7A, 417 (1972).

[9]M. Yamada, Nuovo Cimento 4A, 866 (1971).

[10]K. Kawarabayashi, S. Kitakado, and H. Yabuki, Phys. Letters 28B, 432 (1969).

[11]One can see explicitly that the term

$$\frac{\Gamma(1-\alpha_{K^*}(u))\Gamma(1-\alpha_\rho(q^2))}{\Gamma(1-\alpha_{K^*}(u)-\alpha_\rho(q^2))}$$

is actually very small because of the denominator $\Gamma(1-\alpha_{K^*}(u)-\alpha_\rho(q^2))$. For particles on mass shell, one has $1-\alpha_{K^*}(u)-\alpha_\rho(q^2)\simeq\alpha_{K^*}(t)-0.5$. Then for t small [in the region of K_{l3} decays, $0<t<(m-\mu)^2$] $\alpha_{K^*}(t)-0.5$ is nearly zero.

[12]C. Y. Chien *et al.*, Phys. Letters 35B, 261 (1971). It should also be noted that a recent very-high-statistics experiment by V. Bisi *et al.* [Phys. Letters 36B, 533 (1971)] gives $\lambda_+=0.023\pm0.005$. It seems that the value of λ_+ measured by an experiment depends on the variation of its geometrical acceptance over the t range.

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Can the Pion's Charge Radius be Large? *

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Analyticity has been used to derive dispersive inequalities which bound the spacelike behavior of the pion's charge form factor in terms of the timelike variation of the modulus of the form factor and the p-wave $\pi\pi$ scattering phase shift. The large charge radius, suggested by the Serpukhov-UCLA measurement of πe scattering, is only compatible with timelike data which involve a large p-wave $\pi\pi$ phase shift just above threshold.

I. INTRODUCTION

In recent years there has been a rapid accumulation of data on the behavior of the pion's charge form factor, $F(t)$.[1] Colliding beam measurements[2] of $\sigma(e^+e^-\to\pi^+\pi^-)$ at Novosibirsk, Orsay, and Frascati have furnished information on the size of $|F(t)|$ for timelike momentum transfer: $16m_\pi^2\lesssim t\lesssim 4.4$ GeV^2. On the elastic cut, $t_0\equiv 4m_\pi^2\leq t\leq 16m_\pi^2$, the phase of $F(t)$ is equal to the $J=T=1$ $\pi\pi$ phase shift, $\delta_1(t)$ (modulo π). Chew-Low extrapolation techniques[3] can be used to infer $\delta_1(t)$ from observations of $\pi N\to\pi\pi N$. On the spacelike interval, $-1.2\lesssim t\lesssim -0.2$ GeV^2, the behavior of the form factor has been extracted from pion electroproduction experiments.[4] In addition, the recent Serpukhov-UCLA measurements[5] of pion-electron scattering have provided *direct* access to $|F(t)|$ at small spacelike momentum transfer, $-0.04\lesssim t\lesssim -0.02$ GeV^2.

These experimental results in the spacelike and timelike regions should be correlated by the analyticity of the form factor. The standard method of displaying that correlation is to write an ordinary dispersive equality[6]; i.e., Cauchy's theorem is used to express $F(t)$ at spacelike momentum transfer in terms of a polynomial ("subtractions") and an integral of Im$F(t)$ over the timelike domain. Thus, the computation of $F(t)$ at spacelike t requires knowledge of the following input: the number and size of subtraction constants as well as the behavior of the modulus and phase of $F(t)$ over the entire timelike cut. In the case of the problem considered here, this approach has the disadvantage that it is usually necessary to construct a model in order to estimate the subtraction constants[7] and to estimate the phase of the form factor for $t>16m_\pi^2$.

The correlation of spacelike and timelike experiments can be expressed in a more model-independent way by using analyticity and a smaller amount of timelike information to derive *bounds* on the spacelike form factor. The resulting dispersive inequalities[8-10] typically take the following form: Knowledge of $|F(t)|$ or an upper bound on $|F(t)|$ on the entire timelike cut is used to put upper and lower limits on the value of $F(t)$ for spacelike momentum transfer. This technique has several important advantages over ordinary dispersive equalities. First of all, it is not necessary to build models for the (experimentally inaccessible) phase of $F(t)$ for $t>16m_\pi^2$. Secondly, the input information includes only an upper bound on (not the value of) $|F(t)|$ in the timelike region. This is significant since two-photon effects[11] may be sizable at high timelike t; in that case, colliding beam measurements of $\sigma(e^+e^-\to\pi^+\pi^-)$ determine *only* an upper bound[12] on $|F(t)|$ (provided that the effects of three or more photons are negligible). Finally, the evaluation of dispersive inequalities does not require knowledge of the size of subtraction constants. Several authors[9,10] have computed these bounds using the available data on $\sigma(e^+e^- \to \pi^+\pi^-)$ (for $16 m_\pi^2 \lesssim t \lesssim 4.4$ GeV^2), supplemented with the reasonable extrapolation of this data into unexplored intervals ($t_0 \equiv 4m_\pi^2 \lesssim t \lesssim 16 m_\pi^2$, 4.4 GeV$^2 \leq t < \infty$). They conclude that the pion's charge radius (r_π) must be smaller than the naive vector-dominance value[13]; i.e., $r_\pi^2/r_{\rm VD}^2 \lesssim 1$. This bound is strongly violated by the value of r_π^2 indicated by a preliminary analysis of the Serpukhov-UCLA experiment[5]: $(r_\pi^2/r_{\rm VD}^2)_{\rm expt} \approx 2$.

In this paper we assume that the earlier numerical calculations erred in supposing that the timelike data could be "smoothly" extrapolated into the experimentally so far inaccessible regions: ($4m_\pi^2 \leq t \lesssim 16 m_\pi^2$ and 4.4 GeV$^2 \leq t < \infty$). We seek to learn what sort of timelike data will bring the bounds on the radius into agreement with the experimental result. This program is expedited by deriving a modified dispersive inequality: Knowledge of both the phase of $F(t)$ [the $T=J=1$ $\pi\pi$ phase shift, $\delta_1(t)$] on $4m_\pi^2 \leq t \leq 16 m_\pi^2$ and an upper bound on $|F(t)|$ on the *inelastic* cut ($16m_\pi^2 \leq t \leq \infty$) is used to put upper and lower limits on the spacelike form factor. Notice that the input information now involves the behavior of $\delta_1(t)$ [but not $|F(t)|$] on the elastic cut, $4m_\pi^2 \leq t \leq 16 m_\pi^2$. This is useful since $\delta_1(t)$ [but not $|F(t)|$] has been experimentally determined on this interval.[14] In addition, the shape of $\delta_1(t)$ [as opposed to $|F(t)|$] on this interval is meaningfully related to current theories of strong interaction dynamics. Therefore, the modified dispersive inequality is more ideally suited to exploit maximally the present collection of experimental and theoretical facts. A numerical evaluation of these bounds leads to the conclusion that the large experimental value of r_π^2 is consistent with the timelike data *only if $\delta_1(t)$ is quite large on the elastic cut.*

The analytical expressions for the modified bounds are stated and discussed in Sec. II. Numerical results are presented in Sec. III. Section IV contains the statement and discussion of stronger bounds, which follow from additional assumptions about the number and positions of zeros of the form factor. Section V summarizes the major conclusions of this paper. Appendixes A and B outline the proofs of the analytical results which were quoted in Secs. II and IV, respectively.

II. STATEMENT OF DISPERSIVE INEQUALITIES

The derivation of the bounds on the spacelike form factor exploits the following assumptions:

1. $F(\xi)$ is analytic in the complex ξ plane with a cut at $t_0 \equiv 4m_\pi^2 \leq \xi = \text{real} \leq \infty$. In addition, $F(\xi)$ must be polynomially bounded at infinity.
2. $F(\xi)$ is "real" in the sense that

$$F^*(\xi^*) = F(\xi) \ . \tag{1}$$

3. Let t_1 be any number greater than t_0. Then, we suppose that the phase[15] of $F(t)$, $\delta_F(t)$, is known on the interval $t_0 \leq t \leq t_1$. On that interval $F(t)$ must have no poles which are superimposed on the cut.

4. It is assumed that we also know a finite, positive definite function $w(t)$ (defined on the cut) which bounds $|F(t)|$ from above,

$$|F(t)| \leq w(t), \tag{2}$$

on $t_1 \leq t \leq \infty$ and which behaves as a power of t at infinity,

$$w(t) \underset{t\to\infty}{\sim} (\text{const})t^n \ , \tag{3}$$

where n is any real number. Actually this condition can be replaced by a weaker one which simply proscribes the exponential behavior of $w(t)$ as $t \to \infty$.

