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Analytic Structure of Multiparticle Amplitudes in Complex Helicity

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By studying the partial-wave expansions of multiparticle amplitudes we argue that analytic properties in complex helicity are just a reflection of the familiar analytic structure in angular momentum. We give a criterion which determines when an asymptotic behavior in an azimuthal angle (conjugate to the helicity) can be reached in a physical process. Our discussion centers around the five- and six-point functions; the latter, being relevant for single-particle inclusive processes, is considered in detail. One of the interesting features of analytic structure in λ is that it depends in detail on what other variables one chooses in addition to the azimuthal angle conjugate to it. That singularity structure is found by examining the partial-wave analysis appropriate to the chosen variables. Finally, a discussion of signature in many-particle amplitudes is given.

I. INTRODUCTION

In the study of the asymptotic behavior of hadron amplitudes, it is possible to isolate processes in which one of the "external" objects is a Reggeon; namely, a "particle" both off the mass shell $p^2 = m^2$ and off the spin shell $\alpha(p^2)$ = integer or half-integer. The simplest scattering, of course, in which a Reggeon makes its appearance is elastic or quasi–two-body scattering. Here one measures a Reggeon–two-particle vertex function as the factorized residue of a pole in the complex J plane. In processes involving more particles, one can discuss Reggeon-particle scattering and production.[1] A degree of freedom suppressed in elastic processes, the helicity of the Reggeon begins to play a role in multiparticle problems. One may view its appearance either as reflecting the nontrivial dependence on azimuthal angles which enters in five-, six-, ... point amplitudes, or one may recall that in four or more line amplitudes involving particles with spin, the dependence on helicity becomes significant.

These azimuthal degrees of freedom, ϕ, invite one to inquire into the behavior of multiparticle amplitudes as some $\cos\phi$ becomes asymptotically large.[2] Such behavior will be governed by the analytic structure in the variable conjugate to ϕ; namely, the helicity. One is led thereby to investigate the singularity properties, poles and cuts especially, in complex helicity. From the outset it is clear that singularities in the helicity must be thought of as on a somewhat different footing from those in angular momentum or invariant energies. This difference comes from our understanding of particles as being classified according to irreducible representations of the Poincaré group. Under such a classification the spin J and (mass)$^2 = p^2$, apart from internal quantum numbers, are sufficient to specify a state. When we consider S-matrix singularities in J or p^2, or together as for Regge poles with $J=\alpha(p^2)$, we remain within this Poincaré invariant scheme. However, helicity has quite a different character in the classification of states. It labels the components of a representation and under a Lorentz transformation can change or be mixed up with other helicities. In short it is not a quantity that provides a Lorentz-frame-independent characterization of a state, and to regard singularities in helicity variables as somehow "dynamic" necessitates a major reorientation in our views of what constitutes a particle. We will argue in this paper that such a drastic move is not called for, and that, indeed, *singularities in helicity are kinematic reflections of familiar analytic structure in angular momentum*. The way in which this comes about will be given in detail in the discussion of the five-point function found in Sec. II. The relevant feature is the isolation of a $\Gamma(\lambda - J)$ in the double Sommerfeld-Watson transform in angular momentum J and helicity λ. This factor will ensure that a pole, say, at $J=\alpha$ in the angular momentum is a series of poles in λ at $\lambda = \alpha, \alpha-1, \ldots$. In this way we see directly the "kinematic" manner in which J-plane structure goes over into λ-plane structure.

We will then argue that the isolation of these kinematic Γ functions is enough to determine the analytic structure in λ in multiparticle amplitudes. In particular we will study the six-point function in a configuration appropriate for learning about the three-Reggeon vertex,[3] and during this study we will develop a criterion for deciding when a certain asymptotic azimuthal angle limit can be reached in the physical region of an S-matrix element. This becomes particularly important in the investigation of inclusive processes.

It has been known to many people[4] that there are singularities in λ at $\alpha, \alpha - 1, \ldots$ and, in a sense, our discussion of that point is meant to give a stronger motivation than we have found in the literature. Particularly relevant to the present work are the papers of White[5] and Weis,[6] the latter of which has certainly stimulated many of our ideas here. Beyond this pedagogical contribution, the discussion of more general configurations than five- or six-point functions and the criterion for physical region asymptotic behavior in azimuthal angles may have some value in further study of multiparticle production. One of the additional points we will emphasize is that the detailed structure in the λ plane will depend on exactly what other variables one chooses in addition to the ϕ conjugate to λ. The selection of those variables will be connected with various multiple partial-wave expansions whose significance will be given by the kind of physical information one wishes to extract from the multiparticle amplitude in the $\cos\phi \to \infty$ limit.

II. RELATING ANGULAR MOMENTUM AND HELICITY STRUCTURE

In this section we will first give a heuristic discussion of the manner in which certain kinematic factors in partial-wave expansions enable one to determine where singularities in helicity λ lie when one has specified the analytic structure in the angular momentum J. Our procedure will be to consider in detail the five-point function in the kinematic configuration shown in Fig. 1. All external particles are spinless, and for simplicity we will take them to have equal mass, m.

We want to make a partial-wave decomposition of this amplitude which enables us to look at the analytic properties of the helicity associated with a Reggeon of mass $t_1 = Q_1^2 = (p_1 + p_3)^2$. To make this partial-wave analysis let us sit in a frame where

$$p_4 = (m, 0, 0, 0)\,, \tag{1}$$

and the other vectors are chosen to be

$$Q_i = \sqrt{t_i}\ (\cosh\psi_i, 0, 0, \sinh\psi_i)$$

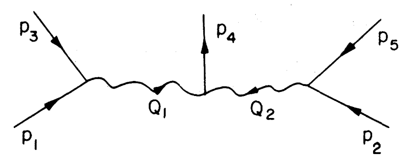

FIG. 1. The tree graph appropriate for the partial-wave analysis of the five-point amplitude A_5. The asymptotic limit of A_5 in the angle between the planes formed by $p_1 p_3$ and $p_2 p_5$ is governed by the singularities in the helicity attached to the $(Q_1 p_4 Q_2)$ vertex.

$$= B_z(\psi_i)(\sqrt{t_i}\ , 0, 0, 0)\,, \quad i = 1, 2 \tag{2}$$

$$p_1 = B_z(\psi_1)(E_1, p_1 \sin\theta_1 \cos\phi_1, p_1 \sin\theta_1 \sin\phi_1, p_1 \cos\theta_1)\,, \tag{3}$$

and

$$p_2 = B_z(\psi_2)(E_2, p_2 \sin\theta_2 \cos\phi_2, p_2 \sin\theta_2 \sin\phi_2, p_2 \cos\theta_2)\,, \tag{4}$$

$$p_i = (\tfrac{1}{4} t_i - m^2)^{1/2} \quad \text{and} \quad E_i = \tfrac{1}{2}\sqrt{t_i}\ , \tag{5}$$

and

$$\cosh\psi_1 = (m^2 + t_1 - t_2)/2m\sqrt{t_1}\ , \tag{6}$$

$$\cosh\psi_2 = (m^2 + t_2 - t_1)/2m\sqrt{t_2}\ . \tag{7}$$

We have chosen the Q_i timelike so we may make an ordinary O(3) partial-wave analysis. The $B_z(\psi)$ is a z boost through the indicated angle.

These kinematics define a set of six variables, $\cos\theta_1$, $\cos\theta_2$, t_1, t_2, $\phi = \phi_1 - \phi_2$, and $\phi_1 + \phi_2$, on which the amplitude may depend. Rotational invariance of the scalar amplitude forbids the appearance of the angle $\phi_1 + \phi_2$, so we have the five-point function given in terms of the first five variables. We will for the moment pretend that ϕ_1 and ϕ_2 may be treated independently and will impose this important constraint soon. Suppressing all variables except θ_1 and ϕ_1 we exhibit the dependence of the five-point amplitude on them by writing the partial-wave expansion

$$A_5(\cos\theta_1, \phi_1) = \sum_{J_1=0}^{\infty} \sum_{\lambda_1=-J_1}^{+J_1} (2J_1+1)\tilde{P}_{J_1}^{\lambda_1}(\cos\theta_1) e^{i\lambda_1\phi_1} \frac{\Gamma(J_1+\lambda_1+1)}{\Gamma(J_1-\lambda_1+1)} M_{J_1\lambda_1}\,, \tag{8}$$

where

$$\tilde{P}_J^{\lambda}(x) = \frac{\Gamma(J-\lambda+1)}{\Gamma(J+\lambda+1)}\, P_J^{\lambda}(x) \tag{9}$$

$$= \frac{(1-x^2)^{\lambda/2}}{2^{\lambda}\Gamma(\lambda+1)}\ {}_2F_1(J-\lambda, J+\lambda+1; \lambda+1; \tfrac{1}{2}(1-x)) \quad \text{for } \lambda \geq 0\,, \tag{10}$$

and $P_J^{\lambda}(x)$ is the usual associated Legendre polynomial. The purpose in taking out the designated Γ functions is most apparent in Eq. (10) because one can see from known properties of the hypergeometric function that there are no associated J, λ singularities in $\tilde{P}_J^{\lambda}(x)$.[7]

Furthermore, using the orthogonality properties of the $\tilde{P}_J^\lambda$ and the normalization integral

$$\int_{-1}^{+1} dx\,[\tilde{P}_J^\lambda(x)]^2 = \frac{2}{2J+1}\,\frac{\Gamma(J-\lambda+1)}{\Gamma(J+\lambda+1)}\,, \tag{11}$$

we find

$$M_{J_1\lambda_1} = \tfrac{1}{2}\int_{-1}^{+1} dx\,\tilde{P}_{J_1}^{\lambda_1}(x)\int_0^{2\pi} d\phi_1 e^{-i\lambda_1\phi_1} A_5(x,\phi_1)\,, \tag{12}$$

where $x=\cos\theta_1$. The important point to notice is that no associated J_1 and λ_1 singularities are present in the partial-wave amplitude so defined. They all reside in the explicit Γ functions.

