parabolic mass trajectory cases will be contrasted.

II. SUMMING THE RESONANCES

Start with the partial-wave expansion of f_{I} (drop the subscript thereafter)

$$f(s,t) = \frac{1}{2k}\sum_J (2J+1) f_J(s) P_J(\cos\theta), \tag{2.1}$$

$s \equiv 4(k^2 + M^2)$, $t \equiv -2k^2(1-\cos\theta)$, where we shall treat spinless external particles of equal mass M for mathematical simplicity, but expect the resulting s and t dependence to be realistic. Now assume that f is the sum of the nonexotic resonances of a recurrence family of signature τ:

$$f_J(s) = \theta_J \frac{\Gamma_J}{\mu_J - i\Gamma_J/2 - \sqrt{s}}, \tag{2.2}$$

where μ_J and Γ_J are mass and (total) width of the resonance of spin J, and

$$P_J(\cos\theta) \to \tfrac{1}{2}[P_J(\cos\theta) + \tau P_J(-\cos\theta)]$$

in (2.1). Γ_J and θ_J have a tacit s dependence which will *usually* be suppressed for notational simplicity.

The ansatz (2.2) has Breit-Wigner form if $\theta_J = (x_{J\alpha}x_{J\beta})^{1/2}$, where $x_{J\alpha} \equiv \Gamma_{J\alpha}/\Gamma_J$, $\Gamma_{J\alpha} \equiv$ partial width for channel α, and α and β are entrance and exit two-body channels.[5,6] If $\alpha = \beta$, $\theta_J = x_{J\alpha}$ is called the "elasticity" for the elastic channel α. But we leave the precise form of θ_J open for these highly relativistic hadron resonances. The assumption of a Breit-Wigner denominator for each resonance at any energy, while admittedly crude, is explicitly made here for simplicity and because we can do no better.

Next one must assume a relation between the mass and width and the spin. Following Harari, we take

$$\mu_J{}^2 = 4(J+\tfrac{1}{2})^2/R^2 + 4M^2 \Longleftrightarrow J+\tfrac{1}{2} = k_J R \qquad (2.3)$$

for some "radius of interaction" R. k_J is the c.m. momentum associated with $s_J \equiv \mu_J{}^2$ and external masses M. The usual linear trajectory is treated briefly in Sec. III.

To elucidate the width-spin relation, we first treat $\Gamma_J \equiv \Gamma$ as J-independent. This preliminary is a convenient introduction to the determination of the correct J dependence.

Replacing the sum over J by a contour C enclosing the points $J = 0, 1, 2, \ldots$ by the SW method,[7] we get

$$f = \frac{i}{8k}\int_C dJ\,(2J+1)\,\theta(J,s) \times \frac{\Gamma}{D(J)}\left[\frac{P_J(-\cos\theta)+\tau P_J(\cos\theta)}{\sin\pi J}\right], \qquad (2.4)$$

$$D(J) \equiv \mu(J) - \tfrac{1}{2}i\Gamma - \sqrt{s}\,.$$

Distort C to the contour $C' \equiv$ the line $\mathrm{Re}J = -\tfrac{1}{2}$ plus the infinite semicircle in a right half plane. One picks up the pole $J=\alpha$ at the simple zero of $D(J)$, which gives[8] $f = f_{\mathrm{pole}} + f_{\mathrm{background}}$, with

$$f_{\mathrm{pole}} = \frac{-\pi}{4k}\,\Gamma(2\alpha+1)R(\alpha)\theta(\alpha,s) \times\left[\frac{P_\alpha(-\cos\theta)+\tau P_\alpha(\cos\theta)}{\sin\pi\alpha}\right]. \qquad (2.5)$$

Here the pole α and residue $R(\alpha)$ are

$$\alpha \approx kR - \tfrac{1}{2} + i\Gamma R\omega/4k \equiv \alpha_R + i\alpha_I, \quad \omega \equiv (k^2+M^2)^{1/2} \qquad (2.6)$$

$$(2\alpha+1)R(\alpha) \approx \tfrac{1}{2}R^2\mu(\alpha) \equiv \tfrac{1}{2}R^2(\sqrt{s}+\tfrac{1}{2}i\Gamma), \qquad (2.7)$$

with the approximation[9]

$$|(\Gamma\omega/2k^2)^2| \ \text{ neglected.} \qquad (2.8)$$

The background integral will be briefly discussed below. From now on f shall mean f_{pole}.

