Comment on the "New" Equation of Motion for Classical Charged Particles

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The new equation of motion proposed by Mo and Papas is solved exactly for the case of a charge moving in a constant magnetic field. It is shown that within the realm of classical electrodynamics the new equation gives the same observable results as that of the Lorentz-Dirac equation.

In a recent article Mo and Papas[1] proposed a new equation of motion for classical charged particles:

$$\dot{u}^{\mu}=\frac{e}{mc}F^{\mu\lambda}u_{\lambda}+\frac{2e^3}{3mc^3}\left(\frac{1}{mc}F^{\mu\lambda}\dot{u}_{\lambda}-\frac{1}{mc^3}F^{\lambda\alpha}\dot{u}_{\lambda}u_{\alpha}u^{\mu}\right). \quad (1)$$

It differs from the Lorentz-Dirac equation

$$\dot{u}^{\mu}=\frac{e}{mc}F^{\mu\lambda}u_{\lambda}+\frac{2e^3}{3mc^3}(\ddot{u}^{\mu}-\dot{u}_{\lambda}\dot{u}^{\lambda}u^{\mu}) \quad (2)$$

in that the radiation reaction force is expressed in terms of the external electromagnetic field $F^{\mu\nu}$ instead of $\dot{u}$ and $\ddot{u}$. By eliminating $\ddot{u}$ the new equation avoids certain difficulties inherent in the Lorentz-Dirac equation, such as runaway solutions and preaccelerations. However, a new term must be added to the energy-momentum tensor in order to satisfy the conservation laws.

Several basic physical situations had been studied by Mo and Papas in Ref. 1. For the special case of motion in a uniform magnetic field, solutions of both Eq. (1) and Eq. (2) are solved by treating the radiation reaction as a small perturbation. Their results show that the new equation predicts a faster inward spiraling than does the Lorentz-Dirac equation by a deviation of the order of δ^2, where

$$\delta=\tfrac{2}{3}\gamma^2\frac{e^3B}{mc^4}, \quad (3)$$

and $\gamma=(1-\beta^2)^{-1/2}$. Therefore, they concluded, for electrons of energy 10^n BeV in a field of 10^b G the deviation is large and measurable if

$$b+2n\geqslant 10\ . \quad (4)$$

The relevant parameters in present electron synchrotrons are far below the requirement of Eq. (4). However, recent developments on the technique of flux compression have created transient magnetic fields with intensities up to 10^7 G.[2] On the other hand, electron beams of energy $\gtrsim$100 BeV are already available from the National Accelerator Laboratory. One can readily see that the requirement $2n+b\geqslant 10$ is more than adequately satisfied in experiments using the megagauss pulses as targets for the high-energy electron beams. A series of such experiments have already been proposed to test the Lorentz-Dirac equation.[2,3]

Since intuitive assumptions are unavoidable in the proposition of any fundamental equation, the ultimate judge on the validity of the "new" equation can only be by experimental test. Therefore, it would be very interesting if the deviation between the trajectory calculated from Eq. (1) and Eq. (2) is indeed of the order of δ. Unfortunately, as we shall show, this is not the case. The deviation is much smaller. In fact, within the realm of classical electrodynamics Eq. (1) and Eq. (2) predict the same physical results under any experimental conditions.

In Ref. 1, Mo and Papas solved Eq. (1) by treating the radiation reaction force as a small perturbation. It can be easily seen that the ratio of the reaction force to the Lorentz force is just δ; therefore, when $\delta\to 1$, the solutions obtained by Mo and Papas are no longer valid. It is, however, not difficult to find an *exact* solution for the new equation of motion.

Let us choose $\vec{B}=B\vec{e}_z$ and $\vec{v}=v_0\vec{e}_x$ at $t=0$. Equation (1) becomes

$$\dot{u}_1=\omega_H u_2+(\omega_H/\omega_0)\left[\dot{u}_2-\frac{1}{c^2}(\dot{u}_1u_2-u_1\dot{u}_2)u_1\right], \quad (5)$$

$$\dot{u}_2=-\omega_H u_1+(\omega_H/\omega_0)\left[-\dot{u}_1-\frac{1}{c^2}(\dot{u}_1u_2-u_1\dot{u}_2)u_2\right], \quad (6)$$

$$\dot{u}_3=0\ , \quad (7)$$

$$\dot{u}_4=-(\omega_H/\omega_0)(\dot{u}_1u_2-u_1\dot{u}_2)u_4/c^2\ , \quad (8)$$

where $\omega_H=eB/mc$ and $\omega_0=3mc^3/2e^2$. Notice now that $\delta=\gamma^2\omega_H/\omega_0$. Subtracting $u_1\times(6)$ from $u_2\times(5)$ and utilizing the relation $u^{\mu}\dot{u}_{\mu}\equiv 0$ we have

$$\dot{u}_1u_2-u_1\dot{u}_2=\gamma^2\omega_H c^2(1+\dot{\gamma}/\gamma\omega_0-\gamma^{-2})\ . \quad (9)$$