Now we make a heuristic Sommerfeld-Watson transform ignoring for the moment all questions of signature.[8] Write (8) as

$$A_5(\cos\theta_1,\phi_1) = \left(\sum_{\lambda_1=0}^{\infty}\sum_{J_1=\lambda_1}^{\infty} + \sum_{\lambda_1=-\infty}^{-1}\sum_{J_1=-\lambda_1}^{\infty}\right)(2J_1+1)e^{i\lambda_1\phi_1}\tilde{P}_{J_1}^{\lambda_1}(\cos\theta_1)\frac{\Gamma(J_1+\lambda_1+1)}{\Gamma(J_1-\lambda_1+1)}\,M_{J_1\lambda_1}\,, \tag{13}$$

to separate $\lambda_1 \geq 0$ and $\lambda_1 < 0$. In order to handle $\lambda_1<0$, note that

$$\tilde{P}_J^\lambda(x) = \frac{\Gamma(J-\lambda+1)}{\Gamma(J+\lambda+1)}\,\tilde{P}_J^{-\lambda}(x) \tag{14}$$

$$= \frac{\Gamma(J-\lambda+1)}{\Gamma(J+\lambda+1)}\,{}_2F_1(J+\lambda, J-\lambda+1; -\lambda+1; \tfrac{1}{2}(1-x))\,\frac{(1-x^2)^{-\lambda/2}}{2^\lambda\Gamma(-\lambda+1)} \quad \text{for } \lambda<0\,, \tag{15}$$

and defining

$$M_R(J,\lambda) = \tfrac{1}{2}\int_{-1}^{+1} dx\,\tilde{P}_J^\lambda(x)\int_0^{2\pi}\frac{d\phi_1}{2\pi}\,e^{-i\lambda\phi_1}A_5(x,\phi_1) \tag{16}$$

for the regime $\lambda \geq 0$, and for $\lambda<0$,

$$M_L(J,\lambda) = \tfrac{1}{2}\int_{-1}^{+1} dx\,\tilde{P}_J^{-\lambda}(x)\int_0^{2\pi}\frac{d\phi_1}{2\pi}\,e^{-i\lambda\phi_1}A_5(x,\phi_1)\,, \tag{17}$$

neither of which has explicit associated J, λ singularities, we may write

$$\begin{aligned} A_5(\cos\theta_1,\phi_1) = {} & \sum_{\lambda_1=0}^{\infty}\sum_{J_1=\lambda_1}^{\infty}(2J_1+1)\tilde{P}_{J_1}^{\lambda_1}(x)e^{i\lambda_1\phi_1}M_R(J_1,\lambda_1)\frac{\Gamma(J_1+\lambda_1+1)}{\Gamma(J_1-\lambda_1+1)} \\ & + \sum_{\lambda_1=-\infty}^{-1}\sum_{J_1=-\lambda_1}^{\infty}(2J_1+1)\tilde{P}_{J_1}^{-\lambda_1}(x)e^{i\lambda_1\phi_1}M_L(J_1,\lambda_1)\frac{\Gamma(J_1-\lambda_1+1)}{\Gamma(J_1+\lambda_1+1)}\,. \end{aligned} \tag{18}$$

Now we make a double Sommerfeld-Watson transform in λ_1 and J_1,

$$\begin{aligned} A_5(\cos\theta_1,\phi_1) = {} & \int_{C_{\lambda_1}}\frac{d\lambda_1}{2\pi i}\,\frac{\pi}{\sin\pi\lambda_1}\int_{C_{J_1}}\frac{dJ_1}{2\pi i}\,\frac{\pi}{\sin\pi(J_1-\lambda_1)}\,(-e^{i\phi_1})^{\lambda_1}(2J_1+1)\tilde{P}_{J_1}^{\lambda_1}(-x)M_R(J_1,\lambda_1)\frac{\Gamma(J_1+\lambda_1+1)}{\Gamma(J_1-\lambda_1+1)} \\ & - \int_{C'_{\lambda_1}}\frac{d\lambda_1}{2\pi i}\,\frac{\pi}{\sin\pi\lambda_1}\int_{C'_{J_1}}\frac{dJ_1}{2\pi i}\,\frac{\pi}{\sin\pi(J_1+\lambda_1)}\,(2J_1+1)\tilde{P}_{J_1}^{-\lambda_1}(-x)M_L(J_1,\lambda_1)\frac{\Gamma(J_1-\lambda_1+1)}{\Gamma(J_1+\lambda_1+1)}\,(-e^{i\phi_1})^{\lambda_1}\,, \end{aligned} \tag{19}$$

where the contours are the standard ones needed to reproduce the sums in (18). Noting now that

$$\frac{-\pi}{\sin\pi(J_1-\lambda_1)} = \Gamma(\lambda_1-J_1)\Gamma(J_1-\lambda_1+1)\,, \tag{20}$$

and

$$\frac{\pi}{\sin[-\pi(J_1+\lambda_1)]} = \Gamma(-\lambda_1-J_1)\Gamma(J_1+\lambda_1+1)\,, \tag{21}$$

we may cast (19) into

$$\begin{aligned} A_5(\cos\theta_1,\phi_1) = {} & -\int_{C_{\lambda_1}}\frac{d\lambda_1}{2i\sin\pi\lambda_1}\int_{C_{J_1}}\frac{dJ_1}{2\pi i}\,\Gamma(\lambda_1-J_1)\Gamma(\lambda_1+J_1+1)(2J_1+1)\tilde{P}_{J_1}^{\lambda_1}(-x)M_R(J_1,\lambda_1)(-e^{i\phi_1})^{\lambda_1} \\ & + \int_{C'_{\lambda_1}}\frac{d\lambda_1}{2i\sin\pi\lambda_1}\int_{C'_{J_1}}\frac{dJ_1}{2\pi i}\,\Gamma(-\lambda_1-J_1)\Gamma(J_1-\lambda_1+1)(2J_1+1)\tilde{P}_{J_1}^{-\lambda_1}(-x)(-e^{i\phi_1})^{\lambda_1}M_L(J_1,\lambda_1)\,. \end{aligned} \tag{22}$$

If there were no other singularities in λ_1, we would now be able to conclude that a pole of $M_R(J_1, \lambda_1)$, say, in J_1 at $\alpha_1(t_1)$ would, through the kinematic Γ functions yield strings of poles at

$$\lambda_1 = \alpha_1(t_1),\ \alpha_1(t_1) - 1, \ldots \tag{23}$$

and

$$\lambda_1 = -\alpha_1(t_1) - 1,\ -\alpha_1(t_1) - 2, \ldots , \tag{24}$$

from the first term in (22). In the second term, which involves the left-hand λ_1 plane, a pole in $M_L(J_1, \lambda_1)$ at $J_1 = \alpha_1(t_1)$ gives rise to singularities in λ_1 integration at

$$\lambda_1 = -\alpha_1(t_1),\ -\alpha_1(t_1) + 1, \ldots \tag{25}$$

and

$$\lambda_1 = \alpha_1(t_1) + 1,\ \alpha_1(t_1) + 2, \ldots . \tag{26}$$

For $\phi_1 \to \pm i\infty$ we want to pick up the poles from the second (first) term of (22) which lie furthest to the left (right) in the λ_1 plane. Since M_R and M_L are the proper functions to be continued in the right (left) half λ_1 planes,[8] this is appropriate.

Because of our construction so far, were there no kinematic constraint on ϕ_1 that it only enter A_5 in the form $\phi = \phi_1 - \phi_2$, we would be strongly motivated to say there are no further singularities from $M_{J_1\lambda_1}$ in λ_1. In the representation (22) of $A_5(\phi_1, \cos\theta_1)$ we would then conclude that the asymptotic behavior in $e^{i\phi_1}$ with the other specified variables fixed is $(\cos\phi_1)^{\alpha_1(t_1)}$ for $\alpha_1(t_1) \geq -\frac{1}{2}$ plus $O((\cos\phi_1)^{\alpha_1(t_1)-1})$.

However, the invariance under z rotations of the scalar function A_5 tells us that if we go back to (8) and restore θ_2 and ϕ_2 and write a double partial-wave expansion to exhibit their dependence also,

$$A_5(\cos\theta_1, \cos\theta_2, \phi_1, \phi_2) = \sum_{J_1=0}^{\infty} \sum_{J_2=0}^{\infty} \sum_{\lambda_1=-J_1}^{+J_1} \sum_{\lambda_2=-J_2}^{+J_2} (2J_1+1)(2J_2+1) e^{i\lambda_1\phi_1 - i\lambda_2\phi_2} \times \tilde{P}_{J_1}^{\lambda_1}(\cos\theta_1)\tilde{P}_{J_2}^{\lambda_2}(\cos\theta_2) F(J_1, J_2, \lambda_1, \lambda_2) \frac{\Gamma(J_1+\lambda_1+1)}{\Gamma(J_1-\lambda_1+1)} \frac{\Gamma(J_2+\lambda_2+1)}{\Gamma(J_2-\lambda_2+1)}, \tag{27}$$

then λ_1 must equal λ_2 so only $\phi = \phi_1 - \phi_2$ appears. This has the implication that singularities in *both* J_1 and J_2, the angular momenta conjugate to θ_1 and θ_2 are transmitted to λ_1 via the kinematic Γ functions we have discussed at length.