Use the identity[10,11]

$$P_\alpha(-\cos\theta) = \cos\pi\alpha\, P_\alpha(\cos\theta) - \frac{2}{\pi}\sin\pi\alpha\, Q_\alpha(\cos\theta) \qquad (2.9)$$

which is convenient *for the s channel* because $Q_\alpha(\cos\theta)$, the (symmetric limit) Legendre function of the second kind, is real for real α. This gives

$$\frac{P_\alpha(-\cos\theta)+\tau P_\alpha(\cos\theta)}{\sin\pi\alpha} = S_\tau(\alpha)P_\alpha(\cos\theta) - \frac{2}{\pi}Q_\alpha(\cos\theta), \qquad (2.10)$$

where the signature factor $S_\tau(\alpha) \equiv (\cos\pi\alpha + \tau)/\sin\pi\alpha$ after a little algebra becomes

$$S_\tau(\alpha) = \frac{\tau\cos\pi\alpha_R + \cosh\pi\alpha_I}{\sinh^2\pi\alpha_I + \sin^2\pi\alpha_R}(\tau\sin\pi\alpha_R - i\sinh\pi\alpha_I)\,. \qquad (2.11)$$

Notice two properties which will be useful later on[12]:

$$\mathrm{Im}S_\tau(\alpha) \sim -1, \quad \mathrm{Re}S_\tau(\alpha) \sim 0, \quad \alpha_I \to \infty\,; \qquad V \equiv \frac{[\mathrm{Im}S_\tau(\alpha)]_{\max}}{[\mathrm{Im}S_\tau(\alpha)]_{\min}} = \coth^2(\tfrac{1}{2}\pi\alpha_I)\,. \qquad (2.12)$$

Regge Limit, *s* Channel

In the Regge limit $\theta \approx \sqrt{-t}/k \to 0$ so that we can use the small-angle approximation[13,14] $P_\alpha(\cos\theta) \sim J_0((\alpha+\tfrac{1}{2})\theta)$. Expand

$$J_0((\alpha+\tfrac{1}{2})\sqrt{-t}/k) \approx J_0((\alpha_R+\tfrac{1}{2})\sqrt{-t}/k) - i\alpha_I\frac{\sqrt{-t}}{k}J_1((\alpha_R+\tfrac{1}{2})\sqrt{-t}/k)\,,$$

where $(\alpha_I\sqrt{-t}/k)^2$ is neglected. Then taking the imaginary part of f, Eq. (2.5), one gets[15] (in a form valid for any trajectory)

$$\mathrm{Im}f \sim -\frac{\pi\Gamma}{4k}(2\alpha+1)R(\alpha)\theta(\alpha,s)\left[\mathrm{Im}S_\tau(\alpha)J_0\!\left(\frac{\sqrt{-t}}{k}(\alpha_R+\tfrac{1}{2})\left(1-\frac{\tau\alpha_I}{\sinh\pi\alpha_I}\,\frac{\sin\pi\alpha_R}{\alpha_R+\tfrac{1}{2}}\right)\right)+\frac{2}{\pi}\,\frac{\alpha_I}{\alpha_R+1}\right]. \qquad (2.13)$$

The last term came from[16]

$$Q_\alpha(1+t/2k^2) \sim -\ln\frac{(\alpha+1)\sqrt{-t}}{2k} - \gamma_0\,, \qquad k\to\infty,\ -t>0 \text{ fixed} \qquad (2.14)$$

with the approximation $\alpha_I/\alpha_R + 1 \ll 1$.

Width and Elasticity

In the resonance region Γ on resonance must be of realistic size [e.g., for $1 \lesssim \mu_J \lesssim 3$ GeV for well-known πN resonances, $\Gamma(\sqrt{s}=\mu_J)$ is typically some hundreds of MeV]. With the typical value $\Gamma = 200$

MeV one gets $\alpha_I = 5\times 0.2/4 = \frac{1}{4}$, or the signature factor $\mathrm{Im}S_\tau(\alpha)$ has oscillations of size $V = \coth^2(\pi/8) \approx 7$, characteristic of the resonance region. But in order that the resonances wash out at high enough energies it is necessary[17] that α_I *increase* with energy, as one sees from (2.11) or (2.12). Since $\alpha_I \sim \frac{1}{4}\Gamma R$ by (2.6), the case Γ_J J-independent is therefore rejected.[18]

But α_I cannot grow too fast with energy because $\alpha = O(k)$ is the distinctive growth law for which the t structure becomes independent of s, as one sees from the small-angle approximation for $P_\alpha(\cos\theta)$. Thus one must have $\alpha_I = O(k^a)$, $0 < a \leq 1$.