Substituting Eq. (9) into Eq. (8) gives

$$\frac{d\gamma}{dt}=-\frac{2e^4B^2(\gamma^2-1)}{3mc^2}[1-(\gamma\omega_H/\omega_0)^2]^{-1}\ . \tag{10}$$

u_μ can be found from Eqs. (5), (6), (9), and (10), and by translating these covariant quantities to the space and time variable of the observable we have

$$v_1=v_0\exp\left(-\frac{\omega_H}{\omega_0}g(t)\right)\cos g(t)\ , \tag{11}$$

$$v_2=v_0\exp\left(-\frac{\omega_H}{\omega_0}g(t)\right)\sin g(t)\ , \tag{12}$$

where

$$g(t)=\int_0^t\frac{\omega_H}{\gamma}\frac{1-2(\gamma\omega_H/\omega_0)^2+(\omega_H/\omega_0)^2}{[1+(\omega_H/\omega_0)^2][1-(\gamma\omega_H/\omega_0)^2]}dt\ . \tag{13}$$

Equations (10)–(12) determine the trajectory, the emission rate, and the radiation spectrum for a charged particle. The corresponding solutions from the Lorentz equation *without* radiation corrections are

$$\frac{d\gamma}{dt}=-\frac{2}{3}\frac{e^4B^2}{mc^2}(\gamma^2-1)\ , \tag{14}$$

$$v_1=v_0\cos\frac{\omega_H t}{\gamma_0}\ , \tag{15}$$

$$v_2=v_0\sin\frac{\omega_H t}{\gamma_0}\ . \tag{16}$$

Exact solutions cannot be found for the Lorentz-Dirac equation, but in an earlier article[3,4] we have solved Eq. (2) in powers of γ^{-1} and $\delta\gamma^{-1}$; the solutions obtained are applicable to cases of strong radiative damping $\delta>1$ provided $\gamma\gg1$.

$$\frac{d\gamma}{dt}=-\frac{2e^4B^2(\gamma^2-1)}{3mc^2}[1-4(\delta/\gamma)^2+O((\delta/\gamma)^4)]\ , \tag{17}$$

$$v_1=v_0e^{-h(t)}\cos f(t)\ , \tag{18}$$

$$v_2=v_0e^{-h(t)}\sin f(t)\ , \tag{19}$$

where

$$h(t)=\frac{\delta\omega_H t}{\gamma_0^{\,3}}\left[1+\frac{\delta\omega_H t}{2\gamma_0}+O((\delta/\gamma)^2)\right]\ , \tag{20}$$

$$f(t)=\frac{\omega_H t}{\gamma_0}\left[\left(1+\frac{\delta\omega_H t}{2\gamma_0}\right)-\left(\frac{\delta}{\gamma}\right)^2\left(\frac{2+6\delta^2}{1+\delta^2}+\frac{\delta\omega_H t}{\gamma_0}\right)+O\left(\left(\frac{\delta}{\gamma}\right)^4\right)\right]. \tag{21}$$

Comparing these three sets of equations we see that the radiation rates differ from each other by the order of $(\delta/\gamma)^2$. As for the trajectory, the new equation does predict an inward spiraling of the order of δ compared to the unperturbed circular orbit, but the spiraling trajectory deviates from that obtained from the Lorentz-Dirac equation only by the order of $\delta/\gamma\sim\gamma e^3B/m^2c^4$, but not by γ^2e^3B/m^2c^4. This result can be seen from the magnitude of the deflection angle θ for an electron after spent a time t in B. The unperturbed deflection is

$$\theta_0(t)=\frac{\omega_H t}{\gamma_0}\ . \tag{22}$$

The deflection predicted by the Lorentz-Dirac equation is

$$\theta_{\text{L-D}}(t)=\theta_0(t)\left[1+\frac{\delta\theta_0}{2}-\left(\frac{\delta}{\gamma}\right)^2\left(\frac{2+6\delta^2}{1+\delta^2}+\delta\theta_0\right)+\theta_0\left(\frac{\delta}{\gamma}\right)^4\right]\ . \tag{23}$$

That predicted by the new equation is[5]

$$\theta_{\text{new}}(t)=\theta_0(t)\left[1+\frac{\delta\theta_0}{2}-\left(\frac{\delta}{\gamma}\right)^2\left(1+\frac{\theta_0}{2\delta}\right)+\theta_0\left(\frac{\delta}{\gamma}\right)^4\right]\ . \tag{24}$$

We have also examined the new equation for the cases of Coulomb interaction and Thomson scattering; the results are similar: The deviation from the result derived from the Lorentz-Dirac equation is of the order δ/γ instead of δ, where δ represents generally the ratio of reaction force to the Lorentz force. It is well known that for a high-energy electron moving in an intense field the quantum corrections are $\sim\gamma Be\hbar/m^2c^3=137\delta/\gamma$, about two orders of magnitude larger than the predicted discrepancy between the two classical equations, and when the electromagnetic field measured in the rest frame of the electron reaches the characteristic quantum electrodynamic value $f_0=m^2c^3/e\hbar=4\cdot4\times10^{13}$ G the classical electrodynamics is no longer applicable.[6] Therefore, we conclude, the new equation cannot lead to results physically distinguishable from the Lorentz-Dirac equation.

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[1]T. C. Mo and C. H. Papas, Phys. Rev. 4, 3566 (1971).
[2]T. Erber, Acta Phys. Austr. Suppl. 8, 329 (1971).
[3]C. S. Shen, this issue, Phys. Rev. D 6, 2736 (1972).
[4]C. S. Shen, Phys. Rev. Letters 24, 410 (1970).

[5]For comparison purposes, the exact solutions of the new equation are expanded in powers of δ/γ.

[6]See, for example, F. Rohrlich, *Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1965).