This lesson is well known,[2] we know, but we have belabored it here to show how it is that the rotational invariance of A_5 or equivalently the covariance of the central $(Q_1 Q_2 p_4)$ vertex in Fig. 1 links together the otherwise independent helicities λ_1 and λ_2. We are informed thereby to think of λ_1 and its associated ϕ_1 as not connected with the external orientation of the plane of p_1 and p_3, but to attach it to the central vertex to exhibit its meaning.

By going to particle poles in t_1 and t_2 in the function A_5, one sees directly that λ_1 and λ_2 are properly interpreted as the helicities of the states with spin J_1, mass $\sqrt{t_1}$, or spin J_2, mass $\sqrt{t_2}$, respectively. The rotational invariance of A_5 informs us that we may not separately continue in λ_1 and λ_2, even though we may, of course, do so in J_1 and J_2. Also by taking, say, just t_2 to a pole of spin J_2, helicity h_2, we see that the continuation of the resulting four-point function $Q_2(\text{spin } J_2, h_2) + p_4 \to p_1 + p_3$ in the angular momentum J_1 does not necessitate, indeed does not allow, a continuation in the helicity λ_1 associated with J_1, for it is constrained to be the external helicity h_2. It is in this manner that we see why one never encounters questions of complex helicity in two-to-two processes.

Returning to Eq. (27), if we define F_R and F_L in analogy with (16) and (17),

$$F_{R,L}(J_1, J_2, \lambda) = \tfrac{1}{2}\int_{-1}^{+1} dx_1 \tilde{P}_{J_1}^{\pm\lambda}(x_1) \tfrac{1}{2}\int_{-1}^{+1} dx_2 \tilde{P}_{J_2}^{\pm\lambda}(x_2) \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i\lambda\phi} A_5(x_1, x_2, \phi), \tag{28}$$

we may write the triple Sommerfeld-Watson transform

$$\begin{aligned} A_5(\cos\theta_1, \cos\theta_2, \phi) = {} & \int_{C_\lambda} \frac{d\lambda}{2i\sin\pi\lambda} \int_{C_{J_1}} \frac{dJ_1}{2\pi i} \int_{C_{J_2}} \frac{dJ_2}{2\pi i} \Gamma(\lambda - J_1)\Gamma(\lambda - J_2)\Gamma(\lambda + J_1 + 1) \\ & \times \Gamma(\lambda + J_2 + 1)(2J_1+1)(2J_2+1)(-e^{i\phi})^{\lambda} F_R(J_1, J_2, \lambda)\tilde{P}_{J_1}^{\lambda}(-x_1)\tilde{P}_{J_2}^{\lambda}(-x_2) \\ & + \int_{C'_\lambda} \frac{d\lambda}{2i\sin\pi\lambda} \int_{C'_{J_1}} \frac{dJ_1}{2\pi i} \int_{C'_{J_2}} \frac{dJ_2}{2\pi i} \Gamma(-\lambda - J_1)\Gamma(-\lambda - J_2)\Gamma(-\lambda + J_1 + 1)\Gamma(-\lambda + J_2 + 1) \\ & \times (2J_1+1)(2J_2+1)(-e^{i\phi})^{\lambda} \tilde{P}_{J_1}^{-\lambda}(-x_1)\tilde{P}_{J_2}^{-\lambda}(-x_2) F_L(J_1, J_2, \lambda) . \end{aligned} \tag{29}$$

Now we have exhibited the dependence on θ_1, θ_2 and $\phi = \phi_1 - \phi_2$ and have extracted all the kinematic Γ functions from the partial-wave amplitudes $F_{R,L}(J_1, J_2, \lambda)$ which may be continued in the right-half (left-half) λ plane. We are strongly urged to assume that these functions have no singularities in λ and thus learn that *the asymptotic behavior in* ϕ *is completely determined from the "dynamical" poles and cuts in* J_1 *and* J_2.[9] This assumption, which is very natural in the light of our remarks about the Poincaré group above, is borne out in model calculations where the simultaneous x_1, x_2, and ϕ asymptotic behavior with t_1, t_2 fixed has been studied.[2]

Perhaps it is worthwhile once more to repeat the procedure we have followed before going on to more complicated, albeit physically more interesting examples. We chose from the outset a kinematic configuration indicated by the "tree" graph of Fig. 1 and designated more precisely by the kinematics (1)–(7) in the rest frame of particle 4. We then argued at length that to find the singularities in the helicity λ_1, conjugate to ϕ_1, which determine the asymptotic behavior of A_5 as $e^{i\phi_1} \to \infty$, one must write a multiple partial-wave expansion which exhibits *all* the constraints on λ_1 coming from the Lorentz invariance of A_5. The partial-wave expansion is, of course, suggested directly by both the tree graph and the kinematics and must be carried out in a frame which guarantees that θ_1, θ_2, ϕ_1, ϕ_2 have their interpretation as polar and azimuthal angles, so we are confident that their conjugate variables are angular momentum and helicity.

This last remark is relevant for the question: What is the behavior of A_5 as $\cos\phi \to \infty$, with $\cos\theta_1$, $\cos\theta_2$, t_1, and t_2 fixed? This limit is not accessible in any physical region of the five-point function, as we will discuss at some length below, but one may ask it. If we define the energy variables $s = (p_1 + p_2)^2$, $s_1 = (p_3 - p_4)^2$, and $s_2 = (p_4 - p_5)^2$, then this limit corresponds to $s \to \infty$ while s_1, s_2, t_1, and t_2 are fixed. From the point of view of the s-, t-channel invariants, our question would seem to have no answer for why should $(\cos\phi)^{\alpha(t_1)}$ appear rather than $(\cos\phi)^{\alpha(s_1)}$ or even $(\cos\phi)^{\alpha(u_1)}$ where $u_1 = (p_1 - p_4)^2$? That is, why, from the point of view of channel invariants is the tree graph of Fig. 1 relevant to the limit $s \to \infty$, t_1, t_2, s_1, s_2 fixed?[9]

Our answer to this question is that in the limit $\cos\phi \to \infty$, $\cos\theta_1$, $\cos\theta_2$, t_1, t_2 fixed, the four invariant dot products $p_1 \cdot p_2$, $p_1 \cdot p_5$, $p_3 \cdot p_2$, and $p_3 \cdot p_5$ all become infinite, while the other six possible inner products among the momenta remain finite. The only choice of tree graph for which ϕ remains an azimuthal angle and for which these, and only these, inner products are infinite in this limit are Fig. 1 and its trivial variations gotton by interchanging p_1 and p_3 or p_2 and p_5 or both. To be more precise in what we mean by tree graph, let us say that the crucial feature is that it defines a way of choosing kinematics so that if p_i and p_j as a pair connect to $Q_{ij} = p_i - p_j$, then we make a partial-wave expansion in the polar and azimuthal angles θ_{ij} and ϕ_{ij} of the plane of p_i and p_j and look for poles in the conjugate variable J_{ij} of θ_{ij} and $J_{ij}(Q_{ij}^2)$.

If for A_5 we had chosen ϕ to be the angle between the planes of $(p_1 p_3)$ and, say, $(p_4 p_5)$, then in the limit $\cos\phi \to \infty$, we are not picking out $p_1 \cdot p_2$, $p_2 \cdot p_3$, $p_1 \cdot p_5$, and $p_3 \cdot p_5 \to \infty$ as before. So a partial-wave expansion which had this interpretation would be inappropriate for the limit we desire, and we return to Fig. 1 as the only available tree graph (including again trivial $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_5$ permutations).

Another observation in this regard is that with our parametrization, when $\phi \to \infty$ so p_1 and p_3 "move away" from the cluster $p_2 p_5 p_4$, the fact that $p_4 \cdot p_1$, $p_4 \cdot p_3$ and $(p_2 - p_5) \cdot p_1$ and $(p_2 - p_5) \cdot p_3$ remain finite, singles out the pair $p_2 p_5$ and the single particle p_4 as the correct subclustering. Again we are led to Fig. 1.

Such a line of thought leads us to expect that as $e^{i\phi} \to \pm\infty$ with $\cos\theta_1$, $\cos\theta_2$, t_1, and t_2 fixed, the function A_5 behaves as

$$A_5(\cos\theta_1, \cos\theta_2, t_1, t_2, \phi) \underset{\substack{\phi \to \infty \\ \cos\theta_i, t_i \text{ fixed}}}{\sim} (-e^{i\phi})^{\alpha_1(t_1)} f_1 + (-e^{i\phi})^{\alpha_2(t_2)} f_2 , \tag{30}$$

where the f_i are functions of the fixed variables while $\alpha_i(t_i) > 0$, are the rightmost poles in the J_i as they appear in the representation of A_5 by Eq. (27). This suggestion is rather hard to verify in models of particle production since the limit in question does not occur in the physical region. (The dual-resonance model may provide a useful testing ground.) When we come to the six-point function, however, the limit analogous to this can occur in the physical region and Eq. (39) then has physical content. With that we close our discussion of the five-point function and proceed.