To determine the correct width trajectory, we proceed as follows. In the pole equation $D(J) \equiv \mu(J) - \frac{1}{2}i\Gamma(J,s) - \sqrt{s} = 0$, the width term should furnish a small correction to the zero order equation with this term absent; therefore replace J in the width by the zero-order root $J = \alpha_0 \equiv kR - \frac{1}{2}$.

Now guided by experiment[19] we make the basic assumption

$$\Gamma_J(\sqrt{s} = \mu_J) = 2\gamma k_J, \tag{2.15}$$

with γ a well-determined small dimensionless constant; this choice will be discussed further below. Notice that we are only assuming a form for the width *on resonance*. For the energy dependence away from resonance we can take any of several favorite phenomenological forms [e.g., Eq. (3.43) of Ref. 6 with Γ_R given by our (2.15)]. But the general energy dependence is largely irrelevant for the results of this paper.

But in virtue of the assumed mass trajectory (2.3), (2.15) is just sufficient to determine the quantity of interest $\Gamma(\alpha_0, s)$. *Proof:* $\sqrt{s} = \mu_J$ has the solution $J = kR - \frac{1}{2} \equiv \alpha_0$ if we extend (2.15) to continuous J values. Hence the extended (2.15) can be written

$$\Gamma(\alpha_0, s) = 2\gamma k. \tag{2.16}$$

Q.E.D.

When (2.16) is put into the pole equation the solution is

$$\alpha \approx kR - \tfrac{1}{2} + \tfrac{1}{2}i\gamma R\omega, \quad \omega \equiv (k^2 + M^2)^{1/2} \tag{2.17}$$

with the approximations γ^2 neglected relative to 1 and ω^2/k^2 not too different from 1. The residue, in which it is sufficient to neglect also $\frac{1}{2}\gamma$ relative to 1, is

$$(2\alpha+1)R(\alpha) \approx \tfrac{1}{2}R^2\sqrt{s}. \tag{2.18}$$

The upshot is that $\alpha_I = \frac{1}{2}\gamma R\omega \sim O(k)$, which is the maximum growth rate ($a = 1$) allowed for s-independent t structure, as argued above.

Next, restricting ourselves to elastic scattering, let us determine the elasticity $\theta_J \equiv x_J$. From Eq. (2.13), using the expressions just found for α, $\Gamma(\alpha, s)$, and $R(\alpha)$, we see that (1) $x(\alpha, s)$ must grow asymptotically like $1/\sqrt{s}$.[3] Other conditions can be imposed on $x_J(s)$: (2) it should make the partial-wave series converge at all angles; (3) it should make the infinite semicircle contribution in the SW transform vanish; (4) $x_J(s) \leq 1$ for all J and s. These do not fix it uniquely, but for example

$$x_J(s) = \frac{2M}{\sqrt{s}}\,\frac{\cosh 1}{\cosh(\mu_J^2/s)} \tag{2.19}$$

does satisfy (1)–(4). The dimensional constant $2M$ stands for the threshold value of $\sqrt{s}$ in the general case; this value is determined essentially by (4). Then (2.19) gives

$$x(\alpha, s) \sim 2M/\sqrt{s}, \quad s \to \infty \tag{2.20}$$

since $\mu^2(\alpha) \sim s$ if $\frac{1}{2}\gamma \ll 1$.

With the substitution of (2.16), (2.17), (2.18), and (2.20) into (2.13) and neglect[20] of $\frac{1}{2}\gamma$ relative to 1 the amplitude becomes

$$\mathrm{Im}f \sim -\tfrac{1}{2}\pi R^2\gamma M\,\mathrm{Im}S_\tau(\alpha) \times J_0\left(R\sqrt{-t}\left(1 - \frac{\tau\alpha_I}{\sinh\pi\alpha_I}\,\frac{\sin\pi(kR - \frac{1}{2})}{kR}\right)\right), \tag{2.21}$$

where $\alpha_I = \frac{1}{2}\gamma R\omega$ and $\mathrm{Im}S_\tau(\alpha)$ is given by (2.11). Keeping only the leading term in energy, one gets

$$\mathrm{Im}f \sim \tfrac{1}{2}\pi R^2\gamma M J_0(R\sqrt{-t}). \tag{2.22}$$

This reproduces the desired form (1.2) with "J_0" $= J_0$ and $\beta(s) \sim$ const. Q.E.D. Hence the range parameter R, introduced originally in the mass trajectory, has the size[1,2] $R \approx 10^{-13}$ cm.

