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# Pion-Pion Scattering Based on Current Algebra, Analyticity, and Unitarity*

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Using SU(2) × SU(2) current algebra and pion-pole dominance, we derive from the Ward identities an exact crossing-symmetric expression for the $\pi\pi$ scattering amplitude. We make approximations which are suitable at low energy for those three- and four-point functions of the problem which cannot be determined from the constraints of current algebra. We parametrize these functions in terms of propagators and polynomials exhibiting the correct analyticity properties. Form factors, analytic in the cut plane, are expressed in effective-range form, and the $s$- and $p$-wave amplitudes are constructed in terms of them. The existence of resonances in the $\pi\pi$ system is not assumed, and soft-pion estimates are not used. Instead all the parameters are free to be varied. We determine all the free parameters of the problem self-consistently by imposing the constraints that follow from elastic unitarity. The scheme predicts all the features of low-energy $\pi\pi$ scattering, the only input parameters being $m_\pi$ and $F_\pi$, the pion mass and decay constant. Among our principal results are the $s$- and $p$-wave scattering lengths, the corresponding phase shifts, and the determination of an important parameter which measures the isospin $T=2$ component of the $\sigma$ commutator, $\sigma^{ab}$. The details of the method predispose scattering lengths to be small. We find that unitarity prefers the $T=2$ component of $\sigma^{ab}$ to be small relative to the $T=0$ component. As a consequence, our scattering lengths are in excellent agreement with those obtained by Weinberg. The $T=J=1$ phase shift exhibits a $\rho$ resonance around 915 MeV with a width of 210 MeV. The $T=2$, $J=0$ phase shift is small and in agreement with experimental results. The $T=J=0$ phase shift displays acceptable behavior at low energy; we offer physical arguments to say that its higher-energy behavior is less reliable than that of the $p$ wave at the same energies. We discuss our results and analyze the predictive power of the method presented. Finally, we suggest some improvements on our calculations, including possible applications to related problems.

## I. INTRODUCTION

For more than a decade, the problem of determining the amplitude for $\pi\pi$ scattering has presented a challenge for theoretical physics to solve. In the absence of a fully developed theory of hadrons, an ultimate solution continues to be an overly ambitious goal. Many approaches to an approximate solution have evolved, and contributed to the unfolding of several features of the problem. The principles of $S$-matrix theory (including Lorentz invariance, analyticity, unitarity, and crossing symmetry) are cornerstones of hadron dynamics[1] and have long been advocated as the means by which a self-consistent solution to the $\pi\pi$ problem may be found. If hadronic theory is to include, in addition, the content of the algebra of vector and axial-vector currents,[2] then any treatments based purely on $S$-matrix theory are to be viewed as part of the prehistory of the problem. The low-energy pion scattering theorems[3] predicted on the basis of partial conservation of axial-vector current (PCAC) and current algebra are results which pure $S$-matrix theory is at a loss to confirm. On the other hand, to base the formulation of $\pi\pi$ dynamics jointly on current algebra and on $S$-matrix methods would be to prescribe a scheme more powerful than either is separately. An approach of this kind has recently been adopted by Schnitzer[4]; our objective in this work is to provide a thorough analysis of a similar method.

The hard-pion techniques of Schnitzer and Weinberg[5] provide the basic sort of framework for this analysis. In this approach the constraints on the off-shell amplitudes which follow from equal-time current algebra are summarized in a collection of Ward identities. These authors studied the three-point functions of the $A_1\rho\pi$ system and solved for the mass-shell amplitudes in meson-pole-dominated (tree) form. The procedure was extended to four-point functions by Gerstein and Schnitzer.[6]

The system of Ward identities also admits the application of methods for extending the hard-pion analysis of three-point functions beyond the approximation of pole dominance in the squared momentum-transfer variable $t$. The technique[7] treats the cut-plane analyticity of the form factor $f(t)$ and the propagator $\Delta(t)$, and has been applied to the $A_1\rho\pi$ system[7,8] and to the $A_1\sigma\pi$ system.[9-11] The scheme of Ward identities, augmented by the smoothness approximation,[5] leads to an expression in which $\mathrm{Im} f^{-1}$ is a known function for $4m_\pi^2 \leq t \leq 16m_\pi^2$ (the $\pi\pi$ region). If $f(0)$ is given, then an effective-range solution results in which $f$ and $\Delta$ are analytic functions with the $\pi\pi$ cut. The phase of $f$ gives the corresponding $\pi\pi$ phase shift in this interval and the resulting partial-wave amplitude $T(t)$ is unitary there. As a method for calculating $T(t)$, the procedure is incomplete. The partial wave is given by an effective-range formula with one parameter, an effective range, which must be regarded as input. Moreover, the implications of crossing symmetry for the full $\pi\pi$ amplitude $T(s,t)$ are not maintained in that, as an effective-range result, $T(t)$ has no left-hand cuts. Thus the $s$- and $p$-wave amplitudes obtained in this way from the three-point functions are provisional results, and to go beyond them we must turn to an analysis of the four-point functions.

When the Ward-identity procedure for the four-point functions of the $\pi\pi$ problem is carried out,[6] a crossing-symmetric amplitude for $T(s,t)$ results, involving the form factors ($f$'s) and propagators ($\Delta$'s) as functions of each of the channel variables $s$, $t$, and $u$. The partial waves $T(s)$ are projected and the objective is to impose the unitarity constraints,

$$\mathrm{Im}T(s) = \rho(s)\,|T(s)|^2\,,$$

$$\mathrm{Im}f(s) = \rho(s) f^*(s) T(s)\,,$$

$$\mathrm{Im}\Delta(s) = \rho(s)\,|f(s)|^2\,,$$

valid in the $\pi\pi$ region ($\rho$ is the $\pi\pi$ phase-space factor). These amount to constraints on the parameters which include the effective ranges of the three-point function problem. The unitarity constraints can be satisfied only approximately. A thorough evaluation of the solution of this problem is presented. It is evident that this approach permits a self-consistent determination of all the parameters which, at the level of the three-point functions, had to be among the input. Thus, to advance to the level of the four-point functions and impose unitarity is to achieve a closure of the parametrization of the $\pi\pi$ problem. Of course, the only phase shifts which can be obtained in this way are those whose quantum numbers correspond to local operators which have been explicitly introduced in the theory. For our purposes these are the $s$- and $p$-wave phase shifts.

The results also include the other features of interest in the low-energy $\pi\pi$ problem. In particular the method yields the pion electromagnetic form factor and the pion-to-pion matrix element of the $\sigma$ commutator. The latter operator is of considerable concern to us. For complete generality we permit it to have both isospin-0 and -2 components. There is a parameter in the formalism which serves to give a measure of the isospin-2 part. It is remarkable that the unitarity constraints force this parameter to correspond to an isoscalar-dominant $\sigma$ commutator. As a consequence, scattering lengths like those obtained by Weinberg[3] are among the results. It should be noted in advance that, as in Ref. 4, the method does not allow for strong left cuts of the partial waves and therefore is predisposed to give small scattering lengths.

We emphasize that, while the procedure is based in part on analyticity and unitarity, it does not employ partial-wave dispersion relations. The Ward identities themselves provide relations among analytic functions and permit us to implement analyticity locally rather than over the whole cut plane. The program is one of imposing, simultaneously, the constraints of current algebra and of unitarity along lines of the sort recently advanced by Schnitzer.[4] The program is thoroughly analyzed to establish the extent of its predictive power. We depart from Schnitzer in several instances, especially in the treatment of quantities related to the $\sigma$ commutator. No assumptions are made about the occurrence of $\rho$ or $\sigma$ resonances in the $\pi\pi$ system; these are to be in the output of the problem, in all three channels symmetrically, if the scheme predicts them. No use is made of low-energy results as input data. The only input parameters are $m_\pi$ and $F_\pi$, the pion mass and decay constant.

In Sec. II the techniques of hard-pion current algebra are used to formulate the $\pi\pi$ problem. The most laborious details of the construction have been relegated to the appendixes. A smoothness hypothesis is employed and the $s$- and $p$-wave amplitudes are projected. The form taken by the unitarity relation is given. In Sec. III we show how the constraints of analyticity and elastic unitarity are used to determine all the parameters of the problem in a self-consistent way. In Sec. IV we present our results and in Sec. V we conclude with some observations and possible future applications of the scheme presented.

## II. $\pi\pi$ AMPLITUDE AND UNITARITY

The hard-pion methods of Refs. 5, 6, and 12 provide the techniques which we utilize to obtain our starting point. The structure of the current-algebra amplitude for the process $\pi_a(q_1)+\pi_b(q_2)\to\pi_c(q_3)+\pi_d(q_4)$ has been developed in Appendix B wherein we have derived the Ward identities for the four-point functions of the axial-vector currents. The exact crossing-symmetric expression for the $\pi\pi$ scattering amplitude on the mass-shell is given by Eq. (B6) when we take $q_i^2=-m_\pi^2$ ($i=1$ to 4):

$$\begin{aligned} F_\pi^4 T^{abcd}(s,t) = & [q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)]_{q_i^2=-m_\pi^2} \\ & -(2F_\pi^2 f^{abcd}(s)+\Delta^{abcd}(s)+F_\pi^2(\tfrac{5}{2}l_2-m_\pi^2+2s)\delta_{ab}\delta_{cd}+\epsilon_{abe}\epsilon_{cde}(u-t)\{F_\pi^2F(s)-\tfrac{1}{4}[\Delta_V(s)-C_V]\}) \\ & -(2F_\pi^2 f^{acbd}(t)+\Delta^{acbd}(t)+F_\pi^2(\tfrac{5}{2}l_2-m_\pi^2+2t)\delta_{ac}\delta_{bd}+\epsilon_{ace}\epsilon_{bde}(u-s)\{F_\pi^2F(t)-\tfrac{1}{4}[\Delta_V(t)-C_V]\}) \\ & -(2F_\pi^2 f^{adcb}(u)+\Delta^{adcb}(u)+F_\pi^2(\tfrac{5}{2}l_2-m_\pi^2+2u)\delta_{ad}\delta_{bc}+\epsilon_{ade}\epsilon_{cbe}(s-t)\{F_\pi^2F(u)-\tfrac{1}{4}[\Delta_V(u)-C_V]\}). \end{aligned} \tag{1}$$

All quantities appearing in (1) have been defined in the appendixes. In summary: $T^{abcd}(s,t)$ is the on-shell $\pi\pi$ amplitude; $T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)$ is the part of the four-point function of axial-vector currents which is free of vacuum contributions and of pion poles in the mass variables $q_i^2$; $f^{abcd}(s)$ and $\Delta^{abcd}(s)$ are the form factor and propagator of the $\sigma$ commutator (in the notation of Ref. 11); $F(s)$ and $\Delta_V(s)$ are the form factor and propagator of the vector current (in the notation of Ref. 7); $C_V=\Delta_V(0)$. The $\sigma$ commutator, defined by

$$\sigma^{ab}(0)\delta(x)=[A_4^a(x),\partial_\nu A_\nu^b(0)]\delta(x_0), \tag{2a}$$

has in general both isoscalar ($T=0$) and isotensor ($T=2$) components. Therefore, its pion-to-pion matrix element has two form factors $f_T(s)$, $T=0$ and 2; correspondingly, there are two propagators $\Delta_T(s)$, $T=0$ and 2. The important parameter $l_2$ appears in the coupling of the operator $\sigma^{abc}(0)$ to the pion:

$$\sigma^{abc}(0)\delta(x)=-[A_4^a(x),\sigma^{bc}(0)]\delta(x_0), \tag{2b}$$

$$(2\omega_q)^{1/2}\langle\pi(qe)|\sigma^{abc}(0)|0\rangle=F_\pi\sum_{T=0,2}l_T P^T_{eabc}, \tag{2c}$$

in which $P^T$ is the isospin projection operator. The parameters $l_0$ and $l_2$ satisfy[11]

$$2l_0-5l_2=6m_\pi^2. \tag{2d}$$

$l_2$ serves as a measure of the isospin-2 part of $\sigma^{ab}$. Thus it also provides a measure of the departure of the calculated scattering lengths from those obtained by Weinberg.[3] An important feature of this project is that $l_2$ is determinable as a result of applying the unitarity constraints.