III. AZIMUTHAL-ANGLE LIMITS OF THE SIX-POINT AMPLITUDE

We turn now to a discussion of the six-point function, A_6, concentrating on the kinematic configuration in Fig. 2. This will be appropriate for the exposition of the triple-Reggeon vertex[3] and plays a central role in the discussion of single-particle inclusive reactions near the end of the physical region.[10]

Our procedure will be to make a multiple O(3) partial-wave expansion of A_6 and, as we have done for A_5, to write Sommerfeld-Watson transformations to yield integral representations useful for continuation to the crossed channel. Since we encounter for the first time a vertex with three spacelike momenta $(Q_1Q_2Q_3)$ we will have to distinguish between two different kinds of partial-wave expansions depending on the sign of the triangle function

$$\Delta(Q_1^2, Q_2^2, Q_3^2) = (Q_1^2 + Q_2^2 - Q_3^2)^2 - 4Q_1^2Q_2^2 .$$

We shall first discuss the kinematics for the process $p_1 + p_2 + p_3 \to p_1' + p_2' + p_3'$ as indicated in Fig. 2 and then give a heuristic argument as to how we may use the O(3) expansion indicated by Fig. 3 and analytically continue to the reaction under consideration.

There are two cases to be distinguished[3]

$$(1)\quad \Delta(t_1, t_2, t_3) > 0 \tag{31}$$

and

$$(2)\quad \Delta(t_1, t_2, t_3) < 0 , \tag{32}$$

where $t_i = Q_i^2$. If the t_i are either all positive or all negative, we may be in case (1) or case (2). If only one of the t_i is positive or negative, we are fixed in case (1). To see the significance of each case, let us consider them in order. First suppose all $t_i > 0$, and $\Delta(t_1, t_2, t_3) > 0$. Then we may sit in a Lorentz frame where Q_3 is along the time direction:

$$Q_3 = (\sqrt{t_3}, 0, 0, 0) , \tag{33}$$

$$Q_1 = B_z(\eta_1)(\sqrt{t_1}, 0, 0, 0) , \tag{34}$$

$$Q_2 = B_z(\eta_2)(\sqrt{t_2}, 0, 0, 0) , \tag{35}$$

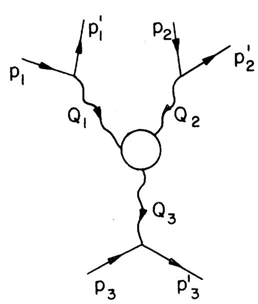

FIG. 2. The tree graph defining the kinematics for the partial-wave analysis of the six-point amplitude A_6 in the regime where the Q_i are spacelike. If $\Delta(Q_1^2, Q_2^2, Q_3^3) < 0$, the asymptotic limit of an azimuthal angle (y-boost angle) associated with the $(Q_1Q_2Q_3)$ vertex can be reached in physical region of A_6; the single-particle inclusive process is an example.

$$\sinh\eta_1 = \frac{[\Delta(t_1, t_2, t_3)]^{1/2}}{2\sqrt{t_1}\sqrt{t_3}} , \tag{36}$$

$$\sinh\eta_2 = -\frac{[\Delta(t_1, t_2, t_3)]^{1/2}}{2\sqrt{t_2}\sqrt{t_3}} . \tag{37}$$

The role played by $\Delta(t_1, t_2, t_3)$ is explicitly shown here. Were it negative, we would not be able to orient the vectors Q_1 and Q_2 in the t-z plane by real z boosts from their rest frames.

The set of vectors (33)–(35) is invariant under a rotation about the z axis, and this will lead to a conservation of the usual helicity at the central vertex. As we have seen in the five-point function of Sec. II this constraint means that analytic structure in, say, λ_1, the helicity of the "state" with momentum Q_1, will be related to the analytic structure in J_2 and J_3, the angular momentum of the states with momenta Q_2 and Q_3, as well as to the analytic structure in J_1. If all the t_i are negative with $\Delta(t_1, t_2, t_3) > 0$, a similar analysis may be presented.[3]

Suppose we are now in case (2). To reach this take all Q_i spacelike, and proceed to a frame where Q_3 is along the z axis

$$Q_3 = (0, 0, 0, \sqrt{-t_3}\,) . \tag{38}$$

We may orient Q_1 and Q_3 in the x-z plane

$$Q_1 = R_y(\theta_1)(0, 0, 0, \sqrt{-t_1}\,) \tag{39}$$

and

$$Q_2 = R_y(\theta_2)(0, 0, 0, \sqrt{-t_2}\,) , \tag{40}$$

where $R_y(\theta)$ is a rotation about the y axis by θ, and

$$\sin\theta_1 = \frac{[-\Delta(t_1, t_2, t_3)]^{1/2}}{2\sqrt{-t_1}\sqrt{-t_3}} , \tag{41}$$

$$\sin\theta_2 = -\frac{[-\Delta(t_1, t_2, t_3)]^{1/2}}{2\sqrt{-t_2}\sqrt{-t_3}} . \tag{42}$$

Because we can choose the orientation of the vectors Q_1 and Q_2 in the x-z plane, the set of momenta (38)–(40) is invariant under a y boost, which is

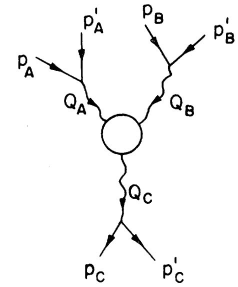

FIG. 3. The tree graph appropriate for the partial-wave expansion of A_6 when Fig. 2 is analytically continued to the regime Q_A^2, A_B^2, $Q_C^2 > 0$.

a noncompact operation, rather than a z rotation, a compact operation, as in case (1). This invariance means that the "boost helicities" λ_i (Ref. 11) conjugate to a y-boost angle will be conserved at the $(Q_1Q_2Q_3)$ vertex, and the analytic structure in the λ_i will reflect the singularities in the J_i entering the vertex.

It is important to note that because in case (2) real y-boost angles have replaced real z-rotation angles as the azimuthal variables, we expect to be able to reach asymptotic limits in a physical region of A_6 by allowing these y-boost angles to become large. The explicit behavior of A_6 in these limits will be determined by the singularity structure in the boost helicities, and that structure is apparent in the multiple partial-wave expansions analogous to (29).

In the following we will make a triple O(3) partial-wave expansion of A_6, choosing the Q_i timelike, and give a heuristic argument as to how this is to be applied in the regime where $\Delta(t_1, t_2, t_3) < 0$ and the $Q_i^2 < 0$. A crossed-channel partial-wave expansion can be given directly for the physical case where the Q_i are spacelike and $\Delta(t_1, t_2, t_3)$ is negative.

First we establish the kinematics for Fig. 2 which are relevant for the three-to-three scattering $p_1 + p_3 + (-p_1') \to (-p_2) + p_3' + p_2'$ whose forward discontinuity in the missing-mass variable $W^2 = (p_1 + p_3 - p_1')^2$ yields the single-particle inclusive cross section for $p_1 + p_3 \to p_1' +$ missing mass W.[12]

All of the Q_i are spacelike for the six-point function described, and in the inclusive process $t_1 = t_2$ while $t_3 = 0$. If we evaluate $\Delta(t_1, t_2, t_3)$ for $t_1 = t_2 < 0$ and $t_3 \to 0$ from below, then

$$\Delta(t_1, t_2, t_3)|_{t_1=t_2} = t_3(t_3 - 4t_1), \tag{43}$$

and we see that the Δ function goes to zero from below, and therefore case (2) is appropriate.

We specify the four-vectors p_i, p_i', and Q_i in a frame F_3, where Q_3 sits along the z axis and Q_1 and Q_2 are in the x-z plane; that is, we employ Eqs. (38)–(42).[3] In F_3 we give p_1 by taking a standard p_1 vector

$$p_1^s = (E_1, 0, 0, q_1) \tag{44}$$

in a frame where $Q_1 = (0, 0, 0, \sqrt{-t_1})$ and parametrize it by the SO(2, 1) little-group element

$$g_1(\chi_1, \xi_1, \phi_1) = B_y(\chi_1)B_x(\xi_1)R_z(\phi_1), \tag{45}$$

which takes it to another frame where Q_1 is solely along the z axis. Then by performing a y rotation by the θ_1 of Eq. (41) we reach F_3; that is,

$$p_1^{F_3} = R_y(\theta_1)g_1(\chi_1, \xi_1, \phi_1)p_1^s. \tag{46}$$

Further, it is easy to see that

$$E_1 = (m^2 - \tfrac{1}{4}t_1)^{1/2}, \quad q_1 = \tfrac{1}{2}(-t_1)^{1/2}, \tag{47}$$

where m is again chosen as the common mass for all external spinless particles.

In exactly the same fashion we parametrize p_3 and p_2,

$$p_2^{F_3} = R_y(\theta_2)g_2(\chi_2, \xi_2, \phi_2)p_2^s \tag{48}$$

and

$$p_3^{F_3} = g_3(\chi_3, \xi_3, \phi_3)p_3^s, \tag{49}$$

with

$$p_i^s = (E_i, 0, 0, q_i) \tag{50}$$

and

$$E_i = (m^2 - \tfrac{1}{4}t_i)^{1/2}, \quad q_i = \tfrac{1}{2}(-t_i)^{1/2}. \tag{51}$$

Since we have spinless external particles, there is no dependence of A_6 on the z-rotation angles ϕ_i. That leaves us with nine variables: t_i, χ_i, and ξ_i, $(i = 1, 2, 3)$ one of which is redundant. Writing out A_6 as a function of the momenta

$$A_6(R_y(\theta_1)B_y(\chi_1)B_x(\xi_1)p_1^s, B_y(\chi_3)B_x(\xi_1)p_3^s, R_y(\theta_2)B_y(\chi_2)B_x(\xi_2)p_2^s), \tag{52}$$

and remembering that it is invariant under y boosts, we see that A_6 depends only on $\chi_1 - \chi_3$ and $\chi_2 - \chi_3$ and not all three χ_i. This is precisely the analog of the restriction on A_5 in Sec. II to depend only on $\phi_1 - \phi_2$.