In Appendix A it is proved that these hypotheses lead to the following rigorous bounds on $F(a)$ in the spacelike region[16] ($a \leq 0$):

$$\begin{aligned} F(a) &\geq \exp[d(a)]\left(\frac{\exp[-d(0)]-x}{1-x\exp[-d(0)]}\right) \ , \\ F(a) &\leq \exp[d(a)]\left(\frac{\exp[-d(0)]+x}{1+x\exp[-d(0)]}\right) \ , \end{aligned} \tag{4}$$

where

$$x \equiv \frac{(t_1-a)^{1/2}-t_1^{1/2}}{(t_1-a)^{1/2}+t_1^{1/2}} \geq 0 \ ,$$

$$d(a) \equiv \frac{(t_1-a)^{1/2}}{\pi}\int_{t_1}^{\infty} dt \frac{\ln w(t)}{(t-a)(t-t_1)^{1/2}} + \frac{(t_1-a)^{1/2}}{\pi}\int_{t_0}^{t_1} dt \frac{\delta_F(t)}{(t-a)(t_1-t)^{1/2}} \ .$$

The above inequalities are transformed into bounds on r_π^2 when $F(0)=1$ is subtracted from each side and each side is divided by a in the limit $a \to 0$:

$$\begin{aligned} \tfrac{1}{6} r_\pi^2 &\leq \frac{\sinh[d(0)]-d(0)}{2t_1} + \bar{J} \ , \\ \tfrac{1}{6} r_\pi^2 &\geq \frac{-\sinh[d(0)]-d(0)}{2t_1} + \bar{J} \ , \end{aligned} \tag{5}$$

where

$$\mathcal{J} \equiv \frac{t_1^{1/2}}{\pi}\int_{t_1}^{\infty} dt\,\frac{\ln w(t)}{t^2(t-t_1)^{1/2}} + \frac{t_1^{1/2}}{\pi}\int_{t_0}^{t_1} dt\,\frac{\delta_F(t)}{t^2(t_1-t)^{1/2}} \quad .$$

Note the following properties of these inequalities:

(a) These bounds are the strongest ones which can be derived from the given input information. Section IV contains a discussion of stronger bounds which follow from additional assumptions about the number and positions of zeros of the form factor. Also, it is clear that these theorems can be easily applied to other form factors [such as the vertex for $e^+e^- \to \pi^0\gamma$ (see Ref. 17)] if the appropriate experimental input is available.

(b) The requirement that the upper bound exceed the lower bound in Eq. (5) implies that the timelike data themselves must obey the consistency condition: $d(0) \geq 0$. For the timelike data used here, $d(0)$ is positive and quite small [$d(0) \approx 0.2 - 0.4$].

(c) In the limit $t_1 \to t_0$, Eqs. (4) and (5) reduce to the dispersive inequalities derived in other papers.[8-10] Notice that a large value of t_1 tends to suppress the contribution of the (experimentally uncertain) high-energy region to Eq. (5). Therefore, choosing $t_1 = 16m_\pi^2$ (see below) instead of $t_1 = t_0 = 4m_\pi^2$ as in previous analyses reduces the uncertainty in the evaluation of Eq. (5) by a factor of 2.

(d) *For all practical purposes we will always choose* $t_1 = 16m_\pi^2$, the threshold of the inelastic cut. Then with one exception, $\delta_F(t)$ can be identified with the experimentally measurable $J = T = 1$ $\pi\pi$ phase shift, $\delta_1(t)$, on the elastic cut ($t_0 \leq t \leq t_1$). If $F(t_0) < 0$, then $\delta_F(t)$ may differ by the amount $\pm\pi$ from $\delta_1(t)$. However, in this case our final bound is in general strengthened; therefore, we will not discuss this possibility here. Note that since $F(0) = 1$, $F(t_0) < 0$ implies that $F(t)$ has a zero point in the interval $0 < t < t_0$. For a related discussion of such timelike zeros, see Sec. IV and Appendix B.

(e) If Eq. (4) or Eq. (5) is violated, it may be that the wrong timelike data [$\delta_F(t)$ and $w(t)$] was used or that the form factor is not analytic in the cut ξ plane. Alternatively, the discrepancy may be attributed to nonpolynomial behavior of $F(t)$ at infinity or to unphysical poles (superimposed on the cut) in $F(t)$ along $t_0 \leq t \leq t_1$.

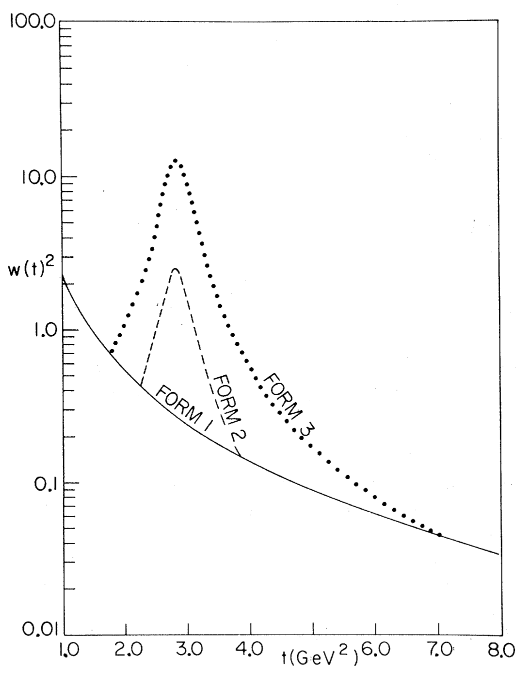

FIG. 1. Trial forms of $w(t)^2$, where $w(t)$ is supposed to bound $|F(t)|$ from above.

III. NUMERICAL EVALUATION OF BOUNDS

A. "Smooth" Timelike Data

The colliding-beam experiments[2] at Novosibirsk and Orsay have measured $|F(t)|$ for $0.34 \leq t \leq 1.04$ GeV^2. The data are reasonably well fitted by the modified, P-wave Breit-Wigner shape[18]:

$$|F(t)|^2 = \frac{0.399}{[(0.592-t)+(1.41)b(t)]^2+(1.99)k^6t^{-1}}\,, \tag{6}$$

where

$$b(t) = k^2[h(t) - 0.504] - (0.0425)(t - 0.592)\,,$$

$$h(t) = \frac{(0.637)k}{t^{1/2}}\ln\left(\frac{t^{1/2}+2k}{0.276}\right)\,,$$

$$k(t) = (0.5)(t - t_0)^{1/2}\,.$$

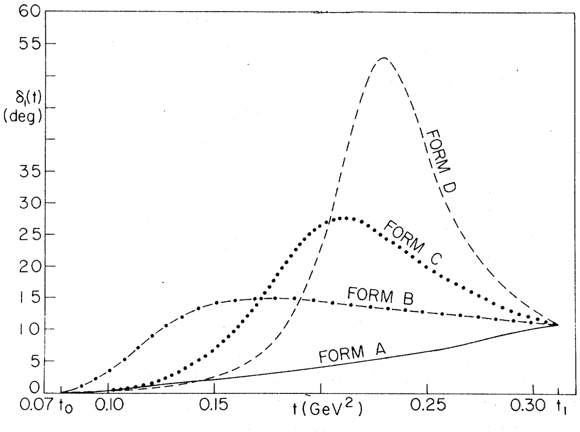

FIG. 2. Trial forms of the $J = T = 1$ $\pi\pi$ phase shift.

We will use the expression in Eq. (6) as an upper bound function, $w(t)^2$, on the interval $t_1 \equiv 16 m_\pi^2 = 0.312\ \mathrm{GeV}^2 \le t \le 1.0\ \mathrm{GeV}^2$. The experiments at Frascati[2] have determined $|F(t)|$ at a few points in the range, $2.0 \le t \le 4.4\ \mathrm{GeV}^2$. These measurements can be represented by a "smooth" curve:

$$|F(t)|^2 = \frac{2.16\ \mathrm{GeV}^4}{t^2} . \tag{7}$$

As a first guess we will use this same form for $w(t)^2$ for $1.0 \le t \le 4.4\ \mathrm{GeV}^2$. For simplicity it will also be assumed that Eq. (7) is an upper bound on $|F(t)|^2$ in the unexplored region, $t > 4.4\ \mathrm{GeV}^2$. The choice of $w(t)$ described in this paragraph corresponds to form 1 in Fig. 1.