Our knowledge of the quantity $T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)$ is quite limited. To proceed we adopt a smoothness hypothesis analogous to that of Ref. 5 in order to express it in terms of propagator functions and appropriately parametrized coupling polynomials. This is the following ansatz:

$$T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)=\Delta^A_{\mu\alpha}(q_1)\Delta^A_{\nu\beta}(q_2)\Delta^A_{\lambda\theta}(q_3)\Delta^A_{\sigma\eta}(q_4)\hat{T}^{abcd}_{\alpha\beta\theta\eta}(q_3;q_1,q_2), \tag{3a}$$

where

$$\begin{aligned} \hat{T}^{abcd}_{\alpha\beta\theta\eta}(q_3;q_1,q_2)= & \epsilon_{abe}\epsilon_{dce}\Gamma^V_{\alpha\beta\gamma}(-q_1,q_2)\Delta^V_{\gamma\delta}(q_1+q_2)\Gamma^V_{\theta\eta\delta}(q_3,-q_4) \\ & +\Gamma^{abij}_{\alpha\beta}(-q_1,q_2)\Delta^{ijkl}(s)\Gamma^{cdkl}_{\theta\eta}(q_3,-q_4)+\delta_{ab}\delta_{cd}\tau_{\alpha\beta\theta\eta} \\ & +\epsilon_{ace}\epsilon_{dbe}\Gamma^V_{\alpha\theta\gamma}(-q_1,-q_3)\Delta^V_{\gamma\delta}(q_1-q_3)\Gamma^V_{\beta\eta\delta}(-q_2,-q_4) \\ & +\Gamma^{acij}_{\alpha\theta}(-q_1,-q_3)\Delta^{ijkl}(t)\Gamma^{bdkl}_{\beta\eta}(-q_2,-q_4)+\delta_{ac}\delta_{bd}\tau_{\alpha\theta\beta\eta} \\ & +\epsilon_{ade}\epsilon_{bce}\Gamma^V_{\alpha\eta\gamma}(-q_1,-q_4)\Delta^V_{\gamma\delta}(q_1-q_4)\Gamma^V_{\theta\beta\delta}(q_3,q_2) \\ & +\Gamma^{adij}_{\alpha\eta}(-q_1,-q_4)\Delta^{ijkl}(u)\Gamma^{cbkl}_{\theta\beta}(q_3,q_2)+\delta_{ad}\delta_{cb}\tau_{\alpha\eta\theta\beta}. \end{aligned} \tag{3b}$$

In (3b) the vertex factors $\Gamma^V_{\alpha\beta\gamma}$ and $\Gamma^{abij}_{\alpha\beta}$ are constructed to be as smooth as possible in their momenta. This is achieved by giving them minimal momentum dependence:

$$C_A{}^2\Gamma^V_{\alpha\beta\gamma}(-q_1, q_2) = \gamma_1\delta_{\alpha\beta}(q_2 - q_1)_\gamma + \gamma_2[\delta_{\alpha\gamma}(q_2+q_1)_\beta - \delta_{\beta\gamma}(q_2+q_1)_\alpha] + \gamma_3(\delta_{\alpha\gamma}q_{2\beta} - \delta_{\beta\gamma}q_{1\alpha}), \tag{3c}$$

$$C_A{}^2\Gamma^{abij}_{\alpha\beta}(-q_1, q_2) = \delta_{\alpha\beta}\sum_{T=0,2}\zeta_T P^T_{abij}. \tag{3d}$$

The construction endows $T^{abcd}_{\mu\nu\lambda\sigma}$ with the cuts which correspond to sigma and to vector propagation. Any residual structure may be taken to be of polynomial form for the purposes of a low-energy analysis. Thus the remaining terms of (3b) with minimal momentum dependence are

$$C_A{}^4\tau_{\alpha\beta\theta\eta} = F_\pi{}^2[\eta_1\delta_{\alpha\beta}\delta_{\theta\eta} + \eta_2(\delta_{\alpha\theta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\theta})]. \tag{3e}$$

In these equations $\gamma_{1,2,3}$, $\zeta_T$ $(T=0,2)$, and $\eta_{1,2}$ are constants. The parametrization is the most general, consistent with crossing symmetry. The constants which appear are among the set of parameters to be determined by application of the unitarity constraints. It will be shown that these constraints relate the coupling parameters of Eqs. (3) to the analogous constants appearing as coupling parameters in the three-point functions.

The smoothness ansatz of Eqs. (3) for the four-point amplitude and of Eqs. (A7) and (A17) for the three-point amplitudes represents the major source of model dependence in this treatment of $\pi\pi$ scattering. The construction specifies the amplitudes to have the relevant cuts, and prescribes the discontinuities on the cuts to be simply proportional to the discontinuities of less complex analytic functions, the propagators. We note that, for the $\pi\pi$ amplitude, Eqs. (1) and (3) imply a suppression of features which correspond to double-spectral functions. This is certainly consistent with our viewpoint of implementing analyticity only locally, and it limits the validity of the calculation to the low-energy region.

When we contract $q_{1\mu}$, $q_{2\nu}$, $q_{3\lambda}$, and $q_{4\sigma}$ into Eq. (3a), use Eqs. (3b)–(3e), and go to the pion mass shell, we obtain

$$\begin{aligned}[q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}T^{abcd}_{\mu\nu\lambda\sigma}(q_3; q_1, q_2)]_{q_i^2=-m_\pi^2} &= \epsilon_{abe}\epsilon_{cde}(t-u)\Delta_V(s)h_1(s) + \sum_{T=0,2}h_T(s)\Delta_T(s)P^T_{abcd} + F_\pi{}^2\delta_{ab}\delta_{cd}\,\Xi(s,t,u)\\ &+ \epsilon_{ace}\epsilon_{bde}(s-u)\Delta_V(t)h_1(t) + \sum_{T=0,2}h_T(t)\Delta_T(t)P^T_{acbd} + F_\pi{}^2\delta_{ac}\delta_{bd}\,\Xi(t,s,u)\\ &+ \epsilon_{ade}\epsilon_{cbe}(t-s)\Delta_V(u)h_1(u) + \sum_{T=0,2}h_T(u)\Delta_T(u)P^T_{adcb} + F_\pi{}^2\delta_{ad}\delta_{bc}\,\Xi(u,t,s),\end{aligned} \tag{4}$$

where

$$h_1(s) = (\tfrac{1}{2}\gamma_{12}s - \gamma_{13}m_\pi{}^2)^2,$$

$$h_T(s) = \zeta_T{}^2 g(s), \quad T=0 \text{ and } 2,$$

$$\Xi(s,t,u) = \eta_1 g(s) + \eta_2[g(t)+g(u)],$$

$$g(s) = (\tfrac{1}{2}s - m_\pi{}^2)^2,$$

$$\gamma_{12} = \gamma_1 - \gamma_2 \text{ and } \gamma_{13} = \gamma_1 + \gamma_3.$$

Expression (4) becomes part of the $\pi\pi$ amplitude, via Eq. (1). The polynomial growth which is apparent in (4), particularly in the $\Xi$ terms, reminds us again that our construction is not expected to apply outside of the low-energy regime.

At this juncture the $\pi\pi$ amplitude (1) has assumed the form of a crossing-symmetric expression involving form factors, propagators, and polynomials. As such, it is a relationship among analytic functions which exhibit all the low-lying cuts. Furthermore, the form factors and propagators themselves can be related one to the other by application of similar techniques. The Ward identities for the three-point functions give equations relating the form factors $F(s)$ and $f_T(s)$ to the corresponding propagator functions $\Delta_V(s)$ and $\Delta_T(s)$, respectively. The derivation of these equations has been sketched in Appendix A; the Ward identities are

$$f_T(s) + l_T = [\Gamma_T(s - 2m_\pi{}^2) - 1]\Delta_T(s)/F_\pi{}^2 \tag{5}$$

and

$$F(s) = [2F_\pi{}^2 - C_V + (1+\Gamma s)\Delta_V(s)]/2F_\pi{}^2. \tag{6}$$

If we rewrite (5) as

$$f_T(s) + l_T = \Gamma_T(s - \alpha_T)\Delta_T(s)/F_\pi{}^2, \tag{7}$$

where

$$\alpha_T = 2m_\pi{}^2 + \Gamma_T{}^{-1},$$

then (6) and (7) imply that we must have

$$F(-\Gamma^{-1}) = 1 - C_V/2F_\pi{}^2 \equiv F_\Gamma \tag{8}$$

and

$$f_T(\alpha_T) = -l_T. \tag{9}$$

We can now realize an expression for $T^{abcd}(s,t)$ which is given entirely in terms of form factors or in terms of propagators.

The next step is to consider the requirements of elastic unitarity. The unitarity constraints are best implemented in terms of partial-wave amplitudes. The $\pi\pi$ scattering amplitude $T^{abcd}(s,t)$ can be decomposed in partial waves as

$$T^{abcd}(s,t)=\frac{16\pi W}{k}\sum_{lT}(2l+1)\exp(i\delta_{lT})\times\sin\delta_{lT}P_l(\cos\theta)P^T_{abcd},$$

where $\delta_{lT}$ is the phase shift and $W$, $k$, and $\theta$ are related to the Mandelstam variables $s$, $t$, and $u$ in the c.m. frame:

$$s=-(q_1+q_2)^2=W^2=4(m_\pi{}^2+k^2),$$
$$t=-(q_1-q_3)^2=-2k^2(1-\cos\theta),$$
$$u=-(q_1-q_4)^2=-2k^2(1+\cos\theta).$$

The partial-wave amplitude $T_{lT}$ is defined by

$$T_{lT}=\exp(i\delta_{lT})\sin\delta_{lT}/\rho_l,$$

where $\rho_l$ is the phase-space factor. At low energy only the $s$- and $p$-waves are important and for these $\rho_0=k/16\pi W$ and $\rho_1=k^3/6\pi W$. The unitarity relations in the elastic region $(4m_\pi{}^2\leqslant s\leqslant 16m_\pi{}^2)$ are

$$\mathrm{Im}T_{lT}=\rho_l|T_{lT}|^2 \quad (10)$$

for the partial waves, and

$$\mathrm{Im}f_T=\rho_0f_T^*T_{0T}\ \text{ and }\ \mathrm{Im}\Delta_T=\rho_0|f_T|^2 \quad (11)$$

as well as

$$\mathrm{Im}F=\rho_1F^*T_{11}\ \text{ and }\ \mathrm{Im}\Delta_V=\rho_1|F|^2 \quad (12)$$

for the form factors and propagators. Equations (6) and (7) then also imply that

$$\mathrm{Im}f_T=\rho_0\Gamma_T(s-\alpha_T)|f_T|^2/F_\pi{}^2 \quad (13)$$

and

$$\mathrm{Im}F=\rho_1(1+\Gamma s)|F|^2/2F_\pi{}^2. \quad (14)$$

We now proceed to construct the partial-wave amplitudes $T_{0T}$ and $T_{11}$ from the full amplitude $T^{abcd}(s,t)$ in (1). To do this we substitute (4) in Eq. (1) and eliminate the propagators in favor of the form factors utilizing Eqs. (6) and (7). When we project the $T=0$ and 2 $s$ waves and the $T=1$ $p$ wave, we obtain

$$T_{0T}=(f_TH_T+\Psi_T)/F_\pi{}^2\ \text{ and }\ T_{11}=(FH_1+\Psi_1)/F_\pi{}^2, \quad (15)$$

where

$$H_T=\frac{h_T-1}{\Gamma_T(s-\alpha_T)}-2\ \text{ and }\ H_1=2\frac{h_1-\frac14}{1+\Gamma s}+1.$$

The $\Psi$ functions in (15) contain the terms having left-hand cuts and polynomials:

$$\begin{aligned}\Psi_0&=I_0+5I_2-J+l_0(H_0+1)\\&\quad+(3\eta_1+2\eta_2)g(s)+\tfrac18(\eta_1+4\eta_2)(s^2+\tfrac13\nu^2),\\ \Psi_2&=\tfrac12(2I_0+I_2+J)+l_2(H_2+1)\\&\quad+2\eta_2g(s)+\tfrac18(\eta_1+\eta_2)(s^2+\tfrac13\nu^2),\\ \Psi_1&=\tfrac12(2I_0'-5I_2'-J')-\tfrac18(\eta_1-\eta_2)s,\end{aligned} \quad (16)$$

in which $\nu=4k^2$ and

$$\begin{aligned}\begin{Bmatrix}I_T\\ J\end{Bmatrix}&=\frac{2}{\nu}\int_{-\nu}^{0}dt\begin{Bmatrix}\tfrac13H_T(t)[f_T(t)+l_T]\\ (4m_\pi{}^2-t-2s)[F(t)-F_\Gamma]2H_1(t)\end{Bmatrix}+\begin{Bmatrix}0\\ s+F_\Gamma(4m_\pi{}^2-3s)\end{Bmatrix},\\ \begin{Bmatrix}I_T'\\ J'\end{Bmatrix}&=\frac{3}{\nu^2}\int_{-\nu}^{0}dt\left(1+\frac{2t}{\nu}\right)\begin{Bmatrix}\tfrac13H_T(t)[f_T(t)+l_T]\\ (4m_\pi{}^2-t-2s)[F(t)-F_\Gamma]2H_1(t)\end{Bmatrix}+\tfrac12\begin{Bmatrix}0\\ 1+F_\Gamma(4H_1-3)\end{Bmatrix}.\end{aligned} \quad (17)$$

Note that we have converted the integration $\int_{-1}^{1}d\cos\theta$ to the form $(2/\nu)\int_{-\nu}^{0}dt$ using $\cos\theta=1+2t/\nu$.