To go from frame F_3 to the regime where the Q_i are timelike we make a complex Lorentz transformation $B_z(\frac{1}{2}i\pi)$ and continue the t_i to positive values[13]:

$$B_z(\tfrac{1}{2}i\pi)Q_3^{F_3} = (\sqrt{t_3}, 0, 0, 0). \tag{53}$$

Under this Lorentz transformation the operations B_y, B_x, R_y necessary for the kinematics in case (2) behave as

$$B_z(-\tfrac{1}{2}i\pi)B_y(\chi)B_z(\tfrac{1}{2}i\pi) = R_x(-i\chi), \tag{54}$$

$$B_z(-\tfrac{1}{2}i\pi)B_x(\xi)B_z(\tfrac{1}{2}i\pi) = R_y(i\xi), \tag{55}$$

and

$$B_z(-\tfrac{1}{2}i\pi)R_y(\theta)B_z(\tfrac{1}{2}i\pi) = B_x(i\theta). \tag{56}$$

This suggests that x-rotation angles play the role of azimuthal angles in the multiple O(3) partial-wave analysis we are now ready to carry out.

With these hints we parametrize the six-point amplitude of Fig. 3 as follows: Work in the frame F_C where

$$Q_C = (\sqrt{t_C}, 0, 0, 0), \tag{57}$$

$$Q_A = B_x(\theta_A)(\sqrt{t_A}, 0, 0, 0), \tag{58}$$

and

$$Q_B = B_x(\theta_B)(\sqrt{t_B}, 0, 0, 0), \tag{59}$$

with

$$\sinh\theta_A = \frac{[\Delta(t_A, t_B, t_C)]^{1/2}}{2\sqrt{t_A}\sqrt{t_C}} \tag{60}$$

and

$$\sinh\theta_B = -\frac{[\Delta(t_A, t_B, t_C)]^{1/2}}{2\sqrt{t_B}\sqrt{t_C}}\ . \tag{61}$$

We parametrize p_A by an O(3) little-group element

$$g_A(\chi_A, \xi_A, \phi_A) = R_x(\chi_A)R_y(\xi_A)R_z(\phi_A), \tag{62}$$

which takes it from a standard vector

$$p_A^s = (E_A, 0, 0, q_A) \tag{63}$$

in a frame where $Q_A = (\sqrt{t_A}, 0, 0, 0)$ to another frame where Q_A is purely along the time axis. To take it to F_C we apply $B_x(\theta_A)$ so

$$p_A^{F_C} = B_x(\theta_A)R_x(\chi_A)R_y(\xi_A)R_z(\phi_A)p_A^s. \tag{64}$$

Clearly we do the same for p_B and p_C finding

$$p_B^{F_C} = B_x(\theta_B)R_x(\chi_B)R_y(\xi_B)R_z(\phi_B)p_B^s \tag{65}$$

and

$$p_C^{F_C} = R_x(\chi_C)R_y(\xi_C)R_z(\phi_C)p_C^s\ , \tag{66}$$

with

$$E_j = \tfrac{1}{2}(t_j)^{1/2}, \quad q_j = (\tfrac{1}{4}t_j - m^2)^{1/2}, \ j = A, B, C\ . \tag{67}$$

Once again the spinlessness of the external particles tells us that A_6 does not depend on the ϕ_j, and the invariance of A_6 under x rotations reminds us that A_6 depends on the eight variables: t_j, ξ_j for $j = A, B, C$ and $\chi_A - \chi_C$ and $\chi_B - \chi_C$.

With these kinematics in hand we can carry out the triple O(3) partial-wave analysis on A_6. The only tricky point is to relate the rotation functions in the basis where J_x is diagonalized on the left and J_z on the right, which is natural for the O(3) labeling $R_x(\chi)R_y(\xi)R_z(\phi)$, to the usual $R_zR_yR_z$ functions. By noting that $R_y(-\frac{1}{2}\pi)R_z(\chi)R_y(\frac{1}{2}\pi) = R_x(\chi)$ we can give the partial-wave expansion

$$\begin{aligned}A_6(&B_x(\theta_A)R_x(\chi_A)R_y(\xi_A)p_A^s, R_x(\chi_C)R_y(\xi_C)p_C^s, B_x(\theta_B)R_x(\chi_B)R_y(\xi_B)p_B^s) \\ &= \prod_{j=A,B,C}\sum_{J_j=0}^{\infty}\sum_{\lambda_j=-J_j}^{+J_j}(2J_j+1)\frac{\Gamma(\lambda_j+J_j+1)}{\Gamma(\lambda_j-J_j+1)}\tilde{P}_{J_j}^{\lambda_j}(\cos(\xi_j+\tfrac{1}{2}\pi)) \\ &\quad\times\exp(i\lambda_A\chi_A - i\lambda_B\chi_B - i\lambda_C\chi_C)M(J_A, \lambda_A, J_B, \lambda_B, J_C, \lambda_C, t_A, t_B, t_C)\,.\end{aligned} \tag{68}$$

In order that A_6 depend only on the differences $\chi_A - \chi_C$ and $\chi_B - \chi_C$, we require $\lambda_C = \lambda_A - \lambda_B$. Remembering from our discussion of A_5 that we must, even beyond considerations of signature, continue separately positive and negative helicities, we divide the λ_j sums in (68) into six regions:

I. $\lambda_A \geq 0, \lambda_B \geq 0, \lambda_C = \lambda_A - \lambda_B \geq 0$; (69)

II. $\lambda_A \geq 0, \lambda_B \geq 0, \lambda_C < 0$; (70)

III. $\lambda_A \geq 0, \lambda_B < 0, \lambda_C \geq 0$; (71)

IV. $\lambda_A < 0, \lambda_B < 0, \lambda_C < 0$; (72)

V. $\lambda_A < 0, \lambda_B < 0, \lambda_C \geq 0$; (73)

and

VI. $\lambda_A < 0, \lambda_B \geq 0, \lambda_C < 0$. (74)

We must define different amplitudes to be continued into the right-half or left-half λ plane for each helicity. We will designate by a subscript R_j or L_j the amplitude continued in the right- or left-half plane for each $j = A, B, C$. Thus the quantity $M_{R_AR_BR_C}$ will be continued into the right-hand plane of λ_A, λ_B, and λ_C; its definition in terms of A_6 is

$$\begin{aligned}M_{R_AR_BR_C}(J_A\lambda_A, J_B\,\lambda_B, J_C\,\lambda_C)\delta_{\lambda_A-\lambda_B,\lambda_C} &= \tfrac{1}{2}\int_{-1}^{+1}dx_A\tilde{P}_{J_A}^{\lambda_A}(x_A)\tfrac{1}{2}\int_{-1}^{+1}dx_B\tilde{P}_{J_B}^{\lambda_B}(x_B)\tfrac{1}{2}\int_{-1}^{+1}dx_C\tilde{P}_{J_C}^{\lambda_C}(x_C) \\ &\quad\times\int_0^{2\pi}\frac{d\chi_A}{2\pi}e^{-i\lambda_A\chi_A}\int_0^{2\pi}\frac{d\chi_B}{2\pi}e^{+i\lambda_B\chi_B}A_6(x_A, x_B, x_C, \chi_A, \chi_B, \chi_C = 0)\,,\end{aligned} \tag{75}$$

where $x_j = \cos(\xi_j + \frac{1}{2}\pi)$. Amplitudes to be defined in left-half planes are defined using $\tilde{P}_J^{-\lambda}$ as in (17) and (28).

We may follow all the steps in the discussion of A_5 above to write Sommerfeld-Watson transforms for each of the six regions. Since there seems to be no particular point in writing out six such long formulas, we will give the transform for region I only, leaving the others to the patient reader. We choose to eliminate λ_C in the writing, and find

$$A_6^{\text{Region I}}(x_A, x_B, x_C, \chi_A - \chi_C, \chi_B - \chi_C)$$
$$= -\int_{C\lambda_A} \frac{d\lambda_B}{2i\sin\pi\lambda_A} \int_{C\lambda_B} \frac{d\lambda_B}{2i\sin\pi\lambda_B} \prod_{j=A}^{C} \int_{C_{J_j}} \frac{dJ_j}{2\pi i}(2J_j+1)\Gamma(\lambda_j - J_j)\Gamma(\lambda_j + J_j + 1)\tilde{P}_{J_j}^{\lambda_j j}(-x_j)$$
$$\times[-e^{i(\chi_A-\chi_C)}]^{\lambda_A}[-e^{-i(\chi_B-\chi_C)}]^{\lambda_B} M_{R_A R_B R_C}(J_A\lambda_A, J_B\lambda_B, J_C\lambda_A - \lambda_B), \tag{76}$$

where λ_C is to be set equal to $\lambda_A - \lambda_B$ in all expressions, and $x_j = \cos(\xi_j + \frac{1}{2}\pi) = -\sin\xi_j$.

The partial-wave coefficients M_R and M_L are taken to have only dynamical poles or cuts in the J_j and to have no further singularities in the λ_j. With this assumption, the asymptotic behavior in, say, χ_A is governed by the singularities in λ_A which reflect, via the Γ functions in (76) and its companions for the other regions, the singularities in J_A, J_B, and J_C.