Chew-Low extrapolation techniques[3] can be applied to the amplitude for $\pi N \to \pi\pi N$ in order to determine $\delta_1(t)$ [and, therefore, $\delta_F(t)$] on the elastic cut: $t_0 \le t \le 16 m_\pi^2 \equiv t_1$. Relatively abundant data[19] of this kind exist for $t \ge 0.25\ \mathrm{GeV}^2$. However, measurements in the low-energy region[20] ($t_0 = 4 m_\pi^2 = 0.078\ \mathrm{GeV}^2 \le t \le 0.25\ \mathrm{GeV}^2$) are quite scarce. All of these experimental results are reasonably well represented by the "smooth" effective-range shape,

$$(t - t_0)^{3/2} \cot\delta_1(t) = \frac{t_0 t^{1/2}}{a_1 m_\pi^3} + f t^{1/2}(t - t_0) , \tag{8}$$

where the scattering length (a_1) and the effective range (f) are given by

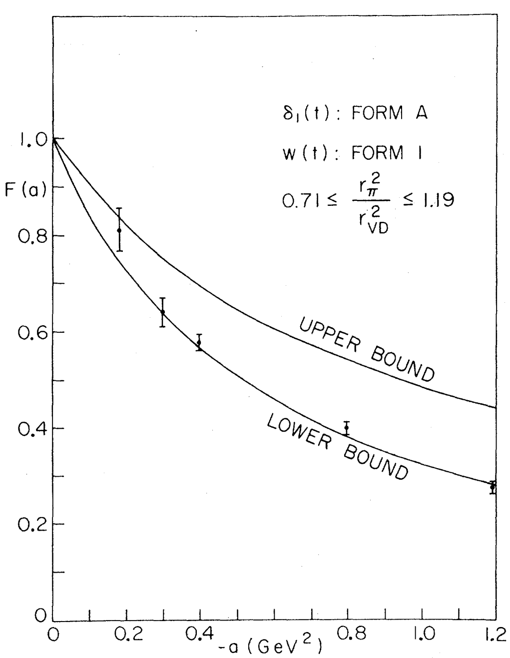

FIG. 3. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 1 of $w(t)$ and form A of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

$$a_1 m_\pi^3 = 0.05 ,$$
$$f = -2.21 .$$

Equation (8) is plotted as form A in Fig. 2.

When form 1 of $w(t)$ and form A of $\delta_1(t)$ are used to evaluate Eqs. (4) and (5), the resulting bounds on $F(a)$ and r_π^2 are those displayed in Fig. 3 and Table I. The upper and lower limits on r_π^2 are close to the ones derived in an earlier calculation.[10] The measurements of the spacelike form factor, extracted from electroproduction experiments at the CEA (Cambridge Electron Accelerator),[4] are also plotted in Fig. 3. There is complete agreement in that each point falls between the upper and lower bounds of the dispersive inequality. A Serpukhov-UCLA collaboration has determined the spacelike form factor at very small momentum transfer ($-0.04 \lesssim t \lesssim -0.02\ \mathrm{GeV}^2$) by observing πe scattering. Eventually, without any further assumptions each finite-t form factor measurement from this experiment can be compared with the bounds listed in Table I. Unfortunately, at the present time these finite-t measurements are not available. Instead the preliminary reports[5] of the experimental results have quoted only an effective radius of the pion; this value is obtained by differentiating a "single-pole" function which fits the whole collection of finite-t form factor measurements.[21] The most cautious interpretation of this fitting procedure indicates a large charge radius[22]:

$$\left[\frac{r_\pi^2}{r_{\mathrm{VD}}^2}\right]_{\mathrm{expt}} = 2.02 \pm 0.58 . \tag{9}$$

Since the bounds of Fig. 3 do not overlap the 1-standard-deviation range of Eq. (9), we are forced to conclude that the wrong timelike data [form 1 for $w(t)$ and form A for $\delta_1(t)$] was used, or that $F(\xi)$ is not analytic in the cut ξ plane, or that the form factor behaves nonpolynomially at infinity. Violations of analyticity and nonpolynomial behavior will not be considered in this paper. Instead we will try to learn what sort of timelike data are consistent with Eq. (9) in the sense that

TABLE I. Form 1 of $w(t)$ and form A of $\delta_1(t)$ produce these bounds on the form factor in the spacelike region explored by the scattering experiments.

$-a$ (GeV^2)	$F(a)$: lower bound	$F(a)$: upper bound
0.01	0.98	0.99
0.02	0.96	0.98
0.03	0.94	0.97
0.04	0.93	0.96
0.05	0.91	0.94

this data produce bounds on r_π^2 which overlap the 1-standard-deviation range of Eq. (9); that is, we require that the "consistent" timelike data yield an upper bound on r_π^2/r_{VD}^2 which is greater than 1.44. Now, Eq. (5) shows that the upper bound on r_π^2 increases monotonically with increases in $w(t)$ or $\delta_1(t)$. Therefore, timelike data which is "consistent" with Eq. (9) will involve a form of $w(t)$ greater than form 1 and/or a form of $\delta_1(t)$ greater than form A.

B. Larger $w(t)$

In this subsection we investigate the possibility that the correct $w(t)$ exceeds the "smooth" shape of form 1 [Eq. (7)] at high momentum transfer ($1.0\ \mathrm{GeV}^2 \leq t \leq \infty$). For example, this might be the case if there exists a high-mass vector meson (ρ'). For $t \approx m_{\rho'}^2$ the modulus of $F(t)$ would be expected to conform to a P-wave Breit-Wigner curve,

$$|F(t)|^2 \approx \frac{R_{\rho'}^2}{(t-m_{\rho'}^2)^2 + \Gamma_{\rho'}^2 k^6 m_{\rho'}^4/k_{\rho'}^6 t}\ , \qquad (10)$$

where

$$k = \tfrac{1}{2}(t-t_0)^{1/2}\ ,$$

$$k_{\rho'} = \tfrac{1}{2}(m_{\rho'}^2 - t_0)^{1/2}\ .$$

Furuichi *et al.*[23] have shown that a ρ' structure in $|F(t)|$ might be compatible with the Frascati data points if the mass, width, and strength parameters are

$$m_{\rho'}^2 = 2.84\ \mathrm{GeV}^2\ ,$$

$$\Gamma_{\rho'} = 0.15\ \mathrm{GeV}\ ,$$

$$R_{\rho'} = 0.4\ \mathrm{GeV}^2\ .$$

The effects of such a resonance can be estimated by choosing $w(t)$ at every point (on $1.0\ \mathrm{GeV}^2 \leq t < \infty$) to be the *larger* of the expressions in Eq. (7) and Eq. (10). The resulting upper-bound function is plotted as form 2 in Fig. 1. When $w(t)$ is given by form 2 and $\delta_1(t)$ is given by form A, the corresponding bounds on r_π^2 and $F(a)$ are those exhibited in Fig. 4. Once again the electroproduction points are compatible with the bounds on $F(a)$, but the Serpukhov-UCLA result violates the bounds on r_π^2.

The situation is not significantly improved even if the strength parameter is increased to the value: $R_{\rho'} = 0.9\ \mathrm{GeV}^2$. In that case the upper-bound function, $w(t)$, is increased to the highest curve (form 3) of Fig. 1. Since form 3 exceeds all Frascati data points by a wide margin, it should constitute a valid upper bound on any reasonable ρ' peak. Form 3 of $w(t)$ and form A of $\delta_1(t)$ produce the bounds listed in Fig. 5. The 1-standard-deviation range of the Serpukhov-UCLA data still does not

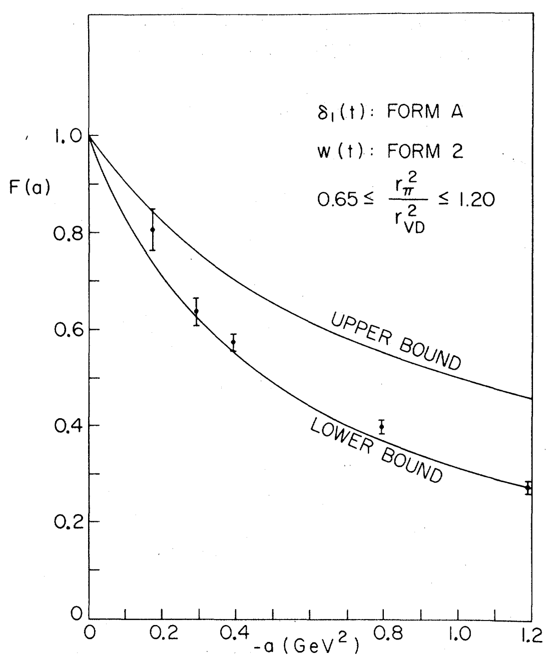

FIG. 4. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 2 of $w(t)$ and form A of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

