PHYSICAL REVIEW D VOLUME 6, NUMBER 11 1 DECEMBER 1972

Strong Isotopic-Spin Breaking and Nonleptonic K Decay

Barry R. Holstein*
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
(Received 2 May 1972)

$\Delta I=\frac{3}{2}$ effects in nonleptonic kaon decay predicted by strong isotopic-spin breaking with an exact $\Delta I=\frac{1}{2}$ weak Hamiltonian are compared with those which follow from an intrinsic $\Delta I=\frac{3}{2}$ Hamiltonian.

I. INTRODUCTION

Several authors have argued for the necessity of nonelectromagnetic isotopic-spin breaking in order to explain electromagnetic mass differences, $\eta\to 3\pi$ decay, and other effects.[1] Recently Baće[2] and Goyal and Li[3] have applied these ideas to the nonleptonic K-decay problem. The latter have concluded that strong isospin-breaking terms in conjunction with an exact $\Delta I=\frac{1}{2}$ weak Hamiltonian are insufficient to explain the magnitude of $\Delta I=\frac{3}{2}$ terms as experimentally found in both $K\to 2\pi$ and $K\to 3\pi$ decays,[4] while Baće has suggested that these effects *may* be large enough. However, both calculations are based on definite phenomenological forms for the weak interaction and assumptions concerning certain matrix elements. Also, Goyal and Li use a pole model for the decay which neglects the existence of contact terms required by current algebra and partially conserved axial-vector current (PCAC), which has had considerable success in this realm.[5]

This note attempts to remove some of the model dependence of the previous calculations. We assume the validity of certain commutation relations of generators with the weak Hamiltonian and show that one may relate certain $\Delta I=\frac{1}{2}$ breaking effects in $K\to 3\pi$ decay to those in $K\to 2\pi$ decay without arbitrary assumptions concerning the size of unknown matrix elements. In the next section we explain our procedure and compare theoretical predictions with experiment.

II. STRONG ISOSPIN BREAKING

We assume here that the strong-interaction Hamiltonian is of the form

$$H_{\rm st}=H_0-\epsilon_0u_0-\epsilon_3u_3-\epsilon_8u_8\,, \tag{1}$$

where H_0 is SU(3)×SU(3)-invariant, and the u's represent a $(3,\bar{3})+(\bar{3},3)$ breaking of the symmetry. Analysis of the pseudoscalar masses by Gell-Mann, Oakes, and Renner suggests that $\epsilon_8/\epsilon_0\cong -1.25$,[6] while examination of mass splittings among members of common isotopic multiplets by Cabibbo and Maiani gives $\epsilon_8/\epsilon_3\cong 10^{-2}$.[7] However, for the present we do not assume any particular values for the ϵ_i.

We suppose the nonleptonic weak decays to be produced as a result of a Hamiltonian

$$H_{\rm wk}=\int d^3x\,\mathcal{H}_{\rm wk}(x)\,,$$

with $\mathcal{H}_{\rm wk}$ such that

$$[F_i,\mathcal{H}_{\rm wk}]=[F_i^5,\mathcal{H}_{\rm wk}]\,,\quad i=1,2,3,8\,, \tag{2}$$

where F_i and F_i^5 are respectively the vector and axial-vector generators of SU(3)×SU(3) transformations. Such a relation follows, for example, if the Hamiltonian is constructed as the product of $V+A$ currents. Also we assume the validity of PCAC for the pion field in the form

$$\partial^\mu A_\mu^i=F_\pi m_\pi{}^2\phi^i\,,\quad i=1,2,3\,,$$

and assume that pion amplitudes can be smoothly extrapolated off the mass shell.

The $\Delta I=\frac{3}{2}$ contributions to the nonleptonic decay $A\to B$ come then from two sources: $\langle B|\mathcal{H}_{\rm wk}^{3/2}|A\rangle$ which represents the effect of an intrinsic $\Delta I=\frac{3}{2}$ component of $\mathcal{H}_{\rm wk}$, and the $\Delta I=\frac{3}{2}$ part of

$$i\epsilon_3\int d^4x\langle B|T(u_3(x)\mathcal{H}_{\rm wk}^{1/2}(0))|A\rangle \tag{3}$$

representing the strong isotopic-spin-breaking correction to $\mathcal{H}_{\rm wk}^{1/2}$.[8]