It is straightforward at this point to impose the unitarity constraint (10) on the partial waves in (15), with the help of Eqs. (13) and (14). We see that over the elastic interval, $4m_\pi{}^2\leqslant s\leqslant 16m_\pi{}^2$, we must have

$$|f_TH_T+\Psi_T|^2=H_T\Gamma_T(s-\alpha_T)|f_T|^2, \quad (18)$$

and

$$|FH_1+\Psi_1|^2=\tfrac12H_1(1+\Gamma s)|F|^2. \quad (19)$$

We can make Eqs. (11)–(14), (18), and (19) consistent in a neighborhood above threshold if we require that

$$H_T=\Gamma_T(s-\alpha_T)\ \text{ and }\ H_1=\tfrac12(1+\Gamma s), \quad (20)$$

*and* that each $\Psi$ function vanishes in a neighborhood of threshold to some leading order in a power series in $\nu$. The $\Psi$ functions of course still give the left-hand cuts of the partial waves in (15) when continued to $s<0$, as demanded by crossing. The relations (20) tell us that several parameters in (4) are now determined. These are $\gamma_{13}=0$, $\gamma_{12}{}^2=\Gamma^2$, and $\zeta_T{}^2=4\Gamma_T{}^2$. Hard-pion physics thus establishes the following forms for the partial waves:

$$T_{0T}=[\Gamma_T(s-\alpha_T)f_T+\Psi_T]/F_\pi{}^2 \quad (21)$$

and

$$T_{11}=[\tfrac{1}{2}(1+\Gamma s)F+\Psi_1]/F_\pi^2\,. \qquad (22)$$

Over the whole elastic interval, Eqs. (18) and (19) now provide a unitarity relation of the form

$$|1+Z_T|^2=1\,, \qquad (23)$$

where

$$Z_T=\Psi_T[\Gamma_T(s-\alpha_T)f_T]^{-1} \quad (T=0 \text{ and } 2) \qquad (24)$$

and

$$Z_1=2\Psi_1[(1+\Gamma s)F]^{-1}\,. \qquad (25)$$

Relation (23) constrains each $Z_T$ as shown in Fig. 1.

This completes the formal development of our problem. The three lowest partial waves (21) and (22) have been extracted from the crossing-symmetric amplitude (1), augmented by the smoothness ansatz (3). The objective now is to determine all the parameters introduced so far and all those to follow by imposing unitarity (23) and by demanding the vanishing of the leading terms in the expansions of each $\Psi$ function in (16). The form factors in (24) and (25) are to be constructed to satisfy (8), (9), (13), and (14). This is by no means the unique solution. It is, however, the one most consistent with the set of constraints (10)–(14).

## III. PARAMETRIZATION

The unitarity equations (23) provide the key constraints which must be satisfied over the whole elastic interval ($4m_\pi^2 \leq s \leq 16m_\pi^2$). In order to utilize them, we first need explicit expressions for the form factors $f_T$ and $F$. Equations (13) and (14) imply that the imaginary parts of $f_T^{-1}$ and $F^{-1}$ are known functions in the elastic interval on the $\pi\pi$ cut. Following the method of Ref. 8 we can construct effective-range solutions for the form factors having the correct cut-plane analyticity. The form factors so constructed must also satisfy Eqs. (8) and (9).

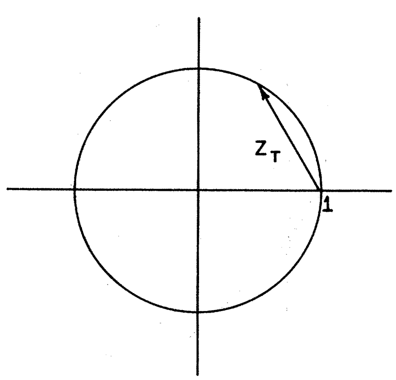


FIG. 1. Argand representation of the unitarity relation $|1+Z_T|^2=1$. The complex function $1+Z_T$ should lie on the unit circle as shown. Note that at threshold $Z_T=0$.

To begin with, we shall express $F$ and immediately make a simplifying observation:

$$F(s)=\left(1+bs+\frac{\pi\nu(1+\Gamma s)Y(s)+4m_\pi^2}{48\pi^2F_\pi^2}\right)^{-1}, \qquad (26a)$$

where $b$ is a constant and $Y(s)$ is the analytic function,

$$Y(s)=\frac{1}{\pi}(\nu/s)^{1/2}\left(\ln\frac{\sqrt{s}+\sqrt{\nu}}{2m_\pi}-\tfrac{1}{2}i\pi\right), \quad \nu\geq 0\,. \qquad (26b)$$

When condition (8) is imposed on (26a), we obtain

$$F_\Gamma=\left(1-\frac{b}{\Gamma}+\frac{m_\pi^2}{12\pi^2F_\pi^2}\right)^{-1}. \qquad (26c)$$

Let us now turn to an examination of the $p$-wave unitarity equation of (23) with $F(s)$ as given in (26). We note that the terms in $|1+Z_1|^2$ growing fastest with $s$ are all accompanied by the factor $\Gamma$. We shall eliminate this rapid growth by setting $\Gamma=0$. It follows from (26c) that $F_\Gamma=0$, i.e., that $C_V=2F_\pi^2$. We hasten to point out that this is not a derivation of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation[13] but rather a reflection of the fact that the $p$-wave unitarity relation of (23) favors a value of $\Gamma$ close to zero. Our choice $\Gamma=0$ is a welcome simplification in that it reduces the number of unknown constants in our problem and, of course, it is also a parametrization of the propagator $\Delta_V$ which has been widely employed. We note that if $\Gamma=0$, then the Schnitzer-Weinberg parameter $\delta=-1$ (see Appendix A). Phenomenologically, this is not the optimum choice[5,7] for the value of $\delta$; nevertheless, it does provide $\rho$ and $A_1$ decay parameters which are not unsatisfactory. From the point of view of dynamics, we are inclined to believe that departures from $\Gamma=0$ are to be understood only in the context of multichannel inelastic unitarity. Once the choice $\Gamma=0$ is made, Eq. (25) becomes

$$Z_1=2\Psi_1/F\,, \qquad (25')$$

and Eqs. (16) and (17) are rewritten in their final form,

$$\begin{aligned}\Psi_0&=I_0+5I_2-J+l_0\Gamma_0(s-2m_\pi^2)+(3\eta_1+2\eta_2)g(s)\\&\quad+\tfrac{1}{8}(\eta_1+4\eta_2)(s^2+\tfrac{1}{3}\nu^2)\,,\\ \Psi_2&=\tfrac{1}{2}(2I_0+I_2+J)+l_2\Gamma_2(s-2m_\pi^2)+2\eta_2 g(s) \qquad (27)\\&\quad+\tfrac{1}{8}(\eta_1+\eta_2)(s^2+\tfrac{1}{3}\nu^2)\,,\\ \Psi_1&=\tfrac{1}{2}(2I_0'-5I_2'-J')-\tfrac{1}{8}(\eta_1-\eta_2)s\,,\end{aligned}$$

where

$$\begin{Bmatrix} I_T \\ J \end{Bmatrix} = \frac{2}{\nu}\int_{-\nu}^{0} dt \begin{Bmatrix} \frac{1}{3}\Gamma_T(t-\alpha_T)f_T(t) \\ (4m_\pi^2 - t - 2s)F(t) \end{Bmatrix} - \begin{Bmatrix} \frac{2}{3}l_T\Gamma_T(\alpha_T + \frac{1}{2}\nu) \\ -s \end{Bmatrix},$$

$$\begin{Bmatrix} I_T' \\ J' \end{Bmatrix} = \frac{3}{\nu^2}\int_{-\nu}^{0} dt \left(1+\frac{2t}{\nu}\right) \begin{Bmatrix} \frac{1}{3}\Gamma_T(t-\alpha_T)f_T(t) \\ (4m_\pi^2 - t - 2s)F(t) \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \frac{1}{3}l_T\Gamma_T \\ 1 \end{Bmatrix}. \quad (28)$$

In the previous section it was shown that our scheme for satisfying elastic unitarity included the requirement that, when each $\Psi$ function is expanded in powers of $\nu$, the leading terms in each expansion should vanish. We shall require that each $\Psi$ vanish to $O(\nu)$. Near threshold, $\nu$ is small and, in the integrals in (28), the integrands are to be evaluated for $t$ close to zero. We therefore can use a linear parametrization of $f_T(t)$ and $F(t)$ under the integrals in (28) for small $\nu$. From soft-pion estimates[11] we expect the values of $\alpha_0$ and $\alpha_2$ to be small also. (Reference 11 supplies the estimate $\Gamma_T^{-1} = -2m_\pi^2$ implying $\alpha_T = 0$.) Thus we parametrize our form factors linearly for $\nu$ near threshold as follows:

$$F(t) = 1 + tF' \text{ and } f_T(t) = -l_T + (t-\alpha_T)f_T'. \quad (29)$$

Note that $f_T(t)$ in (29) conforms to Eq. (9) and that the constant $F'$ in (29) is related to $b$ in (26). There is now a total of eight parameters free to be varied; these are $l_2$, $\alpha_0$, $\alpha_2$, $\eta_1$, $\eta_2$, $f_0'$, $f_2'$, and $F'$. Our requirement that each $\Psi$ vanish to $O(\nu)$ provides six constraints among these eight unknowns, so that two parameters remain free. We take these to be $\alpha_0$ and $\alpha_2$. All the others are given in terms of these as follows:

$$\begin{aligned}
f_0' &= 3m_\pi^2(12m_\pi^2 - 4\alpha_0 - 5\alpha_2) \\
&\quad \times\{(\alpha_0-\alpha_2)[3(4\alpha_0+3\alpha_2+6m_\pi^2) \\
&\qquad + 4m_\pi^4(5\Gamma_0+4\Gamma_2)]\}^{-1}, \\
l_2 &= \frac{3(3\alpha_0 - 4m_\pi^2)}{5\Gamma_2(\alpha_0-\alpha_2)} - \frac{f_0'}{5m_\pi^2}[\alpha_0\alpha_2 - 2m_\pi^2(\alpha_0+\alpha_2)], \\
F' &= f_0'(\Gamma_0\alpha_0^2 - \Gamma_2\alpha_2^2)/12m_\pi^4 - \tfrac{2}{3}f_0'(\Gamma_0-\Gamma_2), \\
\eta_1 &= f_0'[5\alpha_0 + \alpha_2 - 20m_\pi^4(\Gamma_0-\Gamma_2)]/15m_\pi^4, \\
\eta_2 &= f_0'[5\alpha_0 + 7\alpha_2 + 20m_\pi^4(\Gamma_0-\Gamma_2)]/30m_\pi^4, \\
f_2' &= \tfrac{2}{5}f_0',
\end{aligned} \quad (30)$$

in which we recall that $\Gamma_{0,2} = (\alpha_{0,2} - 2m_\pi^2)^{-1}$ and that $2l_0 - 5l_2 = 6m_\pi^2$. The parameter space $\alpha_2$ vs $\alpha_0$ is shown in Fig. 2. All of the six parameters in (30) assume a wide range of values as we vary $\alpha_0$ and $\alpha_2$. We have displayed this by plotting the curves $F' = 0$ and $\infty$, $l_0 = 0$, and $l_2 = 0$.