Crossover and Dips

Equation (2.21) shows that at nonasymptotic energies the "crossover point" (where $\mathrm{Im}f$ vanishes for the first time) is given by

$$(R\sqrt{-t})_{\text{cross}} \approx \zeta_0\left[1 + \frac{\tau\alpha_I}{\sinh\pi\alpha_I}\,\frac{\sin\pi(kR - \frac{1}{2})}{kR}\right], \tag{2.23}$$

where $\zeta_0 = 2.4\cdots$ is the first zero of $J_0(z)$. The first dip (at the first minimum of $\mathrm{Im}f$) is at

$$(R\sqrt{-t})_{\text{dip}} \approx \zeta_1\left[1 + \frac{\tau\alpha_I}{\sinh\pi\alpha_I}\,\frac{\sin\pi(kR - \frac{1}{2})}{kR}\right], \tag{2.24}$$

where $\zeta_1 = 3.8\cdots$ is the first (nonzero) zero of $J_1(z)$. These small corrections to the values $(-t)_{\text{cross}} \approx 0.23$ and $(-t)_{\text{dip}} \approx 0.59$ $(\mathrm{GeV}/c)^2$ given by the qualitative theory (1.2) oscillate with energy around these values and disappear with increasing energy.

More on the Width

(1) Comparing the width formula (2.15) with experiment,[21] one finds that it works fairly well in

view of the large experimental uncertainties, giving a value $\gamma \approx 0.1$ to 0.15. The approximations γ^2 neglected, used in solving for the pole, and $\frac{1}{2}\gamma \ll 1$, used in (2.21), are well satisfied by this value.

(2) The resonances will have vanished when the "variation" V, Eq. (2.12), is about unity. Defining the latter by a 10% variation, say, we set $V = 1.1$, which gives

$$\tfrac{1}{2}\pi\alpha_I \approx 1.8$$

or

$$k_{\text{asy}} \equiv 4\times 1.8/\pi R\gamma \approx 3600 \text{ MeV}/c \qquad (2.25)$$

if we take $\gamma = 0.13$. Or, as in nuclear physics, one could define this energy as that for which the width at resonance becomes equal to the level spacing. In our case the latter is given by $R\Delta k = 2$ or $\Delta k \approx 400$ MeV/c. This gives the same type of criterion as (2.25), but quantitatively on the low side:

$$\Gamma_J(k = k_J) = \Delta k$$

or

$$k'_{\text{asy}} = 1/R\gamma \approx 1500 \text{ MeV}/c\,. \qquad (2.26)$$

The most massive known πN resonances do lie somewhere around the limit (2.25).

A Consistency Check

Our requirements on Γ_J and θ_J have completely determined the asymptotic elastic $\text{Im}f_{\text{I}} = (2.22)$. One can now compare its size forward to that of $\text{Im}f_{\text{II}}$. For the diffractive amplitude we take[22]

$$f_{\text{II}} = i(k/4\pi)\sigma_{\text{tot}}(pp)\exp(\tfrac{1}{4}R^2 t).$$

This gives

$$\frac{\text{Im}f_{\text{I}}(t=0)}{\text{Im}f_{\text{II}}(t=0)} = \frac{2\pi R^2\gamma M}{k\sigma_{\text{tot}}(pp)}\,. \qquad (2.27)$$

According to Harari,[3] this should equal $\eta_0/\nu^{1/2}$, $\nu \equiv$ lab momentum, with $\eta_0 \approx 0.7$ (GeV/c)$^{1/2}$ for πN amplitudes at high energy. So using $k \approx (\frac{1}{2}M_N\nu)^{1/2}$ and replacing $2M \to M_N + \mu_\pi$, we get

$$\eta_0 = \left(\frac{2}{M_N}\right)^{1/2} \frac{\pi^2 R^2\gamma(M_N + \mu_\pi)}{\sigma_{\text{tot}}(pp)} \approx 0.50\ (\text{GeV}/c)^{1/2}$$

as compared to the experimental 0.7. The value $\gamma = 0.13$, suggested by the two series of prominent πN resonances,[21] was used here.