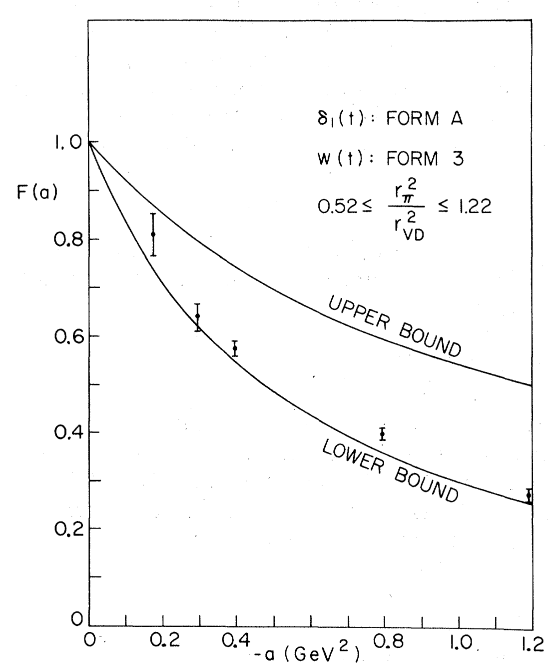

FIG. 5. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 3 of $w(t)$ and form A of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

overlap the bounds on r_π^2. In fact, the inclusion of a ρ' bump hardly increases the upper bound on r_π^2 at all.[24] A dimensional argument supports this observation: the ρ and ρ' contributions to r_π^2 should be proportional to m_ρ^{-2} and $m_{\rho'}^{-2}$, respectively. Since $m_{\rho'}^{-2} \ll m_\rho^{-2}$, the inclusion of a ρ' term is expected to change r_π^2 by only a small amount.

It is also possible that the correct form of $w(t)$ exceeds form 1 in the very high energy region ($t > 4.4$ GeV^2). For example, this might be true if the form factor is actually constant ("pointlike") for $t > 4.4$ GeV^2. In that case $w(t)^2$ should be given by Eq. (7) for $1.0 \leq t \leq 4.4$ GeV^2 and by

$$w(t)^2 = \frac{2.16}{(4.4)^2}$$

for $t > 4.4$ GeV^2. When this upper-bound function is used in conjunction with form A of $\delta_1(t)$, Eq. (5) produces the inequalities

$$0.0 \leq r_\pi^2 / r_{\rm VD}^2 \leq 1.2 \ .$$

Therefore, this method of increasing $w(t)$ does not make the dispersive inequalities consistent with Eq. (9).

The three examples in this subsection clearly demonstrate that *reasonable* increases in $w(t)$ alone will not bring the bounds on r_π^2 into agreement with the Serpukhov-UCLA result. It seems unlikely that $|F(t)|$ is really huge (say, asymptotically rising) at high momentum transfer. Therefore we can draw the following conclusion: *Bounds on r_π^2, which are consistent with the Serpukhov-UCLA result, will be produced only by timelike data, which includes a phase shift exceeding form A.*

C. Larger $\delta_1(t)$

In order to learn how much $\delta_1(t)$ ($t_0 \leq t \leq t_1$) must be increased, consider the following set of peaked phase shifts:

$$\delta_1(t) = \frac{R_B(t-t_0)^{3/2}}{(t-t_B)^2 + \Gamma_B^2} \ , \qquad (11)$$

where R_B, t_B, and Γ_B are phenomenological parameters to be determined. Since there is relatively reliable data[19] on $\delta_1(t)$ for $t \geq 0.25$ GeV^2, we will demand that all phase shifts agree with the experimental measurement at $t = t_1 = 16 m_\pi^2 = 0.312$ GeV^2,

$$\delta_1(t_1) = 11^\circ \ .$$

This single constraint reduces Eq. (11) to a two-parameter family of curves.[25]

A computer was used to evaluate the bounds when $\delta_1(t)$ is given by a representative sample of the above shapes and $w(t)$ is given by form 1. The following parameter choices determine phase shifts which yield bounds on r_π^2 *in marginal agreement with the Serpukhov-UCLA measurement:*

$$\text{form B} \quad \begin{cases} t_B = 0.105 \text{ GeV}^2 \ , \\ \Gamma_B = 0.065 \text{ GeV}^2 \ , \\ (a_1 m_\pi^3) = (10.6)(a_1 m_\pi^3)_{\rm ca} \ , \end{cases}$$

$$\text{form C} \quad \begin{cases} t_B = 0.195 \text{ GeV}^2 \ , \\ \Gamma_B = 0.05 \text{ GeV}^2 \ , \\ (a_1 m_\pi^3) = (1.1)(a_1 m_\pi^3)_{\rm ca} \ , \end{cases}$$

$$\text{form D} \quad \begin{cases} t_B = 0.225 \text{ GeV}^2 \ , \\ \Gamma_B = 0.03 \text{ GeV}^2 \ , \\ (a_1 m_\pi^3) = (0.42)(a_1 m_\pi^3)_{\rm ca} \ . \end{cases}$$

Here $(a_1 m_\pi^3)_{\rm ca} = 0.033$ is the current-algebra value for the scattering length.[26] These phase shifts are graphed as forms B, C, and D in Fig. 2; notice that all of these curves are significantly enhanced relative to form A. The corresponding bounds on r_π^2 and $F(a)$ are displayed in Fig. 6 – Table II,

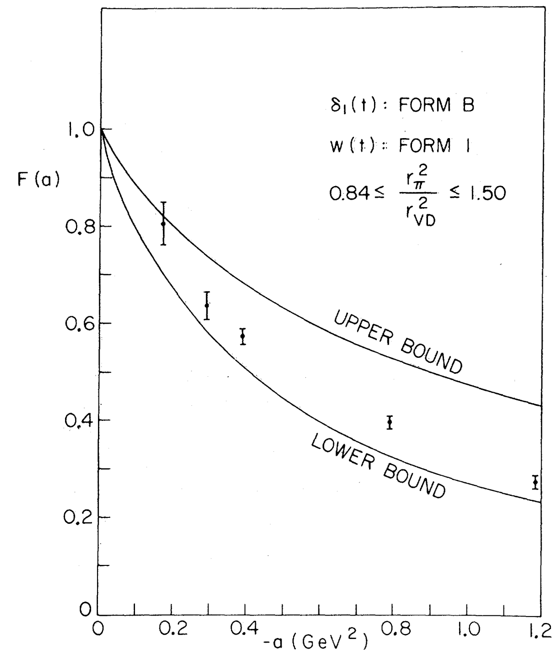

FIG. 6. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 1 of $w(t)$ and form B of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

TABLE II. Form 1 of $w(t)$ and form B of $\delta_1(t)$ produce these bounds on the form factor in the spacelike region explored by the scattering experiments.

$-a$ (GeV^2)	$F(a)$: lower bound	$F(a)$: upper bound
0.01	0.98	0.99
0.02	0.95	0.97
0.03	0.93	0.96
0.04	0.91	0.95
0.05	0.89	0.94

TABLE III. Form 1 of $w(t)$ and form C of $\delta_1(t)$ produce these bounds on the form factor in the spacelike region explored by the scattering experiments.

$-a$ (GeV^2)	$F(a)$: lower bound	$F(a)$: upper bound
0.01	0.98	0.99
0.02	0.95	0.97
0.03	0.93	0.96
0.04	0.91	0.95
0.05	0.89	0.94

Fig. 7 – Table III, and Fig. 8 – Table IV, respectively.

In each case the electroproduction points are still compatible with the bounds on $F(a)$, and the Serpukhov-UCLA result is marginally in accord with the bounds on r_π^2. Since the upper bound on r_π^2 depends monotonically on $\delta_1(t)$, phase shifts which exceed forms B, C, and D will produce bounds which overlap even more with Eq. (9). Conversely, any phase shift which is not as large as forms B, C, and D will give bounds inconsistent with Eq. (9).

D. Discussion

The three preceding subsections show that a large pionic charge radius of the Serpukhov-UCLA variety must be associated with a significantly enhanced low-energy $J=T=1$ $\pi\pi$ phase shift. The only alternatives [violation of analyticity or a huge $|F(t)|$ at high momentum transfer] seem even more unlikely.