To determine the values of $\alpha_0$ and $\alpha_2$ we finally appeal to the unitarity constraints (23). The effective-range constructions of the form factors $f_T$ and $F$ needed in (23) should conform to Eqs. (29). We express our analytic form factors as

$$f_T(s) = \left(-l_T^{-2}[l_T + (s-\alpha_T)f_T'] + \beta_T(1-s/\alpha_T)^2 + \frac{\Gamma_T(s-\alpha_T)[Y(s)-Y(\alpha_T)]}{16\pi F_\pi^2}\right)^{-1} \quad (31)$$

and

$$F(s) = \left(1 + bs + \frac{\pi\nu Y(s) + 4m_\pi^2}{48\pi^2 F_\pi^2}\right)^{-1}, \quad (32)$$

where

$$\beta_T = \frac{\Gamma_T\alpha_T[Y(0)-Y(\alpha_T)]}{16\pi F_\pi^2} - (\alpha_T f_T'/l_T)^2(l_T + \alpha_T f_T')^{-1}$$

and

$$b = -F' - (6\pi F_\pi)^{-2}.$$

Equation (31) appears to be more involved than an effective-range expression ought to be. Actually

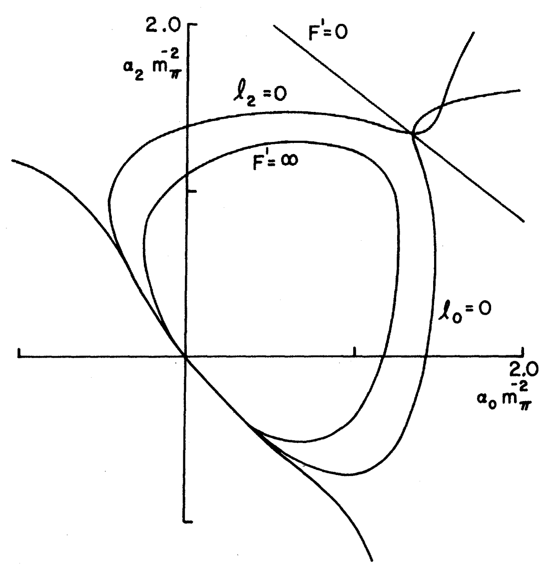


FIG. 2. Parameter space $(\alpha_0, \alpha_2)$. The plotted curves indicate where $l_0 = 0$, $l_2 = 0$, and $F' = 0$ and $\infty$.

its form has been contrived to satisfy the additional requirement that $f_T(0)$ in (31) should match $f_T(0)$ in (29). In doing this we ensure a smooth transition between the linearized form (29) for implementing the conditions on the $\Psi$'s and the analytic form (31) for implementing the unitarity condition over the elastic interval. Expressions (31) and (32) go into (24) and (25′), and the constraints listed in Eqs. (30) are incorporated. A search over the parameter space of Fig. 2 is then to be conducted to find values of $\alpha_0$ and $\alpha_2$ such that Eqs. (23) are satisfied to optimum accuracy. It is clear that by this procedure we can only satisfy the elastic unitarity constraints approximately. Once the parameter search is completed then we may regard the crossing-symmetric expression (1) as an amplitude determined from a self-consistent dynamical method adequate for a description of low-energy $\pi\pi$ scattering.

## IV. RESULTS

Our parameter search to determine $\alpha_0$ and $\alpha_2$ has two phases. The first is a survey and consists of calculating each of the quantities $|1+Z_T|^2$ in the elastic interval $(4m_\pi^2 \leq s \leq 16m_\pi^2)$ for a wide range of points in the parameter space $(\alpha_0, \alpha_2)$. We immediately notice that the departure of $|1+Z_T|^2$ from unity over the whole elastic interval is small only for points in a neighborhood of the line where $F'=0$ (see Fig. 2). The selection of acceptable values of $\alpha_0$ and $\alpha_2$ is further narrowed down when we implement an important constraint imposed by analyticity. We note that form factors $f_T(t)$ and $F(t)$ given by Eqs. (31) and (32) can have poles on the negative $t$ axis and the positions of these poles move as we vary the parameters $\alpha_0$ and $\alpha_2$. These poles of course are physically invalid; they may arise only when the effective-range expressions for the form factors are used far outside of their domain of validity. Thus the parameters must be such that the poles can only occur at locations far removed from this domain. We note that as we increase $\nu$ the integration interval $[-\nu, 0]$ in (28) extends increasingly to negative values of $t$. This situation is illustrated in Fig. 3. A logarithmic branch point can develop in the $\Psi$ functions for positive $\nu$ if the offending poles are not kept far enough to the left on the $t$ axis. We observe that, for those values of $\alpha_0$ and $\alpha_2$ for which $F'$ is not sufficiently positive, there are poles of $f_2(t)$ and $F(t)$ which move to the right on the negative $t$ axis and intervene in the integrations from which the $\Psi$ functions are obtained. This undesirable situation can be avoided if we bound $F'$ from below. We adopt this provision and further reject those values of $\alpha_0$ and $\alpha_2$ for which $|1+Z_T|^2 \geq 1.25$ anywhere in the elastic interval. The second phase of our parameter search consists of calculating the areas under the quantities $|1+Z_T|^2$ over $[4m_\pi^2, 16m_\pi^2]$ for those values of $\alpha_0$ and $\alpha_2$ obtained in the first phase. We then select those parameters which minimize the areas under consideration. The best fit to (23) by this procedure is obtained for parameters $(\alpha_0, \alpha_2)$ in the region $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2}) = (1.07, 1.57)$ to $(1.12, 1.52)$. We find that the intermediate point $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2}) = (1.10, 1.54)$ is the most suitable choice. In Fig. 4 we plot $|1+Z_T|^2$ over the elastic interval $(4m_\pi^2 \leq s \leq 16m_\pi^2)$ for each of the three isospin channels $T=0, 1, 2$. In each case the broken curves correspond to the extreme values of $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2}) = (1.07, 1.57)$ and (1.12 and 1.52); the solid curve corresponds to the ideal choice (1.10, 1.54). For the $T=0$ plot, the broken curves corresponding to the extreme values of $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2})$ almost coincide and therefore only one broken curve is shown in Fig. (4a). We consider it remarkable that unitarity should select such a small region of the $(\alpha_0, \alpha_2)$ space and thereby determine our parametrization of the $\pi\pi$ amplitude so unambiguously.

We take $\alpha_0 = 1.10 m_\pi^2$ and $\alpha_2 = 1.54 m_\pi^2$ to be the values determined in our parameter search and we now proceed to outline the features of our results

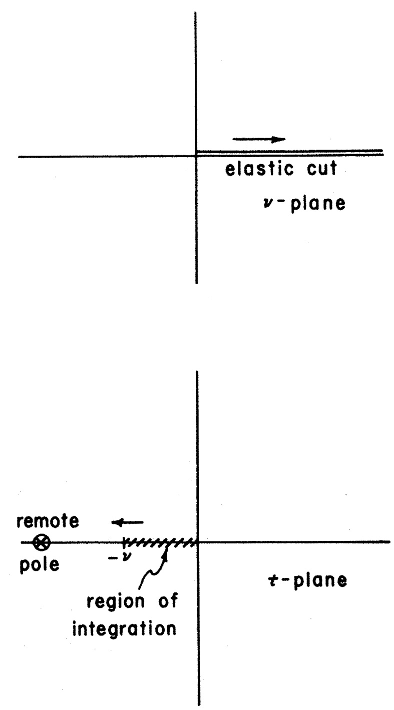


FIG. 3. $\nu$ plane and $t$ plane. For given $\nu$ on the elastic cut, the range of integration in $t$ from Eq. (28) is shown. The form factor pole at ⊗ must be kept remotely to the left of the point $t=-\nu$ as $\nu$ is increased.

for this particular case. We find from Eq. (30) that the slopes of the form factors $f_T$ and $F$ assume the values

$$f_0' = -0.057\,,$$

$$f_2' = -0.023\,,$$

and

$$F' = 0.022 m_\pi^{-2}\,.$$

The coupling parameters $\eta_1$ and $\eta_2$ are

$$\eta_1 = 0.054 m_\pi^{-2} \text{ and } \eta_2 = -0.071 m_\pi^{-2}\,.$$

We can highlight our results by focusing on the scattering lengths and drawing a comparison with the soft-pion values.[3] If we define the scattering lengths $a_{lT}$ by

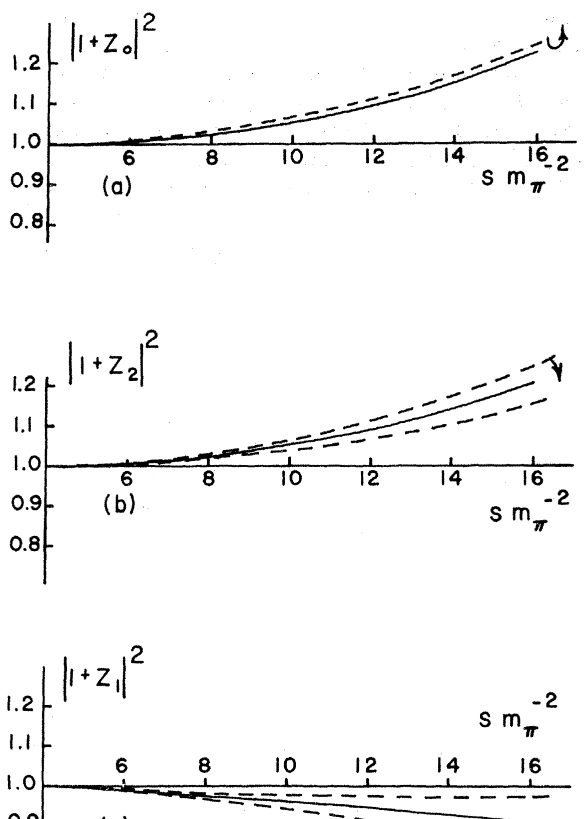


FIG. 4. $|1+Z_T|^2$ vs $s m_\pi^{-2}$ over the elastic interval: (a) $s$ wave $T=0$, (b) $s$ wave $T=2$, and (c) $p$ wave $T=1$. In each case the solid curve corresponds to $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2}) = (1.10,\ 1.54)$, and the broken curves to the extreme values (1.07, 1.57) and (1.12, 1.52). The direction of the arrow indicates how the unitarity fit changes as we vary $(\alpha_0 m_\pi^{-2},\ \alpha_2 m_\pi^{-2})$ in the direction from (1.07, 1.57) to (1.12, 1.52).

$$a_{lT}^{-1} = (k^{2l+1}\cot\delta_{lT})_{\nu=0}\,. \tag{33}$$

Then from Eqs. (21) and (22) we have

$$a_{0T} = \frac{\Gamma_T(4m_\pi^2 - \alpha_T) f_T(4m_\pi^2)}{32\pi m_\pi F_\pi^2}$$

and (34)

$$a_{11} = \frac{F(4m_\pi^2)}{24\pi m_\pi F_\pi^2}\,.$$

The numerical results are

$$a_{0T} = \frac{m_\pi}{32\pi F_\pi^2}\begin{cases} 6.69, & T=0 \\ -2.23, & T=2 \end{cases}$$

and (35)

$$a_{11} = \frac{1}{24\pi m_\pi F_\pi^2}(1.10)\,.$$

These findings are in excellent agreement with those obtained by Weinberg[3]:

$$a_{0T}^W = \frac{m_\pi}{32\pi F_\pi^2}\begin{cases} 7, & T=0 \\ -2, & T=2 \end{cases}$$

and (36)

$$a_{11}^W = \frac{1}{24\pi m_\pi F_\pi^2}\,.$$

From the outset we have dealt with a $\sigma$ commutator $\sigma^{ab}$ which has both $T=0$ and $T=2$ components; results (36) were obtained for $\sigma^{ab}$ isoscalar. Thus the distinguishing feature of the outcome of our analysis is the result we obtain for $l_2$, the parameter which serves as a measure of the $T=2$ component of $\sigma^{ab}$. We get

$$l_2 = -0.48 m_\pi^2 \text{ so that } l_0 = 1.80 m_\pi^2\,. \tag{37}$$

Our conclusion is that unitarity selects $\sigma^{ab}$ to be dominantly isoscalar; if this were not so we would have no reason to expect the agreement we get between (35) and (36). Note that unitarity has selected $\alpha_0$ and $\alpha_2$ to lie in a region near the curve on which $l_2 = 0$ (see Fig. 2).