Following the technique of Nambu and Hara,[9] we may now relate the decay $K\to 3\pi$ to amplitudes for $K\to 2\pi$ transitions. We first parametrize the $K\to 2\pi$ and $K\to 3\pi$ amplitudes keeping in the latter terms linear in the energy:

$$\begin{aligned}A(K^+\to\pi^+\pi^0)&=(\tfrac{3}{10})^{1/2}a_3\,,\\ A(K^0\to\pi^+\pi^-)&=(\tfrac{1}{3})^{1/2}a_1+(\tfrac{1}{15})^{1/2}a_3\,,\\ A(K^0\to\pi^0\pi^0)&=-(\tfrac{1}{3})^{1/2}a_1+2(\tfrac{1}{15})^{1/2}a_3\,,\end{aligned} \tag{4}$$

and

$$A(K^+\to\pi^+\pi^0\pi^0)=-\tfrac{1}{3}\sqrt{2}\{\alpha_1-(\tfrac{1}{2})^{1/2}\alpha_3+[\beta_1-(\tfrac{1}{2})^{1/2}\beta_3]E_+\}-\tfrac{3}{2}(\tfrac{1}{5})^{1/2}\gamma_3(E_+-\tfrac{1}{3}M_K),$$
$$A(K^+\to\pi^+\pi^+\pi^-)=\tfrac{1}{3}\sqrt{2}\{2\alpha_1-\sqrt{2}\,\alpha_3+[\beta_1-(\tfrac{1}{2})^{1/2}\beta_3](E_--M_K)\}+\tfrac{3}{2}(\tfrac{1}{5})^{1/2}\gamma_3(E_--\tfrac{1}{3}M_K),$$
$$A(K^0\to\pi^+\pi^-\pi^0)=\tfrac{1}{3}[\alpha_1+\sqrt{2}\,\alpha_3+(\beta_1+\sqrt{2}\,\beta_3)E_0]+(\tfrac{1}{10})^{1/2}\gamma_3(E_--E_+),\tag{5}$$
$$A(K^0\to\pi^0\pi^0\pi^0)=-\tfrac{1}{3}[3\alpha_1+3\sqrt{2}\,\alpha_3+(\beta_1+\sqrt{2}\,\beta_3)M_K].$$

Here $E_{\pm,0}$ represents the energy of the corresponding pion, and a_i, α_i, β_i, γ_i are constants, $i=1,3$ referring to the $\Delta I=\frac{1}{2},\frac{3}{2}$ contribution to the decay amplitude. Now since

$$\lim_{q_c\to 0}\langle\pi^a_{q_a}\pi^b_{q_b}\pi^c_{q_c}|\mathcal{H}^i_{\rm wk}(0)|K^n_k\rangle=-\frac{i}{F_\pi}\langle\pi^a_{q_a}\pi^b_{q_b}|[F^c(0),\mathcal{H}^i_{\rm wk}(0)]|K^n_k\rangle,\tag{6a}$$

$$\lim_{q_c\to 0}\int d^4x\langle\pi^a_{q_a}\pi^b_{q_b}\pi^c_{q_c}|T(u_3(x)\mathcal{H}^{1/2}_{\rm wk}(0))|K^n_k\rangle$$
$$=-\frac{i}{F_\pi}\int d^4x\langle\pi^a_{q_a}\pi^b_{q_b}|T(u_3(x)[F^c(0),\mathcal{H}^{1/2}_{\rm wk}(0)])|K^n_k\rangle-\frac{1}{\sqrt{3}\,F_\pi}\delta^{c3}\int d^4x\langle\pi^a_{q_a}\pi^b_{q_b}|T([\sqrt{2}\,v_0(x)+v_8(x)]\mathcal{H}^{1/2}_{\rm wk}(0))|K^n_k\rangle\tag{6b}$$

with $v_i(x)$ being the pseudoscalar densities,[6] we can relate the $K\to 3\pi$ amplitudes to $K\to 2\pi$ amplitudes, where energy-momentum conservation is retained in both decays.

We define

$$\int d^4x\langle\pi^+\pi^0|T(u_3(x)\mathcal{H}^{1/2}_{\rm wk}(0))|K^+\rangle=(\tfrac{3}{10})^{1/2}g_3,\tag{7}$$

$$\int d^4x\langle\pi^+\pi^-|T(u_3(x)\mathcal{H}^{1/2}_{\rm wk}(0))|K^0\rangle=(\tfrac{1}{3})^{1/2}g_1+(\tfrac{1}{15})^{1/2}g_3,$$
$$\int d^4x\langle\pi^0\pi^0|T(u_3(x)\mathcal{H}^{1/2}_{\rm wk}(0))|K^0\rangle=-(\tfrac{1}{3})^{1/2}g_1+2(\tfrac{1}{15})^{1/2}g_3,\tag{8}$$
$$-\frac{i}{\sqrt{3}}\int d^4x\langle\pi^a\pi^b|T([\sqrt{2}\,v_0(x)+v_8(x)]\mathcal{H}^{1/2}_{\rm wk}(0))|K^n\rangle=-(\tfrac{1}{3})^{1/2}h_1\delta^{ab}\delta^{n0}.$$