The Background Integral

For completeness one should show that the background integral in the SW transform is negligible compared to f_{pole}. This is (2.4) with $J = -\frac{1}{2} + iy$, $-\infty < y < \infty$, and $\Gamma \to \Gamma_J(s)$. An ansatz for $\Gamma_J(s)$ valid at any energy[23] (and hopefully for the background region $J = -\frac{1}{2} + iy$ as well) should properly be used here. Instead for simplicity we have used $\Gamma_J(s) \to \Gamma_J(\sqrt{s} = \mu_J) =$ Eq. (2.15) here, which is wrong but probably overestimates the integral [because of a barrier factor entering in $\Gamma_J(s)$]. Even more uncertainty attaches to the elasticity, for which we have arbitrarily chosen (2.19). Hence substitute

$$\Gamma(J) = i2\gamma y/R, \quad \mu(J) = 2(-y^2 + M^2R^2)^{1/2}/R\,.$$

After some algebra

$$f_{\text{background}} = \frac{\gamma M}{2k}\int_{-\infty}^{\infty} dy\, \frac{y^2\theta(J,s)}{D(y)} \times\left[S_\tau(J)P_J(\cos\theta) - \frac{2}{\pi}\,Q_J(\cos\theta)\right]\,, \qquad (2.28)$$

$$D(y) \equiv (-y^2 + M^2R^2)^{1/2} + \tfrac{1}{2}\gamma y - \tfrac{1}{2}R\sqrt{s}, \quad J \equiv -\tfrac{1}{2} + iy \qquad (2.29)$$

where $S_\tau(J)$ was given in (2.11).

If (2.19) is now substituted into (2.28), one can prove by a calculation[24] too long to give here that in the Regge limit

$$\text{Im}f_{\text{background}} \sim \gamma^2 g(R\sqrt{-t})\,, \qquad (2.30)$$

where $g(\ )$ is a certain function, independent of s in the limit $\gamma \to 0$. This *is* negligible compared to the pole contribution (2.22) if we neglect γ relative to 1. Q.E.D.

However since $\gamma \approx 0.1$ to 0.2 is not too negligible, a careful evaluation of the background integral with reasonable forms for width and elasticity, along with the terms of $O(\gamma^2)$ we have neglected in (2.21), might be of physical interest.

III. LINEAR TRAJECTORY

It is interesting to see what the conventional linear trajectory yields in the direct channel. The mass and width are given by

$$\mu_J{}^2 = \mu^2(J - a), \quad \Gamma_J = \Gamma\,. \qquad (3.1)$$

The constants μ^2 and a specify slope and intercept; we treat here only the case of constant width. Pole and residue come out to be

$$\alpha \approx a + s/\mu^2 + i\sqrt{s}\,\Gamma/\mu^2, \quad R(\alpha) \approx 2(\sqrt{s} + i\tfrac{1}{2}\Gamma)/\mu^2\,, \qquad (3.2)$$

if $\Gamma^2/4s$ is neglected. These are substituted into the general form (2.5) or, in the Regge limit, into (2.13).[25]

In the present case $\alpha_I \to \infty$ in the Regge limit even though the width is constant. So in that limit, keeping only the leading term, we obtain

$$\mathrm{Im} f \sim \mathrm{const} \times \Gamma s \theta(\alpha, s) J_0(4k\sqrt{-t}/\mu^2). \qquad (3.3)$$

The energy dependence could be made correct by requiring $\theta(\alpha, s) = O(s^{-1})$. But the basic defect of the linear trajectory is the s dependence of the t structure as witnessed by the argument of the Bessel function.

IV. CROSSING AND DUALITY QUESTIONS

Crossing Properties

If one tries to continue the amplitude developed in Sec. II to the t channel [by considering $t \geq 4M^2$, $s \leq 0$ in Eqs. (2.5), (2.6), and (2.7)[9] in the parabolic case, where $\Gamma(\alpha, s)$ is either a constant or $2\gamma k$ and $\theta(\alpha, s)$ is derived from (2.19) or something similar] nonsensical physical behavior is obtained for either parabolic or linear mass trajectories. There is usually blowup with energy squared t as $t \to \infty$ and complicated, unrealistic momentum transfer s dependence. Since this theory presumably applies best to entrance channels like πN, KN, etc., the t channel describes boson-boson reactions. Although there is no experimental data for the latter, we suspect that they could never be as pathological as given by this crossed f_{I}.

Now although there are several conventional reasons[26] why this crossed f_{I} need not be physical, the main source of error is probably in the functional forms used for the trajectory α and for $\mu(\alpha)$, $\Gamma(\alpha, s)$, and $\theta(\alpha, s)$.[27] That is, this unphysical behavior suggests that *these functions of s are only s-channel approximations, not suitable for analytic continuation to other (different) channels.*

Duality

In this paper we have shown that one can reproduce Harari's amplitude (1.2) by summing an infinite number of direct-channel resonances. However, this success required parabolic mass and width trajectories and a certain elasticity. The total and elastic widths, Γ_J and $\Gamma_{J\mathrm{el}} \equiv x_J \Gamma_J$, respectively, on resonance were characterized by a slow growth with J.