Taking this example further we find that for singularities in J_j at $\alpha_j(t_j) > 0$, there are two terms in the leading asymptotic behavior of A_6 as $\chi_A \to -i\infty$ with x_A, x_B, x_C, t_A, t_B, t_C, and χ_B held fixed; set $\chi_C = 0$. These two terms come from poles in λ_A at $\alpha_A(t_A)$ or at $\alpha_B(t_B) + \alpha_C(t_C)$. So

$$A_6(x_j, t_j, \chi_A, \chi_B) \underset{\substack{\chi_A \to -i\infty \\ x_j, t_j, \chi_B \text{ fixed}}}{\sim} (-e^{i\chi_A})^{\alpha_A(t_A)} F_1(x_j, t_j, \chi_B) + (-e^{i\chi_A})^{\alpha_B(t_B)+\alpha_C(t_C)} F_2(x_j, t_j, \chi_B). \tag{77}$$

The identification of two terms in the asymptotic behavior in an azimuthal angle for A_6 has been made in a paper by Low and coworkers.[14] Their definition of poles in helicity differs somewhat from ours, and their method of derivation is certainly remarkably dissimilar; however, their result is equivalent to (77).

In the limit $\chi_A \to -i\infty$ with χ_B, t_j, and ξ_j fixed, the plane $p_A p_A'$ is "moving away" from the cluster of four momenta p_B, p_B', p_C, and p_C'. Since the inner product of p_A or p_A' with any of these vectors is becoming infinite, while any of the inner products among these vectors is remaining finite, one may properly inquire why the tree graph of Fig. 3 should be considered in this limit. That is, why not take a tree configuration where p_B and p_C and p_C' and p_B', say, define a set of pairs for a partial-wave expansion and thus encounter $(-e^{i\chi_A})^{\alpha((p_B-p_C)^2)}$ in the limit. The key to the answer is that as $\chi_A \to -i\infty$, χ_B, ξ_j, t_j fixed, the quantities $p_A\cdot(p_B - p_B')$, $p_A'\cdot(p_B - p_B')$, $p_A\cdot(p_C - p_C')$, and $p_A'\cdot(p_C - p_C')$ remain fixed. This requirement among dot products singles out the pairing $(p_B p_B')$, $(p_C p_C')$ of Fig. 3.

We include just a few words about the results in this section before we proceed to the single-particle inclusive process. The limit (77) is the same as the limit in Eq. (30) if we choose $\alpha_2(t_B) = 0$; that is if we take the residue of A_6 at a spin-zero pole in t_B. We know that this limit on A_5 does not occur in a physical region of A_5 since once we take $t_B = m^2$ to reach the spin-zero pole, we have $\Delta(t_A, t_B, t_C) > 0$ and cannot make it negative by continuing in t_A and t_C to negative values. When we let t_A, t_B, and t_C be continued to negative values such that $\Delta(t_A, t_B, t_C) > 0$, then by our construction, the Sommerfeld-Watson transform of (76) etc. is useful for yielding the asymptotic behavior in the azimuthal angles χ_A and χ_B continued to the y-boost angles χ_1 and χ_2 encountered in (52). *The criterion for an azimuthal-angle asymptotic limit to occur in a physical region of a multiparticle S-matrix element is that for some tree-graph configuration there be a vertex of three spacelike momenta Q_1, Q_2, and Q_3 such that $\Delta(Q_1^2, Q_2^2, Q_3^2) < 0$.* The asymptotic limit of an azimuthal angle associated with this vertex is governed by singularities in the conjugate helicity as given by multiple Sommerfeld-Watson transforms such as (76).

Finally, let us mention an elementary reason why there are two terms in the leading behavior in $\chi_A \to -i\infty$ as in (77). If we consider the vertex corresponding to a particle of mass $\sqrt{t_A}$, spin J_A decaying at rest to $\sqrt{t_B}, J_B + \sqrt{t_C}, J_C$ moving along the

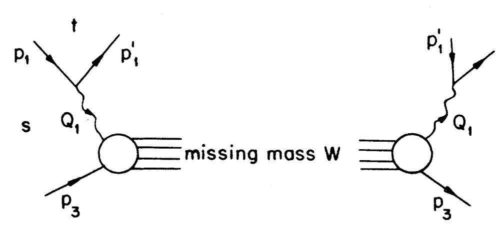

FIG. 4. This shows the single-particle inclusive process as the W^2 discontinuity of A_6 at $t_1 = t_2 = t$, $t_3 = 0$. The limit $s \to \infty$, t, W^2 fixed for this cross section involves only a y-boost angle becoming infinite.

z axis, then the helicity λ_A is restricted to be less than the smaller of J_A or $J_B + J_C$ by conservation of angular momentum. The Γ functions in the decay matrix element which yield this restriction, when continued in helicity and angular momentum, result in precisely the two terms of (77). Said in other words, one of the lessons of the multiple partial-wave analyses is that *the singularities in complex helicity are bounded by the maximum sense values allowed to ordinary helicity.*

IV. AZIMUTHAL-ANGLE LIMITS IN THE SINGLE-PARTICLE INCLUSIVE REACTION

We now propose to take the formalism we have built up for finding the location of helicity singularities in Sommerfeld-Watson transforms of multiparticle amplitudes and apply it to an analysis of the single-particle inclusive distribution for $p_1 + p_3 \to p_1' +$anything. The regime of interest to us will be when the initial energy $s=(p_1+p_3)^2 \to \infty$ while the momentum transfer $t=(p_1-p_1')^2$ and the missing mass $W^2=(p_1+p_3-p_1')^2$ are held fixed. We will demonstrate first that this limit of the forward A_6 can only be reached by taking an azimuthal angle to infinity. As usual we will encounter from each of the six regions of the Sommerfeld-Watson transform two terms in the asymptotic behavior of A_6 in this limit. One term in each limit will be shown to have no dependence on the missing mass, and thus only one of the possible terms from each region will contribute to the inclusive cross section which is extracted from the forward A_6 by taking the absorptive part in W^2. We will show that the term which survives in the $s\to\infty$, t-, W^2-fixed limit describes a Reggeon-particle absorptive part with maximum helicity flip in the "crossed" (with respect to W^2) channel.[10,15]

In order to discuss the kinematics of A_6 appropriate to the inclusive reaction we must take $t_1=t_2=t$ and then let $t_3\to 0$ from below. At the same time we must set $p_1'=p_2$, and it will follow that $p_3'=p_3$ and $p_2'=p_1$ after that.

We choose $\chi_3=0$, as we always may, and letting $t_1=t_2=t$, the y-rotation angles (41) and (42) which orient Q_1 and Q_2 in F_3 become

$$\sin\theta_1=\left(1-\frac{t_3}{4t}\right)^{1/2}=-\sin\theta_2\,, \tag{78}$$

so

$$\theta_1=-\theta_2=\theta\,. \tag{79}$$

Writing out the vectors p_1' and p_2 we see

$$p_1'=(E\cosh\xi_1\cosh\chi_1,\, E\sinh\xi_1\cos\theta-q\sin\theta,\, E\cosh\xi_1\sinh\chi_1,\, -q\cos\theta-E\sinh\xi_1\sin\theta) \tag{80}$$

and

$$p_2=(E\cosh\xi_2\cosh\chi_2,\, E\sinh\xi_2\cos\theta-q\sin\theta,\, E\cosh\xi_2\sinh\chi_2,\, q\cos\theta+E\sinh\xi_2\sin\theta)\,, \tag{81}$$

where $E=(m^2-\frac{1}{4}t)^{1/2}$ and $q=(-\frac{1}{4}t)^{1/2}$. Equating these vectors yields

$$\chi_1=\chi_2=\chi\,, \tag{82}$$

$$\cosh\xi_1=\cosh\xi_2\,, \tag{83}$$

$$\cos\theta\sinh\xi_1=\cos\theta\sinh\xi_2\,, \tag{84}$$

and

$$2q\cos\theta=-E\sin\theta(\sinh\xi_1+\sinh\xi_2) \tag{85}$$

as $t_3\to 0$, $\cos\theta$ goes to 0 and the requirement that (84) hold on the way to the limit [that is, (84) and its derivative with respect to θ at $\theta=0$] means

$$\xi_1=\xi_2=\xi \tag{86}$$

and

$$\sinh\xi=-\frac{q}{E}\cot\xi\,, \tag{87}$$

which implies that in the limit $t_3\to 0$, $\xi\to 0$.

We have three variables left in the forward limit of A_6. They are χ, an azimuthal boost angle, ξ_3, a polar boost angle, and t, a momentum transfer.