Then we find

$$\alpha_3=-i\tfrac{1}{2}(\tfrac{3}{10})^{1/2}\frac{f_3}{F_\pi}+\epsilon_3\left[(\tfrac{2}{3})^{1/2}\frac{h_1+g_1}{F_\pi}+(\tfrac{3}{10})^{1/2}\frac{g_3}{6F_\pi}\right],$$
$$\beta_3M_K=i\,\tfrac{5}{2}\,(\tfrac{3}{10})^{1/2}\frac{f_3}{F_\pi}+\epsilon_3\left[-2(\tfrac{2}{3})^{1/2}\frac{h_1+g_1}{F_\pi}-(\tfrac{3}{10})^{1/2}\frac{5g_3}{6F_\pi}\right],\tag{9}$$
$$(\tfrac{1}{10})^{1/2}\gamma_3M_K=i\,\tfrac{3}{2}\,(\tfrac{3}{5})^{1/2}\frac{f_3}{F_\pi}-\epsilon_3(\tfrac{3}{5})^{1/2}\frac{g_3}{2F_\pi},$$

where $f_3=(\frac{10}{3})^{1/2}\langle\pi^+\pi^0|\mathcal{H}^{3/2}_{\rm wk}|K^+\rangle$. The dominant contributions to α_1 and β_1 come, of course, from $\langle\pi^a\pi^b\pi^c|\mathcal{H}^{1/2}_{\rm wk}|K^n\rangle$, and we find[5]

$$\alpha_1\cong -i\frac{\sqrt{3}}{2}\frac{f_1}{F_\pi},\quad \beta_1M_K\cong i\sqrt{3}\,\frac{f_1}{F_\pi},\tag{10}$$

where $f_1=-\sqrt{3}\langle\pi^0\pi^0|\mathcal{H}^{1/2}_{\rm wk}(0)|K^0\rangle$.

Parametrizing the $K\to 3\pi$ amplitudes

$$\mathfrak{A}(K\to 3\pi)=A\left(1+\lambda 2M_K\frac{E_0-\frac{1}{3}M_K}{m_\pi^{\;2}}\right),\tag{11}$$

where E_0 is the energy of the "odd" pion, A is the average amplitude, and λ is the slope of the Dalitz plot, we find as tests for $\Delta I=\frac{3}{2}$ terms

$$-\tfrac{1}{2}\frac{\lambda_{+00}}{\lambda_{++-}}=1+9(\tfrac{1}{10})^{1/2}\frac{\gamma_3}{\beta_1}\cong 1+\tfrac{27}{2}(\tfrac{1}{5})^{1/2}\frac{f_3}{f_1}+\tfrac{9}{2}(\tfrac{1}{5})^{1/2}i\epsilon_3\frac{g_3}{f_1},$$

$$\frac{\lambda_{+-0}}{\lambda_{+00}}=1+\frac{3}{\sqrt{2}}\frac{\beta_3}{\beta_1}-\tfrac{9}{2}(\tfrac{1}{10})^{1/2}\frac{\gamma_3}{\beta_1}-\frac{3}{\sqrt{2}}\frac{\alpha_3+\beta_3\frac{1}{3}M_K}{\alpha_1+\beta_1\frac{1}{3}M_K}\cong 1\,, \tag{12}$$

$$2\frac{A_{+-0}}{A_{++-}}=1+\frac{3}{\sqrt{2}}\frac{\alpha_3+\beta_3\frac{1}{3}M_K}{\alpha_1+\beta_1\frac{1}{3}M_K}\cong 1-\frac{3}{\sqrt{5}}\frac{f_3}{f_1}+2i\epsilon_3\frac{h_1+g_1}{f_1}-i\epsilon_3(\tfrac{1}{5})^{1/2}\frac{g_3}{f_1}\,,$$

and for the ratio of $\Delta I=\frac{3}{2}$ to $\Delta I=\frac{1}{2}$ amplitudes in $K\to 2\pi$ we predict

$$\frac{a_3}{a_1}=\frac{f_3}{f_1}+i\epsilon_3\frac{g_3}{f_1}\,. \tag{13}$$

Experimentally we have

$$(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1}=0.032\pm0.001\,,\quad \frac{\lambda_{+-0}}{\lambda_{+00}}=0.85\pm0.09\,,\quad -\tfrac{1}{2}\frac{\lambda_{+00}}{\lambda_{++-}}=1.34\pm0.20\,,\quad 2\frac{A_{+-0}}{A_{++-}}=0.91\pm0.02\,. \tag{14}$$