The $s$- and $p$-wave phase shifts can be calculated directly from the partial-wave amplitudes (21) and (22). These equations imply that

$$\tan\delta_{0T} = \mathrm{Im} f_T[\mathrm{Re} f_T + \Psi_T/\Gamma_T(s-\alpha_T)]^{-1}$$

and (38)

$$\tan\delta_{11} = \mathrm{Im}F(\mathrm{Re}F + 2\Psi_1)^{-1}\,.$$

We have computed the $s$- and $p$-wave phase shifts well beyond the four-pion threshold for c.m. energy close to 1 BeV. In Fig. 5 we have plotted the phase shifts in bands. The broken curves correspond to the phase shifts for the extreme values of $(\alpha_0 m_\pi^{-2}, \alpha_2 m_\pi^{-2})$: (1.07, 1.57) and (1.12, 1.52), and the solid curve for the ideal value (1.10, 1.54). The $p$-wave phase shift in Fig. 5(b) shows a broad $\rho$ resonance with $m_\rho = 915$ MeV and $\Gamma_\rho = 210$ MeV. The $s$-wave $T=0$ phase shift, $\delta_{00}$, in Fig. 5(a) rises slowly to about 20° and then climbs very sharply implying a very narrow $\sigma$ resonance with $m_\sigma = 865$ MeV and $\Gamma_\sigma = 30$ MeV. The present experimental situation for the $T=J=0$ $\pi\pi$ phase shift is not altogether clear. Experimental analyses exist which favor either the down-up or up-down solutions.[14] The trend of our phase shift $\delta_{00}$ is to resemble the down-up solution although it rises more sharply than the phenomenological data seem to suggest. We shall return to a consideration of these findings in our concluding section. The $s$-wave $T=2$ $\pi\pi$ phase shift, $\delta_{02}$, is featureless and exhibits a weak repulsive force in this channel. This is in agreement with the present experimental results.[15]

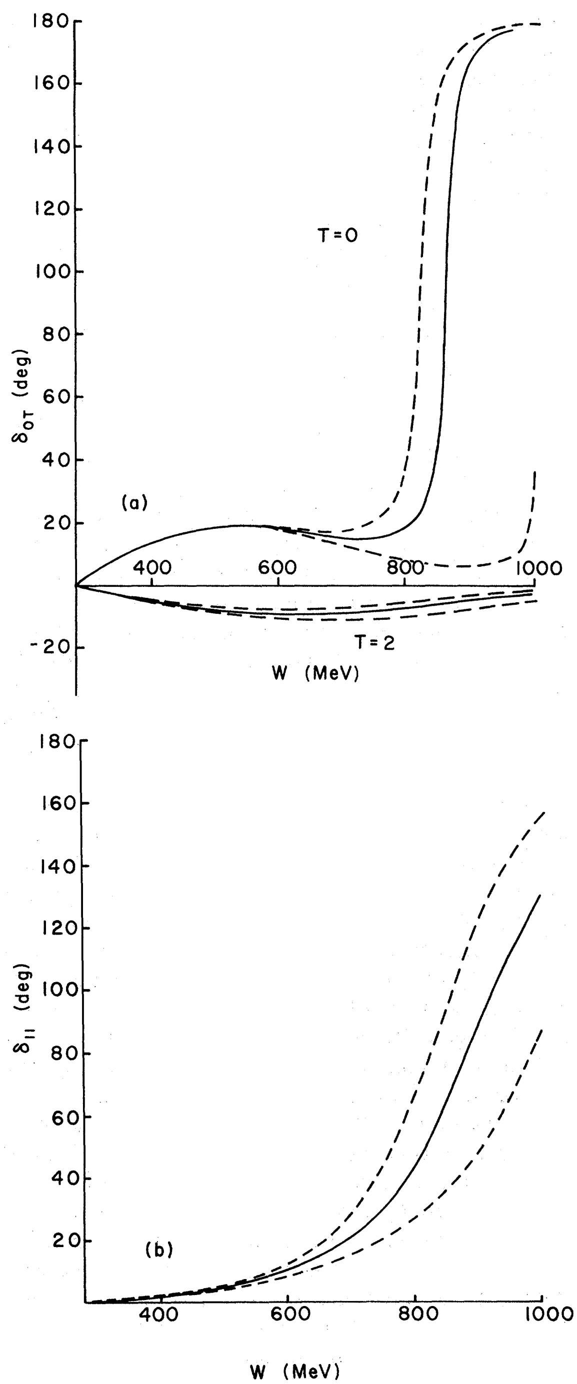


FIG. 5. Phase shifts: (a) the $s$ waves, (b) the $p$ wave.

## V. CONCLUSION

The basic goal of this investigation has been to construct a low-energy $\pi\pi$ scattering amplitude that satisfies the required properties of crossing symmetry, cut-plane analyticity, and elastic unitarity. The approach toward such a construction originates with the Ward identities for the three- and four-point functions of the $\pi\pi$ problem. These expressions contain the dynamical content of current algebra which enters via the equal-time current commutation relations; the Ward identities provide exact representations of the amplitudes under consideration. The construction contains functions such as $F_{\mu\nu\lambda}(q,p)$, $f^{(T)}_{\mu\nu}(q,p)$, and $T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)$ [see Eqs. (A6), (A16), and (1)] which cannot be specified by current algebra alone. These quantities, however, enter only in forms in which they are multiplied with appropriate momentum variables and therefore admit approximations for small values of the momenta. The smoothness approximation is made wherein these functions are parametrized in terms of propagators and polynomials having minimal momentum dependence. The resulting amplitude is then limited in its validity to the low-energy regime. Unitarity provides the additional constraints which we utilize to determine the free parameters of the problem. We have been able to satisfy elastic unitarity approximately for the three partial waves which are important at low energy. From the outset we have incorporated only a minimal number of input data in the problem. Thus the predictive aspects of this scheme become apparent once we have determined all the parameters by imposing the constraints of analyticity and unitarity.

One of the most important of our conclusions is that unitarity prefers the $\sigma$ commutator, $\sigma^{ab}$, to be dominantly isoscalar. Evidence for this feature is the small value we obtain for $l_2$, relative to $l_0$ [see result (37)]. The remarkably close agreement between our scattering lengths and those of Weinberg is attributable to this. Weinberg's results were obtained under the assumption that $\sigma^{ab}$ is purely isoscalar whereas we allow $\sigma^{ab}$ to have a $T=2$ part as well. *A priori*, confirmation of the Weinberg scattering lengths need not have followed had $l_2$ not been small.

When we turn to our predictions for the phase shifts, we are reminded that only elastic unitarity has been invoked. Phenomenological phase shift data do not exist below 600 MeV, so, in the inelastic region where we can compare with data, we must question the extent to which our extrapolated phase shifts are reliable. Where data are available, the $T=J=1$ phase shift, $\delta_{11}$, and the $T=2$, $J=0$ phase shift, $\delta_{02}$, are in qualitative agreement with the experimental results. Our $T=J=0$ phase shift, $\delta_{00}$, does not rise above 20° in the elastic interval; it then rises very sharply around 850 MeV. This rapid increase is even faster than that of the down-up phase shift solution for $\delta_{00}$. Our low-energy construction is expected to break down with increasing energy. The rapid change in $\delta_{00}$ has its origin in polynomial growth, not in any physical effect that has been incorporated in our treatment of the problem. It should be noted that, for the $s$-waves, $\Psi_0$ and $\Psi_2$ [see Eq. (27)] grow as $s^2$, while for the $p$-wave, $\Psi_1$ grows as $s$. Thus, arithmetically, we cannot trust our $s$ waves as far out in energy as we can our $p$ wave. There is a far more important physical consideration to add to this argument. Inelastic contributions to the unitarity relation become appreciable with increasing energy. The first two-particle effect to become important would be the contribution from $K\bar{K}$ intermediate states. Phase-space considerations tell us that $\delta_{00}$ should be strongly influenced because the $K\bar{K}$ effect is in an $s$ state; much less so is this the case for $\delta_{11}$ where the $K\bar{K}$ effect enters in a $p$ state. Of course, $\delta_{02}$ is unaffected by the opening of the $K\bar{K}$ channel. We therefore consider our predicted $T=J=1$ phase shift more reliable at higher energy than that for $T=J=0$. The strong effect in the $s$ wave due to the presence of the $K\bar{K}$ threshold around 980 MeV has also been reported[16,17] experimentally. The experimental analysis of Flatté *et al.*[17] suggests that the opening of the $K\bar{K}$ channel is responsible for a rise in the phase shift $\delta_{00}$ from 90° to about 180° between 900 and 990 MeV. We therefore do not believe that our calculation reliably predicts $\delta_{00}$ to have a narrow $\sigma$ resonance; an analysis of this effect should await a calculation that incorporates inelastic channels, especially $K\bar{K}$. In the elastic interval, of course, we consider our phase shifts as definite predictions. To confront these results for $\delta_{00}$ with data, it appears that a Pais-Treiman analysis[18] of $K_{l4}$ decay[19] is necessary. On the other hand, for the $p$ wave, the arguments presented above permit us to accept the validity of our phase shift, $\delta_{11}$, well into the inelastic region. This amounts to a prediction of the $\rho$ resonance in the $T=J=1$ channel with $m_\rho = 915$ MeV and $\Gamma_\rho = 210$ MeV. Readjustment of the mass and width of the $\rho$ to slightly lower values is expected to follow if the $K\bar{K}$ effect could be included in the calculation. Our $T=2$, $J=0$ phase shift, $\delta_{02}$, lacks any significant structure and is in reasonable agreement with the experimental result.[15] We consider it noteworthy that in the elastic interval our phase shifts for all three partial waves agree quite well with those of Brown and Goble.[20] This is not unexpected because theirs is a calculation based on the unitarization of Weinberg's soft-pion $\pi\pi$ amplitude.[3]

It is clear from the discussion of our results that some improvements on our calculation suggest themselves. We recall from Fig. 4 that we have been able to satisfy the elastic unitarity constraints of Eq. (23) within a departure which becomes as much as 20%. We could improve upon this by means of a more extensive parametrization in which $\Gamma$ is not taken to be zero and the smoothness approximation of Eq. (A17) for $f_{\mu\nu}^{(T)}(q,p)$ contains terms in addition to the one proportional to $\delta_{\mu\nu}$. This would result in an increased number of free parameters in the problem and would allow us to implement the vanishing of the $\Psi$ functions to $O(\nu^2)$. It is clear that this would improve the agreement with unitarity and extend the validity of our calculation. Furthermore, the physics of the problem demands that we ultimately incorporate the effects of inelastic channels such as $K\bar{K}$ to enable reliable predictions to be made for the $T=J=0$ phase shift in the inelastic region.