On the other hand, the idea of duality is that the *same* amplitude should be obtainable as the sum of an infinite number of exchanges of resonances with t-channel quantum numbers. Further work, to be reported in detail later, has shown that we can indeed get essentially Harari's amplitude this way, but *with some important qualitative changes.* Namely: (1) the mass trajectory must be linear instead of parabolic; (2) the total widths on resonance, $\Gamma_J(\sqrt{s} = \mu_J)$, should *vanish* as $J \to \infty$ rather than grow as k_J; (3) the elastic widths contain a "barrier factor" $(4k^2/M^2)^J$ at all energies, which leads to the exceedingly rapid growth $\propto J^J$ with J on resonance. These changes amount to saying that the partial waves for these t-channel resonances have field-theory Feynman-graph form rather than Breit-Wigner form. (See the remarks on the Van Hove–Durand model below). In addition, in accord with Harari and the many absorptive models, the first kR partial waves[28] must be almost completely absorbed out, which goes beyond perturbation theory.

These theoretical findings seem to tie in beautifully with what we find on examining the Particle Data Group Tables. For since the amplitude (1.2), from which one gets the whole complex amplitude f_{I} by affixing the signature factor

$$S_\tau(\alpha) \equiv \frac{1 + \tau e^{-i\pi\alpha}}{\sin \pi\alpha}, \qquad (4.1)$$

is found experimentally to describe elastic and inelastic two-body reactions with baryon ($\pi N, KN$) quantum numbers, the direct-channel resonances should be baryon resonances of this type. These resonances should then obey the trajectory assumptions made above, which do seem to be supported by experiment, in particular the slowly growing width trajectory (2.15).[21]

On the other hand, the t-channel resonances have boson ($\pi\pi, \pi V, K\bar{K}, KK$, etc.) quantum numbers. These in fact do seem to conform to our t-channel resonance trajectory assumptions. The linear mass trajectory for bosons has been in use a long time; and *in particular, widths which* $\to 0$ *as J increases are observed for boson resonances.*[19]

Earlier work by Van Hove and Durand[29,30] similar in spirit to ours illuminates both questions of crossing properties and duality. If one interchanges the names s and t in their work, one can say that they derived a physically sensible result (single-Regge-pole amplitude) in the t channel by continuing an amplitude given as an infinite sum of "narrow" resonances with linear mass trajectory in the s channel. In more detail: their starting point was Feynman graphs in field theory; we can recover Durand's assumptions[31] by setting $\Gamma_J = 0$ *in the denominator* of (2.2) and replacing in the numerator

$$\theta_J \Gamma_J \to g^2 \frac{k}{\sqrt{s}} \left(-\frac{4k^2}{M^2}\right)^J (2J+1)^{-1} C_J^{-1}, \qquad (4.2)$$

where C_J is the leading coefficient of $P_J(z)$. The prescription (4.2) gives Durand's amplitude F, which is connected to the c.m. amplitude f by

$$F = \frac{8\pi\sqrt{-t}}{M^2} f\,.$$

If we now apply the SW method, we obtain for F the expression (2.5), with $\theta(\alpha, s)\Gamma$ replaced by the right-hand side of (4.2) with $J \to \alpha$ and pole and residue $\alpha = a + s/\mu^2$, $R(\alpha) = 1/\mu^2$. Then crossing to the t channel and going to the t-channel Regge limit $t \to \infty$, $s < 0$ fixed, a little calculation gives

$$F \sim -g^2 \frac{\pi}{2\mu^2} S_\tau(\alpha)\left(\frac{2t}{M^2}\right)^\alpha, \quad \alpha = a + s/\mu^2 \tag{4.3}$$

where $S_\tau(\alpha) \equiv$ Eq. (4.1), or conventional Regge-pole behavior. Q.E.D.

This result might be confusing, because we found no sensible crossing properties above, even with the linear mass trajectory. The answer lies in the different width and elasticity forms assumed. The vanishing total width in the Breit-Wigner denominator and elastic width as given by (4.2) give very different results, when analytically continued, from the constant width assumed in Sec. III [even with $\theta(\alpha, s)$ unspecified, as there].

The Van Hove–Durand calculation serves as a simplified illustration of the "further work" on summing t-channel exchanges reported above. It lacks the ingredient "absorption," or cuts in the J plane, hence does not reproduce the Bessel-like function of $R\sqrt{-t}$ in (1.2).