Now previous workers have, in their particular models, argued about the size of ϵ_3/ϵ_8 and of matrix elements such as $\langle\pi^0|u_3|\eta\rangle$ or $\langle\pi\eta|\mathcal{H}_{\rm wk}|K\rangle$ in order to evaluate the credibility of the model. Our procedure can avoid some of the uncertainty, at least as regards the slope predictions. We find that if the only $\Delta I=\frac{3}{2}$ terms are a result of strong isospin breaking

$$-\tfrac{1}{2}\frac{\lambda_{+00}}{\lambda_{++-}}=1+\tfrac{9}{2}(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1}\cong 1.15\,, \tag{15}$$

while if the $\Delta I=\frac{3}{2}$ components emanate purely from an intrinsic $\Delta I=\frac{3}{2}$ term in the Hamiltonian we have

$$-\tfrac{1}{2}\frac{\lambda_{+00}}{\lambda_{++-}}=1+\tfrac{27}{2}(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1}\cong 1.45\,. \tag{16}$$

Unfortunately both models predict $\lambda_{+-0}/\lambda_{+00}=1$.

In order to go further, we note that if we assume single-particle dominance of $\sqrt{2}\,v_0+v_8$ we have[10]

$$h_1\cong -i\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}\,\epsilon_0-\epsilon_8}f_1\,. \tag{17}$$

Thus we predict that if $\Delta I=\frac{3}{2}$ terms are due to strong isospin breaking

$$2\frac{A_{+-0}}{A_{++-}}=1-3(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1}+\frac{\sqrt{3}\,\epsilon_3}{\sqrt{2}\,\epsilon_0-\epsilon_8}=0.91+\frac{\sqrt{3}\,\epsilon_3}{\sqrt{2}\,\epsilon_0-\epsilon_8}\,, \tag{18}$$

where we have used the tree-graph result $g_1=-(\frac{1}{5})^{1/2}g_3$ [cf. Eq. (20)], while if an intrinsic $\Delta I=\frac{3}{2}$ component is the cause

$$2\frac{A_{+-0}}{A_{++-}}=1-3(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1}=0.91\,. \tag{19}$$

TABLE I. Values of parameters violating the $\Delta I=\frac{1}{2}$ rule in various models.

	Expt.[a]	$\frac{\epsilon_3}{\epsilon_0}\cong-\frac{1}{18}$ [b]	$\frac{\epsilon_3}{\epsilon_0}\cong-\frac{1}{40}$ [c]	$\frac{\epsilon_3}{\epsilon_0}\cong-\frac{1}{137}$ [d]	$\mathcal{H}_{\rm wk}^{3/2}$
$-\frac{1}{2}\frac{\lambda_{+00}}{\lambda_{++-}}$	1.34 ± 0.20	1.15	1.15	1.15	1.45
$\frac{\lambda_{+00}}{\lambda_{+-0}}$	1.15 ± 0.09	1.00	1.00	1.00	1.00
$2\frac{A_{+-0}}{A_{++-}}$	0.91 ± 0.02	0.87	0.89	0.91	0.91

[a] B. Aubert, Ref. 4.
[b] Value suggested by R. Oakes, Ref. 1.
[c] Value suggested by N. Cabibbo and L. Maiani, Ref. 1.
[d] Value suggested by R. Gatto, G. Sartori, and M. Tonin, Ref. 1.

Our predictions are summarized in Table I. We see that the experimental slope data do not rule out either model. More accurate data on the τ slope would enable us to make a more definite statement. The discrimination as far as the amplitudes are concerned depends on ϵ_3/ϵ_0 and ϵ_8/ϵ_0. If we assume $\epsilon_8/\epsilon_0 \cong -1.25$ as suggested by Gell-Mann, Oakes, and Renner then for the values of ϵ_8/ϵ_0 suggested by various authors we find the results in Table I. No model is clearly favored or disfavored by the amplitude data.