Finally, we turn to a consideration of problems peripheral to the present work. One example deserving immediate attention is to investigate the effect of $\pi\pi$ scattering in the $t$ channel of $\pi N$ scattering. The objective would be to examine the possibility that these effects modify the $\pi N$ $\sigma$ term, as evaluated by Cheng and Dashen.[21] The quantity in question is the nucleon-to-nucleon matrix element of $\sigma^{ab}$ in which only $\sigma^{T=0}$ plays a role; knowledge of it from experimental $\pi N$ data bears in an important way on questions pertaining to the breaking of chiral symmetry.[22] The result of Cheng and Dashen suggests that the breaking of SU(2)×SU(2) and the breaking of SU(3) are of comparable order. Brown, Pardee, and Peccei[23] have examined the $\pi N$ amplitude on the pion mass-shell and confirmed the findings of Cheng and Dashen. Reference 23 does not explicitly include features arising from $t$-channel $\pi\pi$ scattering, and Schnitzer[24] has proposed a method for introducing them. Schnitzer's analysis employs the results of his $\pi\pi$ investigation.[4] It would appear that our treatment of the $\pi\pi$

problem could be used for this purpose too. This is unfortunately not the case. Our construction is effectively constrained not to modify the $\pi N$ $\sigma$ term.[25,26] This is apparent if we refer to Eq. (5), regarded as a function of $t$. The desired modification factor, called for by Schnitzer, is just

$$F_\pi^2[f_0(t)+l_0]\Delta_0^{-1}(t)$$

(in our notation), evaluated at the point $t=2m_\pi^2$. In our treatment this result is $-1$, independent of $\Gamma_0$. To examine the question raised by Schnitzer, a more extensive parametrization of $f_{\mu\nu}^{(T)}(q,p)$ would be necessary. This sort of modification has already been suggested in the preceding paragraph and should be carried out.

The generality of our technique suggests that it could be employed to investigate a similar problem, namely $K\pi$ scattering. We note that the $t$ channel of $K\pi$ scattering includes the states $\pi\pi$ and $K\bar{K}$. Thus a thorough treatment of $K\pi$ scattering would include in its analysis the resolution of questions raised earlier about the effects of coupling the $\pi\pi$ and $K\bar{K}$ channels. Moreover, a study of this process along the lines of the present formalism would provide a determination of quantities related to the matrix elements for $K_{l3}$ decay; thus this would provide a new solution to the $K_{l3}$ problem itself, a problem of great theoretical and experimental interest.[27] Needless to say, $K\pi$ scattering would be considerably more involved than the present investigation of $\pi\pi$ scattering.

## APPENDIX A: THREE-POINT FUNCTIONS

The Ward identities for the family of four-point functions of axial-vector currents contain the three- and two-point functions which also enter the problem. These quantities have been studied in the earlier literature.[7,11] In this appendix we shall review those two- and three-point functions of primary interest to us and also show how the parameter $\Gamma$ of our text is related to the Schnitzer-Weinberg[5] parameter $\delta$, without making any reference to $\rho$-meson dominance of the vector current.

The three-point functions for the vector and axial-vector currents of SU(2)×SU(2) are, in the notation of Ref. 7,

$$\begin{aligned} W_\lambda^{abc}(q,p)&=\int dxdy\, e^{-iqx+ipy}\langle 0|T\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)V_\lambda^c(0)|0\rangle,\\ W_{\nu\lambda}^{abc}(q,p)&=\int dxdy\, e^{-iqx+ipy}\langle 0|T\partial_\mu A_\mu^a(x)A_\nu^b(y)V_\lambda^c(0)|0\rangle,\\ W_{\mu\nu\lambda}^{abc}(q,p)&=\int dxdy\, e^{-iqx+ipy}\langle 0|TA_\mu^a(x)A_\nu^b(y)V_\lambda^c(0)|0\rangle, \end{aligned}\tag{A1}$$

in which $a$, $b$, and $c$ are isospin indices. The first of Eqs. (A1) for $c=3$ is related to the off-shell electromagnetic form factor of the pion. On shell, the pion form factor $F(t)$ is defined by

$$(4\omega_q\omega_p)^{1/2}\langle\pi(qa)|V_\lambda^c(0)|\pi(pb)\rangle=-i\epsilon_{abc}F(t)Q_\lambda,\tag{A2}$$

where $Q=p+q$ and $t=-(p-q)^2$. In terms of (A1), we have

$$(4\omega_q\omega_p)^{1/2}\langle\pi(qa)|V_\lambda^c(0)|\pi(pb)\rangle=-F_\pi^{-2}m_\pi^{-4}[(m_\pi^2+q^2)(m_\pi^2+p^2)W_\lambda^{abc}(q,p)]_{p^2=q^2=-m_\pi^2},\tag{A3}$$

where the pion decay constant, $F_\pi=94$ MeV, is defined by

$$(2\omega_p)^{1/2}\langle 0|\partial_\mu A_\mu^a(0)|\pi(pb)\rangle=\delta_{ab}F_\pi m_\pi^2.$$

The spectral representations for the vector and axial-vector propagators are

$$\begin{aligned} &\int dy\, e^{iky}\langle 0|TV_\nu^b(y)V_\lambda^c(0)|0\rangle=-i\delta_{bc}[\Delta_{\nu\lambda}^V(k)-C_V\delta_{\nu4}\delta_{\lambda4}],\\ &\int dx\, e^{-iqx}\langle 0|TA_\mu^a(x)A_\nu^b(0)|0\rangle=-i\delta_{ab}\left(\Delta_{\mu\nu}^A(q)+\frac{F_\pi^2q_\mu q_\nu}{q^2+m_\pi^2}-(C_A+F_\pi^2)\delta_{\mu4}\delta_{\nu4}\right), \end{aligned}\tag{A4}$$

where

$$\Delta_{\mu\nu}^{V,A}(k)=\Delta_{V,A}(-k^2)\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\frac{k_\mu k_\nu}{k^2}C_{V,A},$$

$$\Delta_{V,A}(-k^2)=\int\frac{\rho_{V,A}(x)}{x+k^2}\,dx,$$

and

$$C_{V,A}=\Delta_{V,A}(0)\,.$$

(We assume pion-pole dominance for the spin-zero part of the axial-vector current from the outset.) The spectral functions $\rho_V$ and $\rho_A$ are related to $\langle 0|\,V_\mu^a(x)V_\nu^b(0)|0\rangle$ and $\langle 0|\,A_\mu^a(x)A_\nu^b(0)|0\rangle$ in the usual way.

The three-point functions in (A1) satisfy Ward identities which follow from the current commutation relations. The Schwinger terms which appear in these are assumed to be $c$ numbers throughout this paper. Pion-pole dominance, already manifest in (A4), can be employed for the expressions in (A1) as well. In this way off-shell form factors are introduced. In the notation of Ref. 7, they are denoted by $F_\lambda(q,p)$, $F_{\nu\lambda}(q,p)$, and $F_{\mu\nu\lambda}(q,p)$, where, e.g.,

$$[F_\lambda(q,p)]_{q^2=p^2=-m_\pi^2}=F(t)Q_\lambda\,.$$

The following exact relation was obtained in Ref. 7, with $k=p-q$:

$$F_\pi^2[F_\lambda(q,p)-Q_\lambda]=q_\mu p_\nu F_{\mu\nu\lambda}(q,p)+\frac{1}{2}\left(\frac{p^2-q^2}{k^2}k_\lambda-Q_\lambda\right)[C_V-\Delta_V(-k^2)]\,. \tag{A5}$$

The quantity $i\epsilon_{abc}F_{\mu\nu\lambda}(q,p)$ is the part of $W_{\mu\nu\lambda}^{abc}(q,p)$ with no pion poles in either of the variables $q^2$ or $p^2$. On shell (A5) becomes

$$F_\pi^2[F(t)-1]Q_\lambda=[q_\mu p_\nu F_{\mu\nu\lambda}(q,p)]_{q^2=p^2=-m_\pi^2}+\tfrac{1}{2}[\Delta_V(t)-C_V]Q_\lambda\,. \tag{A6}$$

The smoothness ansatz[5] is adopted for $F_{\mu\nu\lambda}$:

$$F_{\mu\nu\lambda}(q,p)=\Delta_{\mu\alpha}^A(q)\Delta_{\nu\beta}^A(p)\Delta_{\lambda\eta}^V(k)[\Gamma_1\delta_{\alpha\beta}Q_\eta+\Gamma_2(\delta_{\alpha\eta}k_\beta-\delta_{\beta\eta}k_\alpha)+\Gamma_3(\delta_{\alpha\eta}p_\beta+\delta_{\beta\eta}q_\alpha)] \tag{A7}$$

in which $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ are constants. With this hypothesis we obtain

$$[q_\mu p_\nu F_{\mu\nu\lambda}(q,p)]_{q^2=p^2=-m_\pi^2}=C_A^2[\tfrac{1}{2}t(\Gamma_1-\Gamma_2)-m_\pi^2(\Gamma_1+\Gamma_3)]Q_\lambda\Delta_V(t)\,.$$

In (A6) we must have $F(0)=1$, therefore, $\Gamma_1+\Gamma_3=0$. Our final hard-pion expression for $F(t)$ follows:

$$2F_\pi^2[F(t)-1]=(1+\Gamma t)\Delta_V(t)-C_V\,, \tag{A8}$$

where $\Gamma=C_A^2(\Gamma_1-\Gamma_2)$. Equation (A8) appears as Eq. (6) of the text.

It only remains to relate $\Gamma$ to the Schnitzer-Weinberg parameter $\delta$. To do this we multiply (A7) by $k_\lambda$ and consider the case $p^2=q^2$; we get

$$[k_\lambda F_{\mu\nu\lambda}(q,p)]_{p^2=q^2}=C_AC_V\Gamma_1\Delta_A(-q^2)(q_\mu q_\nu-p_\mu p_\nu)\,.$$

But $F_{\mu\nu\lambda}$ satisfies the Ward identity[7]

$$k_\lambda F_{\mu\nu\lambda}(q,p)=\Delta_{\mu\nu}^A(q)-\Delta_{\mu\nu}^A(p)$$

so that in general

$$[k_\lambda F_{\mu\nu\lambda}(q,p)]_{p^2=q^2}=\frac{1}{q^2}[C_A-\Delta_A(-q^2)](q_\mu q_\nu-p_\mu p_\nu)$$

When we compare these two expressions, we see that $\Delta_A(-q^2)$ is pole-dominated:

$$\Delta_A(-q^2)=\frac{g_A^2}{q^2+m_A^2}\,,$$

where $m_A^2=(C_AC_V\Gamma_1)^{-1}$ and $g_A^2=(C_V\Gamma_1)^{-1}$. The parameter $\delta$ is defined by[5]

$$\Gamma_2=\Gamma_1(2+\delta)$$

so that

$$\Gamma=-C_A^2\Gamma_1(1+\delta)$$
$$=-\left(1-\frac{F_\pi^2}{C_V}\right)\frac{1+\delta}{m_A^2}\,.$$

In the last step we have used the first Weinberg sum rule[12]:

$$C_V=C_A+F_\pi^2\,.$$

Note that we have nowhere made any reference to the $\rho$ meson.