V. FINAL COMMENTS AND CONCLUSIONS

The generalization to external particles with spin (and different masses) is immediate. One replaces the partial-wave expansion (2.1) by the Jacob-Wick helicity amplitude expansion.[32] Our normalization is such that $f_J(s)$ is then replaced by $f^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s) \equiv \langle\lambda_c\lambda_d|T^J(E)|\lambda_a\lambda_b\rangle$ in their notation, and in the resonance ansatz (2.2) θ_J may bear helicity labels. In the s-channel Regge limit the small-angle approximation[33]

$$d^\alpha_{\lambda\mu}(\theta) \sim J_{\Delta\lambda}((\alpha + \tfrac{1}{2})\theta)\,,$$
$$\Delta\lambda \equiv \lambda - \mu \equiv \lambda_a - \lambda_b - (\lambda_c - \lambda_d)\,, \tag{5.1}$$

can be used. One will then end up with (2.21) or (2.22) in which $J_0 \to J_{\Delta\lambda}$ [in particular if $\theta_J(s)$ and Γ_J are taken exactly as before[34]], which verifies Harari's conjecture (1.2).

It is interesting that the infinite sum of resonances in the direct channel produces by interference exactly the most peripheral partial wave $J \approx kR$ for a parabolic mass trajectory without the aid of "absorption" to cut out low waves or barrier factors to cut out the high ones.[34] For we nowhere had to use any assumptions about Γ_J or θ_J off resonance (except that they be such as to guarantee background and infinite semicircle integrals negligible).

Apparently one needs an exponential t dependence[35] e^{Bt} as well as the Bessel function in (2.22) since $J_{\Delta\lambda}$ does not fall fast enough at small $-t$. Whether this is already contained in the present theory, e.g., in the background integral, is not clear. One *does* obtain it by the t-channel approach described in Sec. IV.

The success of our consistency check [after Eq. (2.27)] encourages the idea that the high-energy amplitude (2.21) with the forms (2.16), (2.20) of width and elasticity on resonance may even be semiquantitatively correct. A good way to check experimentally the growth assumption for the width would be to look for the small energy-dependent oscillations of the crossover and first dip points as predicted in (2.23), (2.24).

Conclusions

The imaginary part of the nondiffractive component of a two-body amplitude can be obtained as a sum over resonances whose masses and widths follow the "parabolic" trajectories $\mu_J \propto J$, $\Gamma_J \propto J$ asymptotically, in agreement with Harari's qualitative theory. This reproduces the characteristic s-independent t structure in Bessel-like functions observed for a very wide class of hadronic two-body reactions at high energies. The elasticity $x(J \approx kR, s)$ is found to fall like $1/\sqrt{s}$ as $s \to \infty$. There is good reason to believe that these trajectory and elasticity functions are approximations suited to the s channel, that is, they do not serve to continue the amplitude to other channels.

A theoretical argument for the striking fact that baryon resonances lie (?) on parabolic mass trajectories $\mu_J \sim J$ and have total widths that grow slowly with J as $J \to \infty$, while boson resonances lie on linear trajectories $\mu_J{}^2 \sim J$ and have total widths that vanish as $J \to \infty$ is found in duality. Duality requires that the two-body amplitudes considered here be simultaneously sums of baryon direct-channel resonances and boson t-channel resonances. The present work plus further work on summing t-channel exchanges shows that the respective resonances *must* have just the above characteristics in order to produce (approximately) one and the same amplitude (1.2).

[1] H. Harari, Ann. Phys. (N.Y.) (to be published); in Proceedings of the 1970 Erice School (to be published).

[2] H. Harari, Phys. Rev. Letters 26, 1400 (1971).

[3] H. Harari, Phys. Rev. Letters 26, 1079 (1971).

[4] More precise forms are given below. Γ_J here means the width at resonance.

[5] A. Barbaro-Galtieri, in *Advances in Particle Physics*, edited by R. Cool and R. Marshak (Wiley, New York, 1968), Vol. 2.

[6] Reference 5, Eqs. (3.39b) and (3.10), (3.26).

[7] R. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[8] Reference 7, Chap. 13.1. We have assumed that $\theta(J,s)$ has no singularities between C and C' and $\to\infty$ at most like a power of J on the infinite semicircle.

[9] We treat Γ, k, ω as complex so that (2.6)–(2.8) may apply also to the t channel.

[10] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea, New York, 1954).

[11] Reference 10, p. 63. N. b., the definition of Q_α, the symmetric limit function, p. 60.

[12] We are using $\sim$ ("is asymptotically equal to") in its precise sense.