Experimental evidence neither supports nor decisively rules out the notion of an enhanced phase shift. The meager experimental measurements[20] of $\delta_1(t)$ for $t<0.25$ GeV^2 indicate a phase shift of form A rather than forms B, C, or D. However, these data may not be completely reliable since they depend on a delicate Chew-Low extrapolation of the amplitude for $\pi N \to \pi\pi N$. From an analysis of K_{e4} decays it is possible to determine the average low-energy value of $\langle \delta_0^0 - \delta_1 \rangle$, where δ_0^0 is the $J=T=0$ $\pi\pi$ phase shift. If $\delta_1(t)$ is negligibly small as in form A the recent K_{e4} experiments[27] give an s-wave scattering length which exceeds the cur-

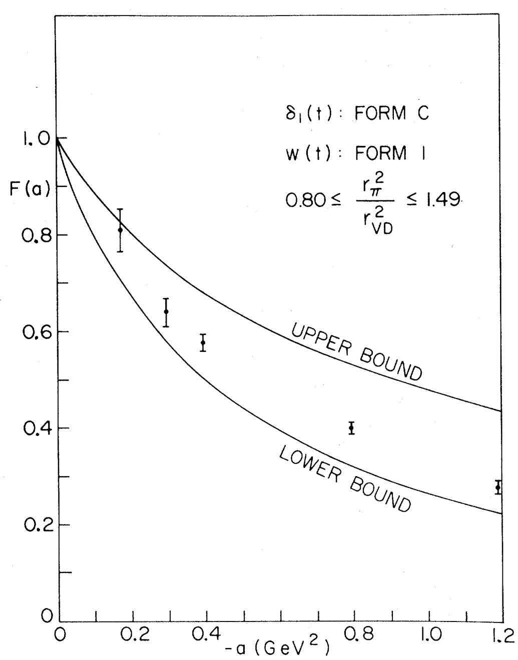

FIG. 7. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 1 of $w(t)$ and form C of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

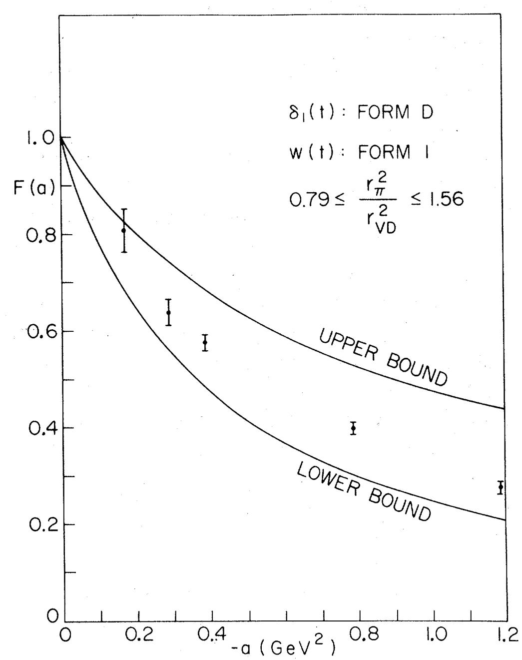

FIG. 8. The bounds on r_π^2 and the spacelike form factor corresponding to the choice of timelike data: form 1 of $w(t)$ and form D of $\delta_1(t)$. The experimental (electroproduction) points are from Ref. 4.

TABLE IV. Form 1 of $w(t)$ and form D of $\delta_1(t)$ produce these bounds on the form factor in the spacelike region explored by the scattering experiments.

$-a$ (GeV^2)	$F(a)$: lower bound	$F(a)$: upper bound
0.01	0.97	0.99
0.02	0.95	0.97
0.03	0.93	0.96
0.04	0.90	0.95
0.05	0.88	0.94

rent-algebra prediction[28] by 2 standard deviations. Therefore, an enhanced $\delta_1(t)$ of types B, C, or D will result in an s-wave scattering length which disagrees even more violently with current algebra.

Most theoretical work indicates that $\delta_1(t)$ takes form A. However, these same theoretical considerations generally involve some sort of "smoothness" assumption which would be violated by a large, rapidly varying phase shift of types B, C, and D. Therefore, these theoretical calculations cannot rule out the existence of an enhanced phase shift; rather, they can only predict some of the properties of $\delta_1(t)$ under the assumption that it is a smooth function. For example, the current-algebra scattering length[26] is much smaller than the scattering length of phase shift B; but, this does not imply that form B is totally out of the question. From the beginning the current-algebra proof assumes a small value of a_1 in that the $\pi\pi$ scattering amplitude is assumed to be "smooth" (have a "weak cut"). Therefore, current algebra cannot be used to demonstrate that the scattering length is small; rather, it can only predict the magnitude of a_1 *under the assumption* that a_1 is small. This point of view has been emphasized by Sucher and Woo.[29] Thus, there are no strong theoretical arguments which show that $\delta_1(t)$ is small. On the other hand, it is certainly difficult to find theoretical models which would explain an enhanced phase shift. For instance, form B, which has a very large positive scattering length, is traditionally associated with a $\pi\pi$ scattering amplitude containing a bound state below threshold.[30] However, from a theoretical point of view, it is very risky to make a mechanical extrapolation of a phase shift formula to the region below threshold. In addition, from a phenomenological point of view, a bound state below threshold would be stable against strong decays and should have been observed already. It should be mentioned that several authors[31] have bounded the hadronic part of $(g_\mu - 2)$ from below in terms of $r_\pi{}^2$ and the p-wave $\pi\pi$ phase shift. These lower bounds exceed the expected (ρ-dominance) contribution to $(g_\mu - 2)$ if $r_\pi{}^2$ is assigned the Serpukhov-UCLA value and the p-wave phase shift is represented by form A. It would be interesting to know how this lower bound changes if the phase shift is increased to form B, C, or D.

In summary: The dispersive inequalities show that the large measured pion radius requires the existence of a large low-energy p-wave $\pi\pi$ phase shift. Although this possibility seems unlikely, current experimental and theoretical evidence does not necessarily exclude such an enhanced phase shift; on the other hand, the origin of the required enhancement is quite puzzling.

IV. ZERO-DEPENDENT BOUNDS

Equation (4) comprises the strongest inequality which can be proved from the four assumptions in Sec. II. However, stronger bounds can be derived if these four input assumptions are supplemented by an additional hypothesis concerning the zeros of the form factor. In order to state that extra assumption, first define the following functions:

$$\Omega(\xi) \equiv \exp\left(\frac{\xi}{\pi}\int_{t_0}^{t_1} dt\, \frac{\delta_F(t)}{t(t-\xi)}\right) ,$$

$$\bar{f}(\xi) \equiv \frac{F(\xi)}{\Omega(\xi)} .$$

Then, the added input assumption is that $\bar{f}(\xi)$ has one and only one zero[32] (at a known location) in the complex ξ plane with a cut at $t_1 \leq \xi = \text{real} \leq \infty$. As shown in Appendix B, it follows that this zero must be positioned on the real axis (say, at $\xi = t_N$). Furthermore, it can be proved that the above assumptions lead to the bounds listed below. For $t_N \leq a < 0$:

$$F(a) \leq \left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1-x}{1+x}\right)\right] ,$$
$$F(a) \geq \left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1+x}{1-x}\right)\right] . \tag{12a}$$

For $a < t_N < 0$:

$$F(a) \leq \left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1+x}{1-x}\right)\right] ,$$
$$F(a) \geq \left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1-x}{1+x}\right)\right] . \tag{12b}$$

For $a < 0$ and $t_N > 0$:

$$F(a) \leq -\left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1-x}{1+x}\right)\right] , \tag{12c}$$
$$F(a) \geq -\left(\frac{\alpha - x}{1-\alpha x}\right) \exp\left[d(a) - [d(0) + \ln|\alpha|]\left(\frac{1-x}{1+x}\right)\right] .$$

Here, a, x, and $d(a)$ are defined as in Sec. II while α is given by

$$\alpha \equiv \frac{(t_1 - t_N)^{1/2} - t_1^{1/2}}{(t_1 - t_N)^{1/2} + t_1^{1/2}} .$$

In the limit that $t_N \to -\infty$ or $t_N \to t_1$, these inequalities become bounds which follow from the assumption that $\bar{f}(\xi)$ has *no* zeros in the cut ξ plane:

$$F(a) \leq \exp\left[d(a) - d(0)\left(\frac{1-x}{1+x}\right)\right] ,$$
$$F(a) \geq \exp\left[d(a) - d(0)\left(\frac{1+x}{1-x}\right)\right] . \tag{13}$$

If we let $a \to 0$, then this gives

$$|F'(0) - d'(0)| \leq \frac{1}{2t_1} d(0) . \tag{13'}$$

The following observations should be made:

(a) Equations (12) and (13) are the strongest bounds which can be proved from the specified input hypotheses. As expected, these inequalities are all more restrictive than Eq. (4). For different values of t_N, Eqs. (12) and (13) describe overlapping sets of bounds, each of which lies completely inside the overall range described by Eq. (4).