We shall also need the two- and three-point functions of the $\sigma$ commutator and the axial-vector currents. Here we adopt the notation of Ref. 11. The operator $\sigma^{ab}$ is defined by

$$[A_4^a(x),\partial_\nu A_\nu^b(0)]\delta(x_0)=\sigma^{ab}(0)\delta(x)\,. \tag{A9}$$

Vector current conservation implies that $\sigma^{ab}$ is symmetric in $ab$ and in general has isospin $T=0$ and $T=2$ parts. Another commutator that plays a role is

$$[A_4^a(x),\sigma^{bc}(0)]\delta(x_0)=-\sigma^{abc}(0)\delta(x)\,. \tag{A10}$$

The Jacobi identity requires that $\sigma^{abc}$ satisfy

$$\sigma^{abc}(x)-\sigma^{bac}(x)=\delta_{bc}\partial_\mu A_\mu^a(x)-\delta_{ac}\partial_\mu A_\mu^b(x)\,.$$

We introduce the constants $l_0$ and $l_2$ defined by

$$(2\omega_q)^{1/2}\langle\pi(qd)|\sigma^{abc}(0)|0\rangle=F_\pi\sum_{T=0,2}l_TP_{dabc}^T\,,$$

where $P^T$ denotes the projection operator for isospin $T$. The Jacobi identity above imposes an important constraint between $l_0$ and $l_2$:

$$2l_0-5l_2=6m_\pi^2\,. \tag{A11}$$

The three-point functions that concern us are

$$\Sigma^{abcd}(q,p)=\int dxdy\, e^{-iqx+ipy} \times\langle 0|T\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)\sigma^{cd}(0)|0\rangle,$$

$$\Sigma_\nu^{abcd}(q,p)=\int dxdy\, e^{-iqx+ipy} \times\langle 0|T\partial_\mu A_\mu^a(x)A_\nu^b(y)\sigma^{cd}(0)|0\rangle, \tag{A12}$$

$$\Sigma_{\mu\nu}^{abcd}(q,p)=\int dxdy\, e^{-iqx+ipy} \times\langle 0|TA_\mu^a(x)A_\nu^b(y)\sigma^{cd}(0)|0\rangle.$$

The first of these is related to the off-shell $\sigma\pi\pi$ form factor $f^{abcd}(q,p)$. On shell, the $\sigma\pi\pi$ form factor $f^{abcd}(t)$ is defined by

$$(4\omega_q\omega_p)^{1/2}\langle\pi(qa)|\sigma^{cd}(0)|\pi(pb)\rangle=-f^{abcd}(t). \tag{A13}$$

Since $\sigma^{cd}$ has $T=0$ and 2 components, we write

$$f^{abcd}(t)=\sum_{T=0,2} f_T(t)P_{abcd}^T. \tag{A14}$$

The two-point functions we need are

$$\Lambda_\nu^{bacd}(p)=\int dy\, e^{ipy}\langle 0|TA_\nu^b(y)\sigma^{acd}(0)|0\rangle =-\frac{p_\nu F_\pi^{\,2}}{p^2+m_\pi^{\,2}}\, l_{bacd}, \tag{A15}$$

$$\Delta^{abcd}(k)=\int dz\, e^{ikz}\langle 0|T\sigma^{ab}(z)\sigma^{cd}(0)|0\rangle =\delta_{ab}\delta_{cd}F_\pi^{\,4}m_\pi^{\,4}(2\pi)^4\delta(k)-i\sum_{T=0,2}\Delta_T(t)P_{abcd}^T,$$

where

$$\Delta_T(t)=\int\frac{dx}{x-t}\rho_T(x)$$

and

$$l_{bacd}=\sum_{T=0,2} l_T P_{bacd}^T.$$

In complete analogy with the treatment of the matrix element of the vector current (A2), we can proceed to generate the Ward identities and to extract all pion poles. This has been done in Ref. 11 and the resulting on-shell exact relation is

$$F_\pi^{\,2}[f_T(t)+l_T]+\Delta_T(t)=[q_\mu p_\nu f_{\mu\nu}^{(T)}(q,p)]_{q^2=p^2=-m_\pi^{\,2}}. \tag{A16}$$

The quantity

$$\sum_{T=0,2} f_{\mu\nu}^{(T)}(q,p)P_{abcd}^T=f_{\mu\nu}^{abcd}(q,p)$$

is the part of $\Sigma_{\mu\nu}^{abcd}(q,p)$ which is free of the vacuum contribution and of pion poles in $q^2$ and $p^2$. Other off-shell form factors,

$$f_\nu^{abcd}(q,p)=\sum_T f_\nu^{(T)}(q,p)P_{abcd}^T$$

and

$$f^{abcd}(q,p)=\sum_T f^{(T)}(q,p)P_{abcd}^T,$$

are related similarly, in Ref. 11, to $\Sigma_\nu^{abcd}(q,p)$ and $\Sigma^{abcd}(q,p)$, respectively. The smoothness ansatz to make is the analog of that of Schnitzer and Weinberg[5] in the case of the vector current:

$$f_{\mu\nu}^{(T)}(q,p)=\Delta_{\mu\alpha}^A(q)\Delta_{\nu\beta}^A(p)\Delta_T(t)(\gamma^{(T)}\delta_{\alpha\beta}). \tag{A17}$$

When we introduce (A17) into (A16) we get

$$F_\pi^{\,2}[f_T(t)+l_T]=[\Gamma_T(t-2m_\pi^{\,2})-1]\Delta_T(t), \tag{A18}$$

where $\Gamma_T=\frac{1}{2}C_A^{\,2}\gamma^{(T)}$. This appears as Eq. (5) in the text.

## APPENDIX B: FOUR-POINT FUNCTIONS

The general structure of the $\pi\pi$ scattering amplitude, satisfying the current-algebra constraints, can be obtained by deriving the Ward identities for the four-point functions of axial-vector currents. In this appendix we show how the Ward identities for the four-point functions are obtained, leading to an exact crossing-symmetric $\pi\pi$ amplitude on the mass shell. A similar derivation has been carried out in the paper of Gerstein and Schnitzer.[6] The development of the formulas is presented here from the beginning, in order to establish our own notation and to coordinate the derivation with that reviewed in Appendix A.

The four-point functions we need are

$$M^{abcd}(q_3;q_1,q_2)=\int dxdydz\, e^{i(q_1x+q_2y-q_3z)}\langle 0|T\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)\partial_\lambda A_\lambda^c(z)\partial_\sigma A_\sigma^d(0)|0\rangle,$$

$$M_\sigma^{abcd}(q_3;q_1,q_2)=\int dxdydz\, e^{i(q_1x+q_2y-q_3z)}\langle 0|T\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)\partial_\lambda A_\lambda^c(z)A_\sigma^d(0)|0\rangle,$$

$$M_{\lambda\sigma}^{abcd}(q_3;q_1,q_2)=\int dxdydz\, e^{i(q_1x+q_2y-q_3z)}\langle 0|T\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^b(y)A_\lambda^c(z)A_\sigma^d(0)|0\rangle, \tag{B1}$$

$$M_{\nu\lambda\sigma}^{abcd}(q_3;q_1,q_2)=\int dxdydz\, e^{i(q_1x+q_2y-q_3z)}\langle 0|T\partial_\mu A_\mu^a(x)A_\nu^b(y)A_\lambda^c(z)A_\sigma^d(0)|0\rangle,$$

$$M_{\mu\nu\lambda\sigma}^{abcd}(q_3;q_1,q_2)=\int dxdydz\, e^{i(q_1x+q_2y-q_3z)}\langle 0|TA_\mu^a(x)A_\nu^b(y)A_\lambda^c(z)A_\sigma^d(0)|0\rangle.$$

As in the case of the three-point functions we can generate a set of Ward identities among the $M$'s of Eqs. (B1) by application of the equal-time commutation relations. These Ward identities relate the quantities in (B1), (A1), and (A12) as follows:

$$q_{4\sigma}M^{abcd}_{\sigma}(q_3;q_1,q_2)=-iM^{abcd}(q_3;q_1,q_2)-[\Sigma^{cbad}(q_3,q_2)+\Sigma^{cabd}(q_3,q_1)+\Sigma^{abcd}(-q_1,q_2)],$$

$$q_{3\lambda}M^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)=-iM^{abcd}_{\sigma}(q_3;q_1,q_2)+\epsilon_{cde}W^{abe}_{\sigma}(-q_1,q_2)-[\Sigma^{bdac}_{\sigma}(-q_2,-q_4)+\Sigma^{adbc}_{\sigma}(-q_1,-q_4)]$$
$$+(2\pi)^4(C_A+F_\pi^2)\frac{m_\pi^4F_\pi^2}{q_1^2+m_\pi^2}\delta_{ab}\delta_{cd}\delta(q_1+q_2)(q_{4\sigma}-q_{44}\delta_{\sigma4}),$$

$$q_{2\nu}M^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)=iM^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)+\Sigma^{cdab}_{\lambda\sigma}(q_3,-q_4)-[\epsilon_{bce}W^{ade}_{\sigma\lambda}(-q_1,-q_4)+\epsilon_{bde}W^{ace}_{\lambda\sigma}(-q_1,-q_3)]$$
$$-i(2\pi)^4(C_A+F_\pi^2)\frac{m_\pi^2F_\pi^2}{q_1^2+m_\pi^2}[\delta_{ad}\delta_{bc}\delta(q_1-q_4)(q_{3\lambda}-q_{34}\delta_{\lambda4})q_{4\sigma}$$
$$+\delta_{ac}\delta_{bd}\delta(q_1-q_3)(q_{4\sigma}-q_{44}\delta_{\sigma4})q_{3\lambda}],\qquad\text{(B2)}$$

$$q_{1\mu}M^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)=iM^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)-[\epsilon_{abe}W^{cde}_{\lambda\sigma\nu}(q_3,-q_4)+\epsilon_{ace}W^{dbe}_{\sigma\nu\lambda}(q_4,q_2)+\epsilon_{ade}W^{cbe}_{\lambda\nu\sigma}(q_3,q_2)]$$
$$-(2\pi)^4(C_A+F_\pi^2)[-\delta_{ab}\delta_{cd}\delta(q_1+q_2)(q_{2\nu}-q_{24}\delta_{\nu4})\hat{\Delta}^A_{\lambda\sigma}(q_3)+\delta_{ac}\delta_{bd}\delta(q_1-q_3)(q_{3\lambda}-q_{34}\delta_{\lambda4})\hat{\Delta}^A_{\nu\sigma}(q_2)$$
$$+\delta_{ad}\delta_{bc}\delta(q_1-q_4)(q_{4\sigma}-q_{44}\delta_{\sigma4})\hat{\Delta}^A_{\nu\lambda}(q_2)],$$

where

$$\hat{\Delta}^A_{\lambda\sigma}(q)=\Delta^A_{\lambda\sigma}(q)+F_\pi^2\frac{q_\lambda q_\sigma}{q^2+m_\pi^2}-(C_A+F_\pi^2)\delta_{\lambda4}\delta_{\sigma4}.$$

The ($c$-number) Schwinger terms in the current commutators are the cause for all the noncovariant terms appearing in (B2). The offending terms in (B2) are proportional to momentum $\delta$ functions. These are exactly canceled by the vacuum intermediate-state contributions in the other terms of (B2). The $\Sigma$'s defined in (A12) have vacuum contributions; so do the $M$'s of Eqs. (B1). These may be split off from the $M$'s as follows:

$$M^{abcd}(q_3;q_1,q_2)=[M^{abcd}(q_3;q_1,q_2)]_{\text{vac}}+\overline{M}^{abcd}(q_3;q_1,q_2)\qquad\text{(B3)}$$

and similarly for the other $M$'s. To give an example of one of the vacuum contributions, we have

$$[M^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)]_{\text{vac}}=(2\pi)^4m_\pi^4F_\pi^2\left(-\delta_{ab}\delta_{cd}\delta(q_1+q_2)\frac{1}{q_1^2+m_\pi^2}\hat{\Delta}^A_{\lambda\sigma}(q_3)\right.$$
$$\left.+\frac{F_\pi^2q_{3\lambda}q_{4\sigma}}{(q_3^2+m_\pi^2)(q_4^2+m_\pi^2)}[\delta_{ac}\delta_{bd}\delta(q_1-q_3)+\delta_{ad}\delta_{bc}\delta(q_1-q_4)]\right).$$

The family of $\overline{M}$ amplitudes, specified as in (B3), can then be used to define a corresponding family of off-shell amplitudes. This is the next task.