The four-vectors p_i and Q_i have become

$$Q_1=(0,\sqrt{-t},0,0)\,, \tag{88}$$

$$Q_2=(0,-\sqrt{-t},0,0)\,, \tag{89}$$

$$Q_3=(0,0,0,0)\,, \tag{90}$$

$$p_1=(E\cosh\chi,\, \tfrac{1}{2}\sqrt{-t},\, E\sinh\chi,\, 0)\,, \tag{91}$$

$$p_2=(E\cosh\chi,\, \tfrac{1}{2}-\sqrt{-t},\, E\sinh\chi,\, 0)\,, \tag{92}$$

and

$$p_3=(m\cosh\xi_3,\, m\sinh\xi_3,\, 0,\, 0)\,. \tag{93}$$

The invariants s and W^2 may be expressed in terms of χ, t, and ξ_3 as

$$\begin{aligned} s&=(p_1+p_3)^2\\ &=2m^2+2mE\cosh\chi\cosh\xi_3-m\sqrt{-t}\,\sinh\xi_3\,, \end{aligned} \tag{94}$$

and

$$\begin{aligned} W^2&=(p_1+p_3-p_1')^2\\ &=m^2+t-2m\sqrt{-t}\,\sinh\xi_3\,. \end{aligned} \tag{95}$$

The limit $s \to \infty$, t, W^2 fixed can clearly be achieved only by $\chi \to -\infty$, ξ_3, t fixed; that is, it is an azimuthal angle limit in which no polar angle ξ becomes large. We have given, if not established, a rule in the discussion around Eq. (30) that one must employ the partial-wave analysis dictated by one's choice of polar and azimuthal angles to locate the singularities in helicity in the Sommerfeld-Watson transform which yield the azimuthal angle limit being considered.

We return, therefore, to the O(3) analysis of A_6 for the tree graph of Fig. 3. The following identification of variables for the forward A_6 is made with the help of (54) and (55):

$$\chi_A = \chi_B = i\chi; \quad \chi_C = 0, \tag{96}$$

$$\xi_A = \xi_B = 0; \quad \xi_C = -i\xi_3 \tag{97}$$

and

$$t_A = t_B = -t; \quad t_3 = t_C = 0. \tag{98}$$

We know that in the Sommerfeld-Watson transform of each of the six regions (69)–(74) the singularities in λ reflect, via the explicit kinematic Γ functions, singularities in the J's. For the case we now adopt of poles in J_j at $\alpha_j(t_j) > 0$, the region which gives the leading asymptotic behavior as $\chi \to -\infty$ is region III. The partial-wave amplitude must be continued into the right-half plane for λ_A and λ_C, and the left, for λ_B. Furthermore, since $\chi_A = \chi_B = i\chi$, the asymptotic behavior in χ is governed by $\lambda_C = \lambda_A - \lambda_B$, so let us eliminate λ_B in making the Sommerfeld-Watson transform which reads with forward kinematics

$$\begin{aligned} A_6^{\text{Region III}} = -\int_{C_{\lambda_A}} \frac{d\lambda_A}{2i\sin\pi\lambda_A} \int \frac{d\lambda_C}{2i\sin\pi\lambda_C} \prod_{j=A}^{C} \int \frac{dJ_j}{2\pi i} & (2J_j+1)\Gamma(\lambda_A - J_A)\Gamma(\lambda_A + J_A + 1)\Gamma(\lambda_C - J_C)\Gamma(\lambda_C + J_C + 1) \\ & \times \Gamma(\lambda_C - \lambda_A - J_B) \\ & \times \Gamma(\lambda_C - \lambda_A + J_B + 1)\tilde{P}_{J_A}^{\lambda_A}(0)\tilde{P}_{J_B}^{-\lambda_B}(0)\tilde{P}_{J_C}^{\lambda_C}(\sin\xi_C) \\ & \times (-e^{i\chi_A})^{\lambda_C} M_{R_A L_B R_C}(J_A, \lambda_A; J_B, \lambda_A - \lambda_C; J_C, \lambda_C). \end{aligned} \tag{99}$$

We have argued that the partial-wave coefficients M defined in this manner contain only poles and cuts in the J_j (dynamical singularities) and are regular in the λ_j.

In the limit $\chi \to -\infty$ there are two contributions to the asymptotic behavior of $A_6^{\text{Region III}}$. The first comes from the λ_C pole in $\Gamma(\lambda_C - J_C)$ which is just the $J_C = \alpha_C(0)$ pole transferred to the λ_C plane. This contribution has its ξ_C behavior explicit since we must do the J_C integral around the $\alpha_C(0)$ pole. The second leading contribution comes from the pair of Γ functions $\Gamma(\lambda_A - J_A)\Gamma(\lambda_C - \lambda_A - J_B)$ and occurs at $\lambda_C = \alpha_A(t_A) + \alpha_B(t_B)$ with $\lambda_A = \alpha_A(t_A)$, $\lambda_B = -\alpha_B(t_B)$. The ξ_3 dependence is not specified here. We then read from (99)

$$A_6^{\text{Region III}}(\chi, \xi_3, t) \underset{\substack{\chi \to -\infty \\ \xi_3, t \text{ fixed}}}{\sim} (-e^{-\chi})^{\alpha_C(0)}\tilde{P}_{\alpha_C(0)}^{\alpha_C(0)}(-i\sinh\xi_3)F_1(t) + (-e^{-\chi})^{\alpha_A(t)+\alpha_B(t)}F_2(\xi_3, t, \lambda_A = \alpha_A, \lambda_B = -\alpha_B), \tag{100}$$

where F_1 and F_2 are unknown functions of the specified variables. From Eq. (10) for $\alpha_C(0) > 0$ we find

$$\tilde{P}_{\alpha_C(0)}^{\alpha_C(0)}(-i\sinh\xi_3) = (\cosh\xi_3)^{\alpha_C(0)}/2^{\alpha_C(0)}\Gamma(\alpha_C(0)+1), \tag{101}$$

so noting

$$e^{-\chi} = s/mE\cosh\xi_3 \tag{102}$$

in the limit we have taken, we learn that the *first term of (100) has no dependence on* ξ_3. Changing over to s, W^2, and t we may rewrite the limit (100) as

$$A_6^{\text{Region III}}(s, W^2, t) \underset{\substack{s \to \infty \\ W^2, t \text{ fixed}}}{\sim} (-s)^{\alpha_C(0)}\hat{F}_1(t)(-s)^{\alpha_A(t)+\alpha_B(t)} \frac{\hat{F}_2(W^2, t, \lambda_A = \alpha_A(t), \lambda_B = -\alpha_B(t))}{[+\Delta^{1/2}(W^2, t, m^2)]^{\alpha_A(t)+\alpha_B(t)}}. \tag{103}$$

When we take the absorptive part in W^2 of this formula to extract the contribution to the inclusive cross section from Region III, we pick up

$$(-s)^{\alpha_A(t)+\alpha_B(t)}\text{Abs}_{W^2}\left\{\frac{F_2(W^2, t, \lambda_A = \alpha_A, \lambda_B = -\alpha_B)}{[\Delta^{1/2}(W^2, t, m^2)]^{\alpha_A+\alpha_B}}\right\}. \tag{104}$$

Every other contribution to A_6 from the other five regions is either of the form $(-s)^{\pm\alpha_C(0)}F(t)$ or it has lower powers of s than (104). We thus reach the important conclusion that the leading contribution to the inclusive cross section in the limit $s \to \infty$, W^2, t fixed (as depicted in Fig. 4) is just (104). Since $\hat{F}_2$ is proportional to the t_3-channel amplitude for Q_1 (helicity α_A) + Q_2 (helicity $-\alpha_B$) $\to p_3+(-p_3)$, this is precisely what we expect.[10]

Our reaching this result lends strong support to three of the basic steps we have been carrying out: (1) Our assumption that the partial-wave coefficients such as $M_{R_A L_B R_C}$ contain only dynamical singularities in the J_j; (2) our argument that the helicity singularities in a multiple Sommerfeld-Watson transform are, therefore, to be read off from the kinematic Γ functions; and (3) our rule that the partial-wave analysis for which one defines the polar and azimuthal angles ξ and χ is the one to employ when the azimuthal angle becomes asymptotic even though no polar angles become large.

V. DISCUSSION AND CONCLUSIONS

By considering the five- and six-point amplitudes in some detail we have developed a set of operational instructions for locating the singularities in helicity appearing in multiple Sommerfeld-Watson transformations of many-particle amplitudes. These instructions have just been repeated at the end of Sec. IV where we also argued that the application of our procedures to the single-particle inclusive process yields the proper answer.

We have also argued that an azimuthal-angle asymptotic limit of a many-particle amplitude can occur in a physical region of that amplitude when one encounters in some tree graph a vertex where three spacelike momenta Q_1, Q_2, and Q_3 meet with $\Delta(Q_1^2, Q_2^2, Q_3^2)<0$. As we showed, following Ref. 3, in this kinematic configuration an azimuthal z-rotation angle which is bounded in physical regions is replaced by a y-boost angle, χ, which may lie anywhere along the real line.

In the two examples, A_5 and A_6, we have treated, the covariance of a vertex with which the azimuthal angle was associated leads to a coupling of the singularities in the helicities entering the vertex. One may ask whether in higher-point functions somehow helicity singularities are not passed from one end of the process to the other, therefore, making the singularity structure in helicity unspeakably complicated? We can see from the tree-graph configuration for A_8 given in Fig. 5 that this will not occur, and any helicity communicates only with the neighboring helicities at any vertex. In the partial-wave expansion of the graph in Fig. 5, one encounters the product of rotation functions

$$d^{J_1}_{0\lambda_1}(\theta_1)d^{J_2}_{0\lambda_2}(\theta_2)d^{J_5}_{\lambda_5,\lambda_5'}(\theta_5)d^{J_3}_{\lambda_3,0}(\theta_3)d^{J_4}_{\lambda_4,0}(\theta_4)\,, \qquad (105)$$

where J_i and θ_i are the angular momentum and polar angle associated with Q_i. Covariance at the vertices I and II requires $\lambda_5=\lambda_1-\lambda_2$, and $\lambda_5'=\lambda_4-\lambda_3$, respectively. However, except in the very special configuration where $\theta_5=0$ (that is, forward internal Reggeon scattering), there is no coupling of λ_1, say, with λ_3 or λ_4.