(b) The requirement that the upper bound exceed the lower bound in Eqs. (12) and (13) leads to the following restrictions on the positions of possible zeros: If $\bar{f}(\xi)$ has one and only one zero and it is located at $\xi = t_N < 0$, then t_N must lie in the following interval:

$$-\infty < t_N \leq \frac{-4t_1 \exp[-d(0)]}{\{1 - \exp[-d(0)]\}^2} . \tag{14}$$

If $\bar{f}(\xi)$ has one and only one zero and it is located at $\xi = t_N > 0$, then t_N is restricted to

$$\frac{4t_1 \exp[-d(0)]}{\{1 + \exp[-d(0)]\}^2} \leq t_N < t_1 . \tag{15}$$

For typical timelike data [e.g., form 1 of $w(t)$, form A of $\delta_1(t)$, and $t_1 = 16m_\pi^2$], these conditions are

$$-\infty < t_N \leq -20 \text{ GeV}^2$$

and

$$0.308 \text{ GeV}^2 \leq t_N < t_1 = 0.312 \text{ GeV}^2.$$

(c) In the limit $t_1 \to t_0$, Eqs. (13) and (13′) reduce to a result derived independently by other authors.[33]

(d) In principle, these stronger inequalities can be used to learn something about the number and positions of zeros of the form factor. For instance, if all the experimental measurements of the spacelike form factor do not lie within the bounds associated with a particular value of t_N, then we could rule out the possibility that $\bar{f}(\xi)$ has a single zero which is located at t_N. In practice, it appears that the present experimental data is not sufficiently accurate to draw such conclusions. As an illustration, suppose that we ignore the Serpukhov-UCLA experiment and use forms 1 and A to represent $w(t)$ and $\delta_1(t)$, respectively. Then it turns out that all of the lower bounds described by Eqs. (12) and (13) for $t_N < 0$ barely lie above the overall lower bound of Eq. (4). Since the electroproduction points in Fig. 3 also fall near the lower bound of Eq. (4), the experimental error bars overlap with the bounds associated with all possible $t_N < 0$. Therefore, we cannot learn anything about possible spacelike zeros other than what we already knew from Eq. (14). Equations (12) and (13) will be useful only when the electroproduction measurements are more accurate or are taken at higher momentum transfer.

V. CONCLUSIONS

Analyticity has been used to derive dispersive inequalities which bound the spacelike behavior of the pion's charge form factor in terms of the timelike variation of the modulus of the form factor and the p-wave $\pi\pi$ scattering phase shift. These techniques provide a clean method of exploiting maximally the current experimental data on the pion's form factor. A large variety of timelike data produces numerical bounds on the spacelike form factor which are compatible with recent pion electroproduction measurements. However, the large charge radius, suggested by Serpukhov-UCLA observations of πe scattering, is only compatible with timelike data which involves a large p-wave phase shift just above threshold. Independent experimental and theoretical considerations neither support nor decisively rule out the existence of such an enhanced phase shift.

Also, if a few points in the spacelike region are accurately measured and if these data are used as input, then we can improve our bounds for other spacelike points by a method similar to the one given in Appendix B. Moreover, if $|F(t)|$ on the interval $t_0 \leq t \leq t_1$ is measured in colliding beam experiments, we can test its consistency with related dispersive inequalities.[16] At any rate, a more precise measurement of $\delta_1(t)$ in the low-energy region or of $|F(t)|$ in the interval $t_0 \leq t \leq t_1$ is clearly desirable.

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APPENDIX A: PROOF OF GENERAL BOUNDS

In the following we outline the method of proof which is used to deduce the dispersive inequalities [Eq. (4)] from assumptions 1–4 in Sec. II.

To begin with, it is necessary to define a few related functions. First, let $\Omega(\xi)$ denote the Omnès-type function:

$$\Omega(\xi) \equiv \exp\left(\frac{\xi}{\pi}\int_{t_0}^{t_1} dt\, \frac{\delta_F(t)}{t(t-\xi)}\right) . \tag{A1}$$

$\Omega(\xi)$ is a "real" function which is analytic and nonvanishing in the complex ξ plane with a cut at $t_0 \le \xi = \text{real} \le t_1$. Hereafter we assume that $\delta_F(t)$ is continuous on the interval $t_0 \le t \le t_1$. If we were to allow $\delta_F(t)$ to be discontinuous by an amount π (or $-\pi$) at a point $t=\lambda$ $(t_0 < \lambda < t_1)$, then $\Omega(\xi)$ could have a pole (or zero point) at $\xi = \lambda$ (in addition to the cut). Anyway, $\Omega(\xi)$ satisfies

$$\Omega(0) = 1 ,$$

$$\Omega(t) \underset{t\to+\infty}{\longrightarrow} \text{finite nonzero constant} , \tag{A2}$$

and also

$$\Omega(t+i\epsilon) = |\Omega(t)|\, e^{i\delta_F(t)} , \tag{A3}$$

where $\epsilon > 0$ and $t_0 \le t \le t_1$. Next, define $\bar{f}(\xi)$:

$$\bar{f}(\xi) \equiv \frac{F(\xi)}{\Omega(\xi)} . \tag{A4}$$

$\bar{f}(\xi)$ is a "real" function which is analytic[34] in the ξ plane with a cut at $t_1 \le \xi = \text{real} \le \infty$. Notice that the cuts of $F(\xi)$ and $\Omega(\xi)$ have "canceled" so that $\bar{f}(\xi)$ is analytic along $t_0 \le \xi = \text{real} \le t_1$. $\bar{f}(\xi)$ obeys the normalization condition $\bar{f}(0) = 1$ and also satisfies (see assumption 4 in Sec. II)

$$|\bar{f}(t)| \le \bar{w}(t) \equiv \frac{w(t)}{|\Omega(t)|} \tag{A5}$$

for all $t \ge t_1$. The asymptotic behavior of $w(t)$ and $\Omega(t)$ [see Eq. (A2) and assumption 4 in Sec. II] guarantees the existence of the exponentiated integral:

$$\bar{G}(\xi) \equiv \exp\left(\frac{-i(\xi - t_1)^{1/2}}{\pi}\int_{t_1}^{\infty} dt\, \frac{\ln \bar{w}(t)}{(t-\xi)(t-t_1)^{1/2}}\right) . \tag{A6}$$

Here the square root is taken to have a branch cut along the positive real axis with positive values on the upper cut. $\bar{G}(\xi)$ is a "real" function which is analytic and nonvanishing in the ξ plane with a cut at $t_1 \le \xi = \text{real} \le \infty$. In addition, it can be easily seen that $\bar{G}(\xi)$ satisfies the "boundary condition"

$$|\bar{G}(t)| = \bar{w}(t) \tag{A7}$$

for all $t \ge t_1$ on the cut. Finally, let $\bar{B}(\xi)$ denote the following ratio:

$$\bar{B}(\xi) \equiv \frac{\bar{f}(\xi)}{\bar{G}(\xi)} . \tag{A8}$$

If follows that $\bar{B}(\xi)$ is a "real" function which is analytic in the ξ plane with a cut at $t_1 \le \xi = \text{real} \le \infty$. It is normalized at the origin such that

$$\bar{B}(0) = \bar{G}(0)^{-1} > 0 \tag{A9}$$

and obeys the simple "boundary condition"

$$|\bar{B}(t)| \le 1 \tag{A10}$$

for all $t \ge t_1$. Because of our assumption of polynomial boundedness for $F(t)$ and of a similar property for $w(t)$, $\bar{B}(\xi)$ is also polynomially bounded at infinity. Then, a version of the Lindelöf-Phragmén theorem due to Nevanlinna[35] may be used to prove that the inequality (A10) is valid at infinity; it follows from the maximum modulus theorem that $|\bar{B}(\xi)| \le 1$ *for all* ξ *in the entire cut plane*. Note that the polynomial boundedness of $\bar{B}(\xi)$ is crucial for the validity of this result. For example, an exponentially increasing function $f(\xi) = \exp[(t_1 - \xi)^{1/2}]$ satisfies $|f(t+i\epsilon)| = 1$ on the entire cut; nevertheless, it leads to $f(\xi) \to \infty$ along the negative axis in contradiction to the last inequality. Finally, observe that the previous definitions[36] imply

$$F(\xi) = \Omega(\xi)\bar{G}(\xi)\bar{B}(\xi) . \tag{A11}$$

Now, suppose a is some spacelike momentum transfer $(a \le 0)$. We want to find bounds on $F(a)$ in terms of $\delta_F(t)$ $(t_0 \le t \le t_1)$ and $w(t)$ $(t_1 \le t \le \infty)$. Since $\Omega(\xi)$ and $\bar{G}(\xi)$ are completely determined by $\delta_F(t)$ and $w(t)$, Eq. (A11) shows that $F(a)$ will be bounded by $\delta_F(t)$ and $w(t)$ if we can find upper and lower limits on $\bar{B}(a)$ in terms of $\delta_F(t)$ and $w(t)$. It is convenient to use the following mapping in order to transfer the problem from the ξ plane (complex t plane) to the z plane:

$$(\xi - t_1)^{1/2} = i t_1^{1/2}\left(\frac{1+z}{1-z}\right) . \tag{A12}$$

This transformation maps the whole ξ plane into the open unit disk in the z plane, the upper and lower cuts in the ξ plane (at $t_1 \le \xi = \text{real} \le \infty$) onto the lower and upper unit semicircles in the z plane, and the points $\xi = 0, t_1, \infty$ into $z = 0, -1, +1$. In the z plane the representative of $\bar{B}(\xi)$ is $\bar{\beta}(z)$:

$$\bar{\beta}(z) \equiv \bar{B}[\xi(z)] . \tag{A13}$$

If we denote x $(0 \le x < 1)$ as the z-plane image of $\xi = a$, then the problem is to bound $\bar{\beta}(x)$ in terms of

$\delta_F(t)$ and $w(t)$. First note that the properties of $\bar{B}(\xi)$ imply that $\bar{\beta}(z)$ is a "real" function which is analytic on the open unit disk, $|z|<1$. Also Eq. (A9), Eq. (A10), and the subsequent remarks require

$$\bar{\beta}(0)=\bar{G}(0)^{-1}>0 \ , \quad |\bar{\beta}(z)|\leq 1 \tag{A14}$$

for all $|z|\leq 1$.