Throughout this paper we consistently employ pion-pole dominance for the spin-zero part of the axial-vector current. The diagonalization of the four-point functions which incorporates this is as follows:

$$\overline{M}^{abcd}(q_3;q_1,q_2)=i\frac{m_\pi^8F_\pi^4}{\prod_{i=1}^4(q_i^2+m_\pi^2)}T^{abcd}(q_3;q_1,q_2),$$

$$\overline{M}^{abcd}_{\sigma}(q_3;q_1,q_2)=\frac{m_\pi^6F_\pi^3}{\prod_{i=1}^3(q_i^2+m_\pi^2)}\left(T^{abcd}_{\sigma}(q_3;q_1,q_2)-\frac{F_\pi q_{4\sigma}}{q_4^2+m_\pi^2}T^{abcd}(q_3;q_1,q_2)\right),$$

$$\overline{M}^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)=-i\frac{m_\pi^4F_\pi^2}{\prod_{i=1}^2(q_i^2+m_\pi^2)}\left[T^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)-\frac{F_\pi q_{3\lambda}}{q_3^2+m_\pi^2}T^{abcd}_{\sigma}(q_3;q_1,q_2)\right.$$
$$\left.-\frac{F_\pi q_{4\sigma}}{q_4^2+m_\pi^2}\left(T^{abdc}_{\lambda}(q_4;q_1,q_2)-\frac{F_\pi q_{3\lambda}}{q_3^2+m_\pi^2}T^{abcd}(q_3;q_1,q_2)\right)\right],\qquad\text{(B4)}$$

$$\overline{M}^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)=-\frac{m_\pi^2F_\pi}{q_1^2+m_\pi^2}\left[T^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)+\frac{F_\pi q_{2\nu}}{q_2^2+m_\pi^2}T^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)\right.$$
$$-\frac{F_\pi q_{3\lambda}}{q_3^2+m_\pi^2}\left(T^{acbd}_{\nu\sigma}(-q_2;q_1,-q_3)+\frac{F_\pi q_{2\nu}}{q_2^2+m_\pi^2}T^{abcd}_{\sigma}(q_3;q_1,q_2)\right)$$

$$-\frac{F_\pi q_{4\sigma}}{q_4^2+m_\pi^2}\left(T^{adcb}_{\lambda\nu}(q_3;q_1,-q_4)+\frac{F_\pi q_{2\nu}}{q_2^2+m_\pi^2}T^{abdc}_{\lambda}(q_4;q_1,q_2)\right)$$
$$+\frac{F_\pi^2 q_{3\lambda}q_{4\sigma}}{(q_3^2+m_\pi^2)(q_4^2+m_\pi^2)}\left(T^{adcb}_{\nu}(q_3;q_1,-q_4)+\frac{F_\pi q_{2\nu}}{q_2^2+m_\pi^2}T^{abcd}(q_3;q_1,q_2)\right)\Bigg],$$

$$\overline{M}^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)=i\Bigg[T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)$$
$$+\frac{F_\pi q_{2\nu}}{q_2^2+m_\pi^2}\left(T^{bacd}_{\mu\lambda\sigma}(q_3;q_2,q_1)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)\right)$$
$$-\frac{F_\pi q_{3\lambda}}{q_3^2+m_\pi^2}\left(T^{cbad}_{\nu\mu\sigma}(-q_1;-q_3,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{acbd}_{\nu\sigma}(-q_2;q_1,-q_3)\right)$$
$$-\frac{F_\pi q_{4\sigma}}{q_4^2+m_\pi^2}\left(T^{dbca}_{\nu\lambda\mu}(q_3;-q_4,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{adcb}_{\lambda\nu}(q_3;q_1,-q_4)\right)$$
$$-\frac{F_\pi^2 q_{2\nu}q_{3\lambda}}{(q_2^2+m_\pi^2)(q_3^2+m_\pi^2)}\left(T^{cbad}_{\mu\sigma}(-q_1;-q_3,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{abcd}_{\sigma}(q_3;q_1,q_2)\right)$$
$$-\frac{F_\pi^2 q_{2\nu}q_{4\sigma}}{(q_2^2+m_\pi^2)(q_4^2+m_\pi^2)}\left(T^{dbca}_{\lambda\mu}(q_3;-q_4,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{abdc}_{\lambda}(q_4;q_1,q_2)\right)$$
$$+\frac{F_\pi^2 q_{3\lambda}q_{4\sigma}}{(q_3^2+m_\pi^2)(q_4^2+m_\pi^2)}\left(T^{cdab}_{\mu\nu}(-q_1;-q_3,-q_4)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{adcb}_{\nu}(q_3;q_1,-q_4)\right)$$
$$+\frac{F_\pi^3 q_{2\nu}q_{3\lambda}q_{4\sigma}}{(q_2^2+m_\pi^2)(q_3^2+m_\pi^2)(q_4^2+m_\pi^2)}\left(T^{dbca}_{\mu}(q_3;-q_4,q_2)+\frac{F_\pi q_{1\mu}}{q_1^2+m_\pi^2}T^{abcd}(q_3;q_1,q_2)\right)\Bigg].$$

In the first of Eqs. (B4) we have defined the off-shell amplitude $T^{abcd}(q_3;q_1,q_2)$ for the process of interest: $\pi_a(q_1)+\pi_b(q_2)\to\pi_c(q_3)+\pi_d(q_4)$.

The next step is to obtain the constraints among the $T$'s which follow from (B2) and the relevant three-point function Ward identities. We find

$$\begin{aligned}
q_{4\sigma}T^{abcd}_{\sigma}(q_3;q_1,q_2)&=F_\pi T^{abcd}(q_3;q_1,q_2)-\frac{1}{F_\pi m_\pi^2}[(q_1^2+m_\pi^2)f^{cbad}(q_3,q_2)+(q_2^2+m_\pi^2)f^{cabd}(q_3,q_1)\\
&\qquad+(q_3^2+m_\pi^2)f^{abcd}(-q_1,q_2)],\\
q_{3\lambda}T^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)&=F_\pi T^{abcd}_{\sigma}(q_3;q_1,q_2)-\epsilon_{abe}\epsilon_{cde}F_\sigma(-q_1,q_2)\\
&\qquad-\frac{1}{F_\pi m_\pi^2}[(q_1^2+m_\pi^2)f^{bdac}_{\sigma}(-q_2,-q_4)+(q_2^2+m_\pi^2)f^{adbc}_{\sigma}(-q_1,-q_4)]-\frac{q_{4\sigma}}{m_\pi^2}f^{abcd}(-q_1,q_2),\\
q_{2\nu}T^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)&=-F_\pi T^{abcd}_{\lambda\sigma}(q_3;q_1,q_2)-\frac{q_1^2+m_\pi^2}{F_\pi m_\pi^2}f^{cdab}_{\lambda\sigma}(q_3,-q_4)+\epsilon_{bce}\epsilon_{ade}F_{\sigma\lambda}(-q_1,-q_4)\\
&\qquad+\epsilon_{bde}\epsilon_{ace}F_{\lambda\sigma}(-q_1,-q_3)+\frac{1}{m_\pi^2}[q_{4\sigma}f^{acdb}_{\lambda}(-q_1,-q_3)+q_{3\lambda}f^{adcb}_{\sigma}(-q_1,-q_4)],\\
q_{1\mu}T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)&=-F_\pi T^{abcd}_{\nu\lambda\sigma}(q_3;q_1,q_2)+\frac{1}{m_\pi^2}[q_{2\nu}f^{cdba}_{\lambda\sigma}(q_3,-q_4)-q_{4\sigma}f^{cbad}_{\lambda\nu}(q_3,q_2)-q_{3\lambda}f^{bdca}_{\nu\sigma}(-q_2,-q_4)]\\
&\qquad-[\epsilon_{abe}\epsilon_{cde}F_{\lambda\sigma\nu}(q_3,-q_4)+\epsilon_{ace}\epsilon_{dbe}F_{\sigma\nu\lambda}(q_4,q_2)+\epsilon_{ade}\epsilon_{cbe}F_{\lambda\nu\sigma}(q_3,q_2)].
\end{aligned}\tag{B5}$$

It is now straightforward to obtain the final Ward identity for the $\pi\pi$ scattering amplitude by contracting $q_{2\nu}$, $q_{3\lambda}$, and $q_{4\sigma}$ into the last of Eqs. (B5) and then using the first three relations in (B5). Our final result for the off-shell $\pi\pi$ amplitude is given by

$$\begin{aligned}
F_\pi^4 T^{abcd}(q_3;q_1,q_2)&=q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}T^{abcd}_{\mu\nu\lambda\sigma}(q_3;q_1,q_2)\\
&\quad+\frac{F_\pi^2}{m_\pi^2}[(q_3^2+q_4^2+m_\pi^2)f^{abcd}(-q_1,q_2)+(q_1^2+q_2^2+m_\pi^2)f^{dcab}(q_4,-q_3)\\
&\quad+(q_2^2+q_4^2+m_\pi^2)f^{cabd}(q_3,q_1)+(q_1^2+q_3^2+m_\pi^2)f^{dbac}(q_4,q_2)\\
&\quad+(q_2^2+q_3^2+m_\pi^2)f^{dacb}(q_4,q_1)+(q_1^2+q_4^2+m_\pi^2)f^{cbad}(q_3,q_2)]
\end{aligned}$$

$$\begin{aligned}
&+\epsilon_{abe}\epsilon_{cde}\{\tfrac{1}{2}F_\pi^{\,2}[(q_4-q_3)_\sigma F_\sigma(-q_1,q_2)-(q_2-q_1)_\nu F_\nu(q_3,-q_4)]\\
&\qquad+\tfrac{1}{4}[C_V(t-u)+(q_2-q_1)_\nu(q_3-q_4)_\sigma\Delta^V_{\sigma\nu}(q_3+q_4)]\}\\
&+\epsilon_{ade}\epsilon_{bce}\{\tfrac{1}{2}F_\pi^{\,2}[(q_3+q_2)_\lambda F_\lambda(-q_1,-q_4)-(q_4+q_1)_\sigma F_\sigma(q_3,q_2)]\\
&\qquad+\tfrac{1}{4}[C_V(s-t)+(q_2+q_3)_\nu(q_1+q_4)_\sigma\Delta^V_{\nu\sigma}(q_2-q_3)]\}\\
&+\epsilon_{ace}\epsilon_{bde}\{\tfrac{1}{2}F_\pi^{\,2}[(q_4+q_2)_\sigma F_\sigma(-q_1,-q_3)-(q_3+q_1)_\lambda F_\lambda(q_4,q_2)]\\
&\qquad+\tfrac{1}{4}[C_V(s-u)+(q_2+q_4)_\nu(q_1+q_3)_\sigma\Delta^V_{\sigma\nu}(q_2-q_4)]\}\\
&-\frac{1}{m_\pi^{\,4}}[(q_1^{\,2}+q_2^{\,2}+m_\pi^{\,2})(q_3^{\,2}+q_4^{\,2}+m_\pi^{\,2})\Delta^{abcd}(s)+(q_1^{\,2}+q_3^{\,2}+m_\pi^{\,2})(q_2^{\,2}+q_4^{\,2}+m_\pi^{\,2})\Delta^{acbd}(t)\\
&\qquad+(q_1^{\,2}+q_4^{\,2}+m_\pi^{\,2})(q_2^{\,2}+q_3^{\,2}+m_\pi^{\,2})\Delta^{adcb}(u)]\\
&+\frac{F_\pi^{\,2}}{m_\pi^{\,2}}\,l_2(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})(\tfrac{3}{2}m_\pi^{\,2}-s-t-u)\\
&-F_\pi^{\,2}[\delta_{ab}\delta_{cd}(2s-m_\pi^{\,2})+\delta_{ac}\delta_{bd}(2t-m_\pi^{\,2})+\delta_{ad}\delta_{bc}(2u-m_\pi^{\,2})]\,,
\end{aligned}\qquad\text{(B6)}$$

where the parameter $l_2$ has been defined in Appendix A, and where the variables $s$, $t$, and $u$ are defined by

$$s=-(q_1+q_2)^2\,,\quad t=-(q_1-q_3)^2\,,\quad u=-(q_1-q_4)^2\,.$$

When Eq. (B6) is taken on the pion mass shells ($q_i^{\,2}=-m_\pi^{\,2}$, $i=1$ to 4), we obtain Eq. (1) of the text.

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