If we are willing to abandon model independence, we may make specific numerical predictions. We work in a tree-graph approximation so that

$$\int d^4x \langle \pi^a\pi^b | T(u_3(x)\mathcal{H}_{\rm wk}^{1/2}(0)) | K^n\rangle = \delta^{a3}\langle \pi^a|u_3|\eta\rangle \frac{i}{m_\pi^{\,2}-m_\eta^{\,2}}\langle \pi^b\eta|\mathcal{H}_{\rm wk}^{1/2}|K^n\rangle + \delta^{b3}\langle \pi^b|u_3|\eta\rangle \frac{i}{m_\pi^{\,2}-m_\eta^{\,2}}\langle \pi^a\eta|\mathcal{H}_{\rm wk}^{1/2}|K^n\rangle ; \tag{20}$$

then

$$g_3 = -\sqrt{5}\, g_1 = i(\tfrac{10}{3})^{1/2}\frac{\langle \pi^0|u_3|\eta\rangle\langle \eta\pi^+|\mathcal{H}_{\rm wk}^{1/2}|K^+\rangle}{m_\pi^{\,2}-m_\eta^{\,2}} . \tag{21}$$

The matrix element is

$$\langle \pi^0|u_3|\eta\rangle = -\tfrac{2}{3}\frac{\epsilon_0}{\epsilon_8}(m_\pi^{\,2}-m_K^{\,2})$$

in the model of Gell-Mann, Oakes, and Renner, while for $\langle \pi^+\eta|\mathcal{H}_{\rm wk}^{1/2}|K^+\rangle$ several models yield[11]

$$\frac{\langle \pi^+\eta|\mathcal{H}_{\rm wk}^{1/2}|K^+\rangle}{\langle \pi^+\pi^-|\mathcal{H}_{\rm wk}^{1/2}|K^0\rangle} = (\tfrac{2}{3})^{1/2} .$$

Hence we predict

$$(\tfrac{3}{10})^{1/2}\frac{g_3}{f_1} = \frac{i}{6\epsilon_0}\left(1+\tfrac{3}{2}\frac{\eta}{1-\eta}\right), \tag{22}$$

where $\eta = m_\pi^{\,2}/M_K^{\,2}$, so that if $I=\frac{3}{2}$ terms are due to strong isospin breaking,

$$(\tfrac{1}{5})^{1/2}\frac{a_3}{a_1} = -i\epsilon_3\frac{g_3}{\sqrt{5}f_1} = \frac{\epsilon_3/\epsilon_0}{6}(\tfrac{2}{3})^{1/2}\left(1+\tfrac{3}{2}\frac{\eta}{1-\eta}\right), \tag{23}$$

which is, even for the large value of ϵ_3/ϵ_0 suggested by Oakes, rather too small to explain the experimental $\Delta I=\frac{1}{2}$ breaking parameters in $K\to 2\pi$ or $K\to 3\pi$. However, this depends critically on the assumed values for $\langle \pi^0|u_3|\eta\rangle\langle \pi^+\eta|\mathcal{H}_{\rm wk}^{1/2}|K^+\rangle$.

We have attempted to show that even without knowledge of the size of these matrix elements or of the specific form of the weak Hamiltonian, one may be able to distinguish the consequences of an intrinsic $\Delta I=\frac{3}{2}$ interaction from strong isospin breaking.

---

*Work supported in part by the National Science Foundation.

[1]N. Cabibbo and L. Maiani, Phys. Letters 28B, 131 (1968); R. Gatto, G. Sartori, and M. Tonin, *ibid.* 28B, 128 (1968); R. Oakes, *ibid.* 29B, 683 (1969).

[2]M. Bače, Phys. Rev. D 4, 2838 (1971).

[3]A. Goyal and L. F. Li, Phys. Rev. D 4, 2012 (1971).

[4]B. Aubert, in *Topical Conference on Weak Interactions, CERN, 1969* (CERN, Geneva, 1969), p. 205.

[5]See, e.g., B. R. Holstein, Phys. Rev. 183, 1228 (1969), and references therein.

[6]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[7]N. Cabibbo and L. Maiani, in *Evolution in Particle Physics*, edited by N. Conversi (Academic, New York, 1970).

[8]There are two types of contributions from this term. One involves the mass renormalization of the K mesons arising from u_3. However, this should be combined with the $\Delta I=\frac{3}{2}$ terms arising from the electromagnetic mass shift. Since we do not know how to handle the latter, we neglect such terms and concentrate on other $\Delta I=\frac{3}{2}$ effects arising from the possibility of the transitions $\eta\to\pi$, $\eta\to 3\pi$, etc. involving u_3.

[9]Y. Nambu and Y. Hara, Phys. Rev. Letters 16, 87 (1966).

[10]We assume, as consistent with Ref. 6, a very massive η' and no η-η' mixing, so that the η' contribution to these decays may be neglected. $\sqrt{2}v_0+v_8$ represents then the emission of a zero four-momentum η.

[11]This result is found, e.g., in the pole model of J. J. Sakurai, Phys. Rev. 156, 1508 (1967), or in the phenomenological weak Hamiltonian due to J. A. Cronin, *ibid.* 161, 1483 (1967).