For the moment suppose that $\bar{\beta}(z)$ is not identically unity on the closed unit disk. Then, the maximal modulus theorem implies that

$$|\bar{\beta}(z)|<1$$

for all $|z|<1$. In that case $\bar{\beta}(z)$ satisfies the hypotheses of the Schwarz lemma,[37] it follows that

$$\left|\frac{\bar{\beta}(z)-\bar{\beta}(0)}{z[1-\bar{\beta}(0)\bar{\beta}(z)]}\right|\leq 1 \tag{A15}$$

for all $|z|<1$. At the real point x, $\bar{\beta}(x)$ is real; therefore, Eq. (A15) becomes

$$-1\leq\frac{\bar{\beta}(x)-\bar{\beta}(0)}{x[1-\bar{\beta}(0)\bar{\beta}(x)]}\leq 1 \tag{A16}$$

or

$$\frac{\bar{\beta}(0)-|x|}{1-|x|\bar{\beta}(0)}\leq\bar{\beta}(x)\leq\frac{\bar{\beta}(0)+|x|}{1+|x|\bar{\beta}(0)} \ . \tag{A17}$$

The last relation also holds even if $\bar{\beta}(z)$ is identically unity on the unit disk; therefore, Eq. (A17) is true *in every case*. Now, Eq. (A14) and (A17) can be combined to give

$$\frac{\bar{G}(0)^{-1}-|x|}{1-|x|\bar{G}(0)^{-1}}\leq\bar{\beta}(x)=\bar{B}(a)\leq\frac{\bar{G}(0)^{-1}+|x|}{1+|x|\bar{G}(0)^{-1}} \ . \tag{A18}$$

Since $\bar{G}(0)$ is known in terms of $\delta_F(t)$ and $w(t)$, we have succeeded in bounding $\bar{\beta}(x)$ or $\bar{B}(a)$ in terms of $\delta_F(t)$ and $w(t)$. Substituting Eq. (A18) into Eq. (A11) gives upper and lower limits on $F(a)$ in terms of $\delta_F(t)$ $(t_0\leq t\leq t_1)$ and $w(t)$ $(t_1\leq t\leq\infty)$. Algebraic manipulation cast these bounds in the form of Eq. (4).

Before closing this section, it is worth making the following technical observations: First, we could have simplified our proof considerably if we had used

$$\hat{\omega}(\xi)=\exp\left(\frac{1}{\pi}(t_1-\xi)^{1/2}\int_{t_0}^{t_1}dt\,\frac{\delta_F(t)}{(t-\xi)(t_1-t)^{1/2}}\right) \tag{A19}$$

instead of $\Omega(\xi)$ as given in Eq. (A1). $\hat{\omega}(\xi)$ has properties similar to those of $\Omega(\xi)$ in addition to the fact that it also satisfies

$$|\hat{\omega}(\xi+i\epsilon)|=1 \tag{A20}$$

on the cut $\xi\geq t_1$. Secondly, if $\bar{f}(\xi)$ should develop a pole at $t=t_0$, it is convenient to use

$$f(\xi)=P(\xi)\bar{f}(\xi) \ , \tag{A21}$$

$$P(\xi)=\left(\frac{(t_1-t)^{1/2}-(t_1-t_0)^{1/2}}{(t_1-t)^{1/2}+(t_1-t_0)^{1/2}}\right)^n \ . \tag{A22}$$

Then, $f(\xi)$ has no pole and satisfies

$$|f(\xi+i\epsilon)|=|\bar{f}(\xi+i\epsilon)| \tag{A23}$$

for $\xi\geq t_1$. On the other hand, if $\bar{f}(\xi)$ has a zero point (instead of pole) at $\xi=t_0$, then n can be taken to be negative in (A22). In that case, the resulting dispersive inequality will be much stronger than previous ones. Finally, if $F(b)$ $(b\leq 0)$ is accurately measured in future experiments and if the data are used as input, then we can improve our inequality for other spacelike points by exploiting the Schwarz lemma again. To see this, let λ be the image of b under the mapping in Eq. (A12), and set

$$\gamma(z)=\frac{1-\lambda^*z}{z-\lambda}\,\frac{\bar{\beta}(z)-\bar{\beta}(\lambda)}{1-\bar{\beta}(\lambda^*)\bar{\beta}(z)} \ .$$

Then, the techniques of this section can be used to show

$$|\gamma(z)|\leq 1$$

for all $|z|\leq 1$. This process can be repeated if more than one spacelike point is to be used as input.

APPENDIX B: PROOF OF ZERO-DEPENDENT BOUNDS

This appendix sketches the proof of the stronger inequality [Eq. (12)], which follows from the original four assumptions of Sec. II supplemented by an additional assumption about the nature of the zeros of the form factor.

The additional hypothesis is that $\bar{f}(\xi)$ [defined in Eq. (A4)] has one and only one zero[32] (at a known location) in the cut ξ plane. Since "reality" implies that this zero must be on the real axis, its position will be denoted by t_N. Let α be the image of $\xi=t_N$ under the transformation, Eq. (A12). Then, in analogy to the proof in Appendix A, we seek to bound $\bar{\beta}(x)$ in terms of α, $\delta_F(t)$ $(t_0\leq t\leq t_1)$, and $w(t)$ $(t_1\leq t\leq\infty)$.

The added assumption mentioned above implies that $\bar{\beta}(z)$ has one and only one zero (located at $z=\alpha$) in the entire open unit disk. In order to exploit this extra information, define a new function $\bar{\beta}_1(z)$:

$$\bar{\beta}_1(z)\equiv\frac{|\alpha|(1-\alpha z)}{\alpha(\alpha-z)}\bar{\beta}(z) \ . \tag{B1}$$

$\bar{\beta}_1(z)$ is a "real" function which is analytic and nonvanishing on the open unit disk. It is normalized such that

$$\bar{\beta}_1(0) = |\alpha|^{-1}\bar{G}(0)^{-1} > 0 \quad \text{(B2)}$$

and obeys the "boundary condition"

$$|\bar{\beta}_1(e^{i\theta})| \leq 1 \ . \quad \text{(B3)}$$

Therefore, the maximum modulus theorem requires that either $|\bar{\beta}_1(z)|<1$ for $|z|<1$ or $\bar{\beta}_1(z) = \bar{\beta}_1(0)$ for all $|z| \leq 1$.

Assume for the moment that $\bar{\beta}_1(z)$ is not identically unity on the unit disk. Then, the above statements imply

$$|\bar{\beta}_1(z)|<1 \quad \text{(B4)}$$

for all $|z|<1$. In this case it is possible to define another function:

$$\bar{\gamma}_1(z) \equiv \frac{\ln\bar{\beta}_1(z) - \ln\bar{\beta}_1(0)}{z[\ln\bar{\beta}_1(z) + \ln\bar{\beta}_1(0)]} \ . \quad \text{(B5)}$$

$\bar{\gamma}_1(z)$ is also analytic on the open unit disk. In addition, a straightforward computation, which utilizes Eqs. (B3) and (B4), shows that

$$|\bar{\gamma}_1(e^{i\theta})| \leq 1 \ . \quad \text{(B6)}$$

Applying the maximum modulus theorem to $\bar{\gamma}_1(z)$, we find that

$$|\bar{\gamma}_1(z)| \leq 1 \quad \text{(B7)}$$

for all $|z|<1$. At $z=x$, this relation becomes

$$-1 \leq \frac{\ln\bar{\beta}_1(x) - \ln\bar{\beta}_1(0)}{x[\ln\bar{\beta}_1(x) + \ln\bar{\beta}_1(0)]} \leq 1 \quad \text{(B8)}$$

or

$$[\bar{\beta}_1(0)]^{(1+|x|)/(1-|x|)} \leq \bar{\beta}_1(x) \leq [\bar{\beta}_1(0)]^{(1-|x|)/(1+|x|)} \ . \quad \text{(B9)}$$

The above inequality also holds if $\bar{\beta}_1(z)$ is identically unity on the unit disk; therefore, Eq. (B9) is valid *in every case*. Equations (B1) and (B2) transform the last equation into

$$[|\alpha|\bar{G}(0)]^{-(1+|x|)/(1-|x|)} \leq \frac{|\alpha|(1-\alpha x)}{\alpha(\alpha-x)}\bar{\beta}(x) \leq [|\alpha|\bar{G}(0)]^{-(1-|x|)/(1+|x|)} \ . \quad \text{(B10)}$$

Since $\bar{G}(0)$ is determined by $\delta_F(t)$ and $w(t)$, Eq. (B10) constitutes the desired bound on $\bar{\beta}(x)$ in terms of α, $\delta_F(t)$, and $w(t)$. Substituting Eq. (B10) into Eq. (A11) gives upper and lower limits on $F(a)$ in terms of α, $\delta_F(t)$, and $w(t)$; the result can be put in the form of the bound to be proved, Eq. (12).