[13] A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.

[14] Reference 13, p. 147, Eq. 10.

[15] The combination $\mathrm{Im}S_T(\alpha)J_0\big((\alpha_R+\frac{1}{2})\sqrt{-t}/k\big) + \mathrm{Re}S_T(\alpha)\alpha_I(\sqrt{-t}/k)J_1\big((\alpha_R+\frac{1}{2})\sqrt{-t}/k\big)$ has been recombined into $J_0(\ )$ of the shown argument. Also the imaginary part of $(2\alpha+1)R(\alpha)\theta(\alpha,s)$ has been neglected.

[16] Reference 13, p. 163, Eq. 12, and $\psi(z)\sim\ln z$. $\gamma_0 \equiv$ Euler's const.

[17] The energy ratio of f_I to f_II ($1/k$ in the elastic case) also helps to wash out the resonances.

[18] The alternative $\Gamma_J(s)$ J-independent but increasing with s is unrealistic and not further considered.

[19] Particle Data Group, Rev. Mod. Phys. 39, 1 (1967); 43, S1 (1971).

[20] We neglect also the term $\gamma\ln(R\sqrt{-t}/2)$ coming from $\mathrm{Re}Q_\alpha(\cos\theta)$ [see (2.14)] even though it blows up forward. The reasoning is that this spurious singularity must be cancelled by other singularities at $t=0$ arising from the background and infinite semicircle integrals. For the whole integral (2.4) equals the partial-wave series, which is well behaved at $\theta=0$ if $\theta_J(s)$ is chosen as in (2.19). A similar discussion applies to the linear trajectory case, where $\Gamma/\sqrt{s}$ is neglected relative to 1, including the term $(\Gamma/\sqrt{s})\ln(2k\sqrt{-t}/\mu^2)$. However, this mathematical point remains obscure.

[21] For the sequence $N^*_{1/2}$ of the πN resonances $\frac{3}{2}^-$, $\frac{7}{2}^-$, $\frac{11}{2}^-$, $\frac{15}{2}^-$ one finds $\gamma\equiv\Gamma_J/2k_J=0.11, 0.11, 0.13, 0.14$, respectively. For the sequence $N^*_{3/2}$ of πN resonances $\frac{3}{2}^+$, $\frac{7}{2}^+$, $\frac{11}{2}^+$, $\frac{15}{2}^+$, $\frac{19}{2}^+$ one finds $\gamma=0.26$, 0.14, 0.13, 0.12, 0.15, respectively. See Ref. 19. The value 0.26 for $N^*_{3/2}(1236)$ might be discarded because the energy is not asymptotic enough.

[22] For pp has no nonexotic resonance contribution.

[23] E.g., Ref. 5, Eq. (3.43) or Ref. 19, Part IV C.

[24] The basic point, responsible for the $O(\gamma^2)$ behavior, is that for $B(y)\equiv$ contents of the bracket [ ] in the integral (2.28), the even and odd parts are real and pure imaginary, respectively. We choose the square root branches so that $\mu(J)$ is real and positive for $-MR<y<MR$.

[25] We neglect $\Gamma/\sqrt{s}$ relative to 1. See Ref. 20.

[26] (1) It could be that only the total amplitude $f_\mathrm{I}+f_\mathrm{II}$ yields a sensible t-channel result when continued. (2) It is only the *imaginary part* of f_I which is supposed to be resonance-dominated. In other words, Re(2.5) may be in considerable physical error. Therefore f_I (or $f_\mathrm{I}+f_\mathrm{II}$) continued and its imaginary part will in general pick up an admixture of this wrong Re(2.5).

[27] Notice only the values of $\Gamma_J(s)$ and $\theta_J(s)$ on this restricted domain are relevant for the amplitude f_I in s *or* t channel.

[28] R here is a true interaction range, rather than a mass trajectory parameter as for the direct-channel resonances.

[29] L. Durand, Phys. Rev. 161, 1610 (1967).

[30] L. van Hove, Phys. Letters 24B, 183 (1967).

[31] Reference 29, Eq. (1). To compare with our work just interchange the labels s and t.

[32] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959), Eq. (31).

[33] A. Dar, T. Watts, and V. Weisskopf, Nucl. Phys. B13, 477 (1969), Eq. (4).

[34] J. Danburg, Phys. Rev. D 4, 1325 (1971), which we noticed after this work was done, has similarities to our work, in particular the total width ansatz (2.15). This reference takes the opposite view that barrier factors are responsible for cutting out the high partial waves $J>kR$.

[35] Reference 2, p. 1401.