There is an amusing point here, however, for the analytic structure in λ_1, say, will reflect through the familiar kinematic Γ functions the singularity structure in $J_1(Q_1^2)$, $J_2(Q_2^2)$, *and* $J_5(Q_5^2)$. Since the line carrying Q_5 or Q_2 could have been composed in a variety of ways from the p_2, p_2', p_3, p_3', p_4, and p_4', and since the pole and cut structure in J_2 or J_5 may depend on this, we see that the analytic structure in λ_1 may vary with the tree graph considered. An example of this occurs when we have internal quantum numbers. Suppose we choose the charges of the spinless particles, call them pions as in Figs. 5 and 6, to be as in those figures. In each case the charge carried by Q_1 is +1, with even G parity. In Fig. 5, Q_2 carries charge −1 with even G parity, and Q_5 carries charge 0 with G even. In Fig. 6, Q_4' carries charge 0 with G odd, and Q_5 carries charge +1, G odd. Clearly the singularities in λ_1 will be different for the two configurations.

A final set of remarks concerns signature, which we have avoided until now in order not to draw attention from the main issue of analytic structure in helicity. The arguments of Goddard and White[2] and Weis[6] tell

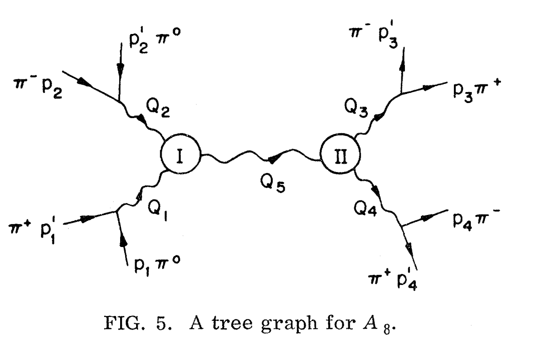

FIG. 5. A tree graph for A_8.

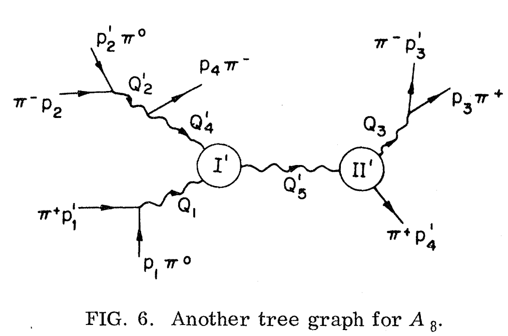

FIG. 6. Another tree graph for A_8.

us to regard the Sommerfeld-Watson transforms we have given as appropriate for signatured amplitudes which have only "right-hand cuts" in the variables $\cos\chi$ and $\cos\xi$, azimuthal and polar cosines. These arguments are exceedingly plausible but rest on an assumed analyticity structure in the cosines for multiparticle amplitudes. That analyticity could prove false.

Let us see, however, what consequence such an addition of signature will have for us. To identify the signatured amplitudes consider, for example, our A_5 as a function of $z=e^{i\phi}$ and $x=\cos\theta_1$. We wish to write a double dispersion relation in z and x and use the definition of M_R and M_L given in (16) and (17) to find partial-wave coefficients which can be continued in J and λ, the conjugate variables to θ_1 and ϕ. We will write the formulas for M_R only. Assuming then sufficient analyticity for A_5 we can use the dispersion relation[2]

$$A_5(x,z)=\int_{-\infty}^{+\infty}dx'dz'\,\frac{\rho(x',z')}{(x'-x)(z'-z)}\,, \tag{106}$$

to write

$$M_R(J,\lambda)=\int_1^\infty dz'\,z'^{-\lambda-1}\int_{x_0}^\infty dx'\,\tilde{Q}_J^\lambda(x') \times\{[\rho(x',z')-(-1)^{J-\lambda}\rho(-x',z')]+(-1)^{\lambda+1}[\rho(x',-z')-(-1)^{J-\lambda}\rho(-x',-z')]\}, \tag{107}$$

where we have started the x' integration at x_0 and noted the symmetry $\tilde{Q}_J^\lambda(-x)=-(-1)^{J-\lambda}\tilde{Q}_J^\lambda(x)$ for the "second-kind functions" $\tilde{Q}_J^\lambda(x)$. We see that it is appropriate to continue separately J even and odd and λ even and odd, so we define a helicity signature $\tau_\lambda=\pm$ and a usual J-signature $\tau_J=\pm$ and signatured partial-wave coefficients

$$M_R^{\tau_J\tau_\lambda}(J,\lambda)=\int_1^\infty dz'\,z'^{-\lambda-1}\int_{x_0}^\infty dx'\,\tilde{Q}_J^\lambda(x') \times\{[\rho(x',z')-\tau_J\tau_\lambda\rho(-x',z')]-\tau_\lambda[\rho(x',-z')-\tau_J\tau_\lambda\rho(-x',z')]\}, \tag{108}$$

which coincides with $M_R(J,\lambda)$ for J even (odd), $\tau_J=\pm1$ and λ even (odd), $\tau_\lambda=\pm1$.

We now perform a Sommerfeld-Watson transform on the sum

$$\sum_{\lambda=0}^\infty\sum_{J=\lambda}^\infty(2J+1)z^\lambda\tilde{P}_J^\lambda(x)M_R(J,\lambda)\,\frac{\Gamma(J-\lambda+1)}{\Gamma(J+\lambda+1)}\,, \tag{109}$$

which becomes

$$-\int_{C_\lambda}\frac{d\lambda}{2i\sin\pi\lambda}\int_{C_J}\frac{dJ}{2\pi i}\,\Gamma(\lambda-J)\Gamma(\lambda+J+1)(2J+1) \times\sum_{\tau_\lambda,\tau_J}\tfrac{1}{2}[(-z)^\lambda+\tau_\lambda(z)^\lambda]\,\tfrac{1}{2}[\tilde{P}_J^\lambda(-x)+\tau_J\tau_\lambda\tilde{P}_J^\lambda(x)]M_R^{\tau_J\tau_\lambda}(J,\lambda)\,. \tag{110}$$

The significant feature of (110) is that the *product* $\tau_J\tau_\lambda$ *of* J *and* λ *signatures appears. In every simultaneous* J *and* λ *continuation, then, we will find the product of* τ_J *and the associated* τ_λ.

If we apply these considerations to the inclusive process discussed in Sec. IV, we see that since $\xi_A=\xi_B=0$ we always encounter $\tilde{P}_J^\lambda(0)(1+\tau_J\tau_\lambda)$ for these and so must have $\tau_\lambda=\tau_J$ for A and B. Writing the full contribution to $A_6^{\text{Region III}}$ which has a W^2 discontinuity, we have in the limit $s\to\infty$, W^2, t fixed

$$A_6^{\text{Region III}}\underset{\substack{s\to\infty\\ W^2,t\text{ fixed}}}{\sim}\sum_{\tau_{J_A}\tau_{J_B}}s^{\alpha_A+\alpha_B}\frac{(\tau_{J_A}+e^{-i\pi\alpha_A})}{\sin\pi\alpha_A}\frac{(\tau_{J_B}+e^{+i\pi\alpha_B})}{\sin\pi\alpha_B}\frac{1}{[\sin(\xi_C+\frac{1}{2}\pi)]^{\alpha_A+\alpha_B}} \times\sum_{\tau_{J_C}}\int\frac{dJ_C}{2\pi i}\,\Gamma(\alpha_A+\alpha_B-J_C)\Gamma(\alpha_A+\alpha_B+J_C+1)(2J_C+1)M_{R_A L_B R_C}^{\tau_{J_A}\tau_{J_B}\tau_{J_C}}(J_C,t) \times[\tilde{P}_{J_C}^{\alpha_A+\alpha_B}(-\cos(\xi_C+\tfrac{1}{2}\pi))+\tau_{J_A}\tau_{J_B}\tau_{J_C}\tilde{P}_{J_C}^{\alpha_A+\alpha_B}(\cos(\xi_C+\tfrac{1}{2}\pi))]\,, \tag{111}$$

where $\alpha_A=\alpha_A(t)$, $\alpha_B=\alpha_B(t)$, and because $\lambda_C=\lambda_A-\lambda_B$ we have noted $\tau_{\lambda_C}=\tau_{\lambda_A}\tau_{\lambda_B}=\tau_{J_A}\tau_{J_B}$ here. Thus we see the result of Einhorn *et al.*[17] that the signature factor for J_C in this limit must be $\tau_{J_C}\tau_{J_A}\tau_{J_B}$. It is curious that because of the special kinematics of the inclusive reaction, one does not employ the full signature structure of the six-point function. In general, for a configuration which is not forward, we now see, that the signature product rule given in Ref. 17 does not apply. When in the J_C integral Eq. (111) a specific pole contribution from $J_C = \alpha_C(0)$ is picked up, the discontinuity in $\cos\xi_C(W^2)$ would contain the factor

$$\Gamma(\alpha_A + \alpha_B - \alpha_C)\sin\pi(\alpha_C - \alpha_A - \alpha_B) = \frac{\pi}{\Gamma(\alpha_C + 1 - \alpha_A - \alpha_B)}\,. \tag{112}$$

This is the famous factor ensuring the vanishing of the "triple-Pomeranchukon" vertex[10] and its presence in our treatment is a consequence of the dynamical assumption we made that $M(J_C, t)$ does not have any further singularities, let alone fixed poles.

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