If $F(\xi)$ has no zero point in the entire cut plane, then we replace $\bar{\beta}_1(z)$ by $\bar{\beta}(z)$ in Eq. (B5). The result is equivalent to letting $t_N \to t_1$ in Eq. (13), as is stated in the text.

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[1]Notation: The pion's form factor is normalized such that $F(0)=1$; the electromagnetic radius $\langle r_\pi^2\rangle$ of the pion is defined as $F'(0) = \frac{1}{6}r_\pi^2$; timelike momentum transfer corresponds to $t>0$.

[2]Novosibirsk: V. Auslander *et al.*, Phys. Letters 25B, 433 (1967); Orsay: J. Augustin *et al.*, Phys. Rev. Letters 20, 129 (1968); Phys. Letters 28B, 508 (1969); Frascati: G. Salvini, in *Particles and Fields-1971*, edited by A. C. Melissinos and P. F. Slattery (A.I.P., New York, 1971); R. Wilson, rapporteur's talk, in *Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, 1970* (Atomizdat, Moscow, 1971). Also, R. Garland, Ph.D. thesis, Columbia University, 1971 (unpublished), measures $\langle r_\pi^2\rangle = 0.208 \pm 0.61$ F^2 from a $\pi^- p \to n e^+ e^-$ experiment.

[3]For a review of meson-meson scattering, see J. L. Petersen, Phys. Reports 2, 155 (1971).

[4]C. N. Brown *et al.*, Phys. Rev. Letters 26, 991 (1971). The form factor can be indirectly deduced from this data with the aid of a model for electroproduction. The uncertainties in any data attributed to Brown *et al.* are experimental statistical uncertainties; the systematic uncertainties are approximately the same size.

[5]P. Shepard, report presented at the 1972 meeting of American Physical Society (unpublished). This report was based on an analysis of 25% of the data.

[6]For a recent calculation of this sort, see C. F. Cho and J. J. Sakurai, Lett. Nuovo Cimento 2, 7 (1971).

[7]One subtraction constant is determined by the normalization condition, $F(0)=1$.

[8]B. V. Geshkenbein, Yad. Fiz. 9, 1232 (1969) [Sov. J. Nucl. Phys. 9, 720 (1969)].

[9]I. Raszillier, Lett. Nuovo Cimento 2, 349 (1971); Institute of Physics (Bucharest) report, 1971 (unpublished).

[10]D. N. Levin, V. S. Mathur, and S. Okubo, Phys. Rev. D 5, 912 (1972).

[11]S. J. Brodsky, T. Kinoshita, and H. Terazawa, Phys. Rev. Letters 25, 972 (1970).

[12]See "Note added" in Ref. 10.

[13]The vector-dominance value for r_π^2 is $r_{VD}^2 = 6/m_\rho^2$.

[14]J. H. Scharenguivel *et al.*, Nucl. Phys. B22, 16 (1970).

[15]The phase of $F(t)$, $\delta_F(t)$, is defined so that

$$F(t+i\epsilon) = |F(t)|e^{i\delta_F(t)} \quad (\epsilon > 0).$$

$\delta_F(t)$ is expressed in radians unless noted otherwise.

[16]The same method of proof can be used to find upper and lower bounds on $F(a)$ for $0 \le a \le t_0$ and on $|F(a)|$ for $t_0 \le a \le t_1$.

[17]Some nonoptimal bounds on the vertex function for $e^+e^- \to \pi^0\gamma$ have been derived in D. R. Palmer, Nuovo Cimento 9A, 212 (1972).

[18]G. Gounaris and J. Sakurai, Phys. Rev. Letters 21, 244 (1968).

[19]J. P. Baton *et al.*, Phys. Letters 33B, 525 (1970); 33B, 528 (1970); S. Marateck *et al.*, Phys. Rev. Letters 21, 1613 (1968).

[20]J. H. Scharenguivel, L. J. Gutay, and D. H. Miller, Nucl. Phys. B22, 16 (1970).

[21]The following argument suggests that this analysis of the data may underestimate r_π^2: If Im$F \ge 0$ on the timelike cut [as in vector-dominance models or N/D models of $F(t)$], then it can be shown that any linear fit to finite-t values of the form factor will underestimate the radius. Since the "single-pole" fit in question is largely linear, it might also lead to an underestimate of r_π^2.

[22]Equation (9) results when the "single-pole" fit is not constrained to satisfy the normalization condition, $F(0) = 1$. It is more realistic to constrain the "single-pole" fit to within 3.7% of this normalization condition. In that case, the Serpukhov-UCLA experiment gives an even larger radius:

$$(r_\pi^2/r_{VD}^2)_{\text{expt}} = 2.27 \pm 0.35.$$

[23]S. Furuichi, H. Kanada, and K. Watanabe, Rikkyo University report, 1972 (unpublished). Also, see A. Bramon, E. Etim, and M. Greco, Phys. Letters 39B, 514 (1972); G. Barbarino *et al.*, Lett. Nuovo Cimento 3, 689 (1972); A. Bramon and M. Greco, *ibid.* 3, 693 (1972).

[24]This is also true for narrower and broader ρ' resonances ($\Gamma_{\rho'} = 0.10$ GeV, $\Gamma_{\rho'} = 0.20$ GeV).

[25]By considering only this restricted set of phase shifts, we are ignoring several other possibilities (such as the existence of a $J = T = 1$ $\pi\pi$ resonance just above threshold).

[26]For Weinberg's value of a_1, see Eq. (5.46) of Ref. 3.

[27]E. W. Beier, report presented at the 1972 meeting of the American Physical Society (unpublished); A. Zylbersztejn *et al.*, Phys. Letters 38B, 457 (1972).

[28]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[29]J. Sucher and C. H. Woo, Phys. Rev. Letters 18, 723 (1967).

[30]See the analogous discussion of s-wave phase shifts in G. Barton, *Dispersion Techniques in Field Theory* (Benjamin, New York, 1965), p. 169. A low-mass bound state would have a large Compton wave length and therefore make a large contribution to r_π^2.

[31]P. Langacker and M. Suzuki, Phys. Rev. D 4, 2160 (1971); G. Auberson and L.-F. Li, *ibid.* 5, 2269 (1972). See also G. Nenciu and I. Raszillier, Bucharest report, 1972 (unpublished); D. R. Palmer, Phys. Rev. D 4, 1558 (1971).

[32]An mth-order zero is counted as m zeros.

[33]I. Raszillier, Institute of Physics (Bucharest) report, 1972 (unpublished); T. Kawai, D. Kiang, and K. Morita, Lett. Nuovo Cimento 3, 609 (1972).

[34]Technically, $\bar{f}(\xi)$ could have a pole at $t = t_0$, but in this note we ignore that possibility since it is physically unlikely.

[35]R. Nevanlinna, *Eindentige Analytisch Funktionen* (Springer, Berlin, 1953), p. 44; E. Hill, *Analytic Function Theory* (Ginn, Boston, 1962), Vol. II, p. 412.

[36]This is essentially the factorization theorem for H^p function ($p > 0$) or more generally for a class N^+ function in the unit disk. See P. L. Duren, *Theory of H^p Spaces* (Cambridge Univ. Press, London, 1970).

[37]G. Sansone and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable* (Wolters-Noordhoff, Groningen, 1969), Vol. II. See also E. E. Radescu, Phys. Rev. D 5, 135 (1972), for applications of this method to the K_{l3} decay problem, and I-Fu. Shih and S. Okubo, *ibid.* 6, 1393 (1972).